Calculus of Several Real Variables
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Module No # 03
Lecture No # 12
Implicit Function Theorem

Welcome to the second lecture of the third week. We are going to talk about something about implicit function theorem. An implicit function theorem is very deep idea in mathematics used in many places used in optimization in differential geometry in solutions of equations many aspects of mathematics is trust by this very simple idea.

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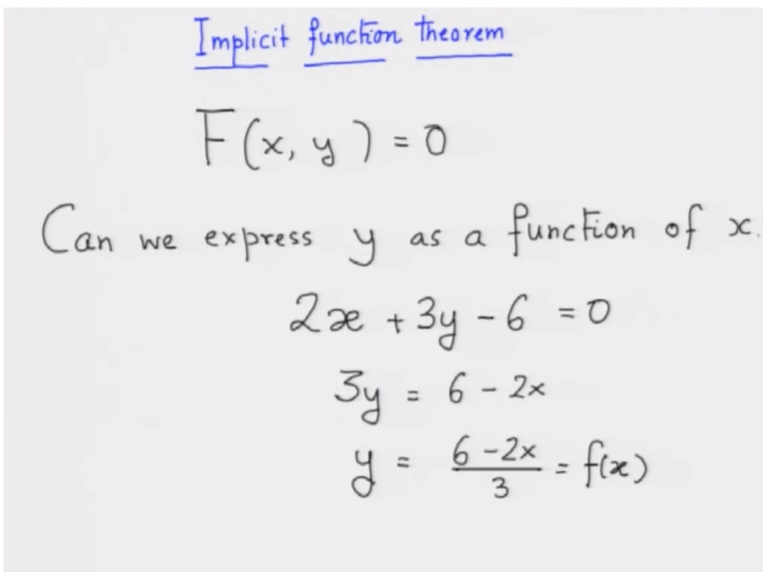

So given a function say of 2 variables suppose I was either equation written like this F of x y = 0 can we express y as a function of x that is the question we asked ourselves. Once I know that there is a basically F x y = 0 is the z = F x y so it is a curve on the plane 2 dimensional plane, and I am asking what does that curve represent. How is y related to x on that curve? For example, if I say if the equation is 2xy + x = 0 then you have x into 2y + 1 = 0 or 2y + 1 = 0 and there is no x.

So we could not represent x as a function of y here. Now for example if I write things like 2x + 3y - 6 = 0 then can I present y as a function of x. In this case it is much more simpler because here I can write 3y = 6 - 2x or y = 6 - 2x / 3. So y represents as a function of x.

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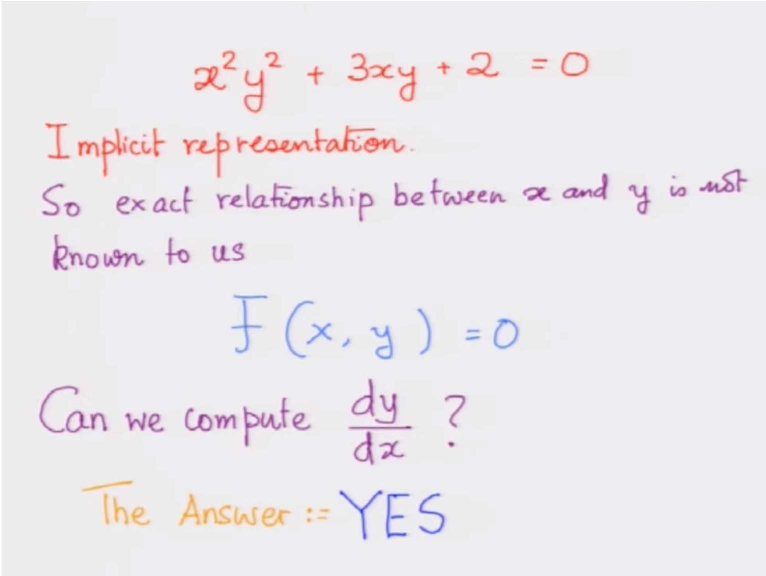

This whole issue is not always possible because for example if I write x y square into y square + 3xy + 2 = 0. Let me try to see what I can do with it. It is ok how do I take the x to the other side. Here I have y so if I would take y outside. So y you cannot even do that you can write it as some kind of product, but y would anyway remain here. So this kind of representation is called implicit representation of a function.

So function of one variable y = f x is implicitly represented by an equation in 2 variable. Implicit representation y is a function of x what I do not know how is it? I do not know the exact relation. So exact relationship between x and y is not clear to us known to us. Suppose I say ok you have a situation like this F of x y = 0 and you know that y must be some function of x but you cannot find the exact relationship.

Then what to do then but you know that the curve is very smooth and nice suppose it is differentiable at every point you can draw a tangent can you find dy/dx. So the question is can you compute this? The answer surprisingly is yes why yes, the answer is yes. So what is that tool which allow us to get that answer and that is the chain rule that you have learned about partial differentiation. Now basically you think x as a function of x itself and y is also a function of x.

Right an f is a function of x y. So what you will do you will apply the chain rule for partial derivatives. So you compute say del F del x del F basically you want to compute chain rule. Now you have f the function F x y right. Now you want to compute something like the derivative of F with respect to x.

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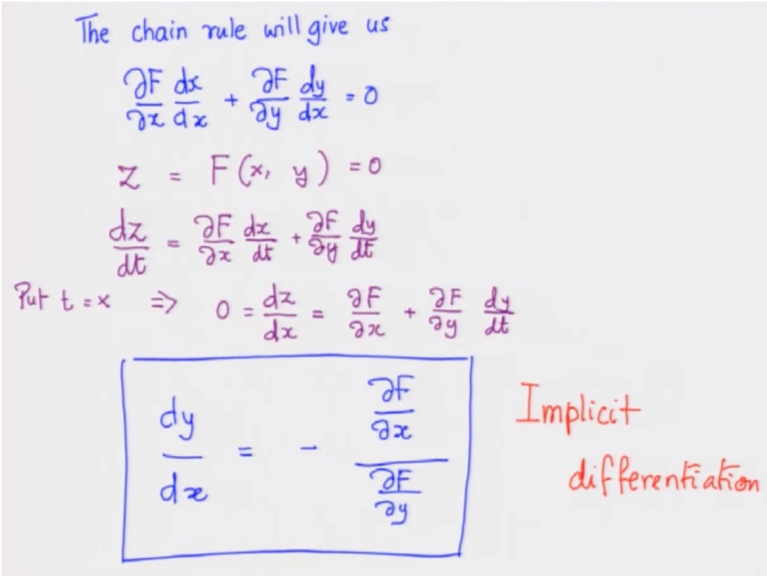

So as you know in the chain rule what you will do the chain rule will be applied here and the chain rule will give us the following. del F del x dx dx + del F del y dy dx and that is equal to 0. What have we actually done? How do we get such a chain rule? What have we actually done?

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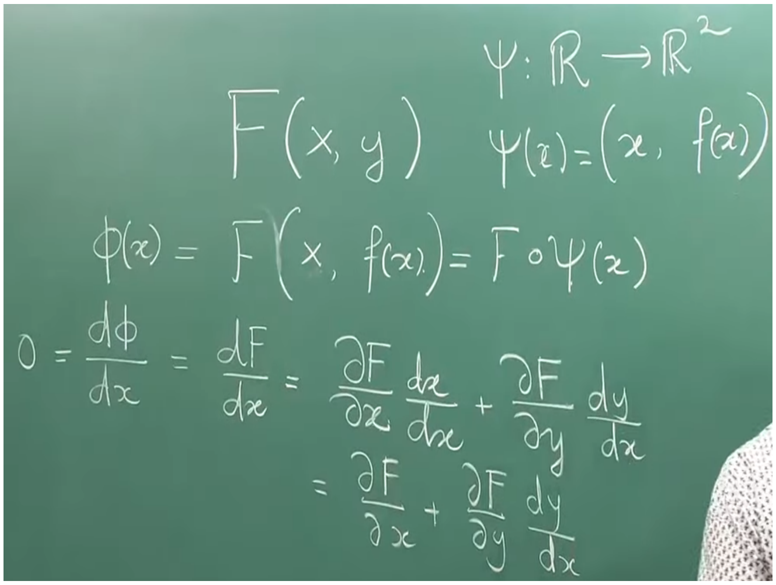

The idea comes from this. The idea is that if I have function of x y and suppose in my mind I write y as f x so this is actually a function of x Phi x. And then I am trying to find d Phi dx which is df dx but how will I do it? So as you have learned in the chain rule first have to now ok here the 2 entries this is the function of x and these are y which is also function of x. So what I will do I have to first take the derivative of this with respect to the x variables. And then derivative of x with respect to the x variable plus you know a partial derivative chain rule.

Basically, it is composition now so I can write this itself as a function. So f can be written as F composed Psi of x. The Psi of x is a function Psi of x is a function from R1 that is R to R2 and that is given as Psi of x = x and f x. That is the way to think of it and (()) (08:46) and put y in place of f x. So del F dx dx so del F so you know what to do so you have the gradient vector into the derivative of this right. Now sorry not the derivative the kind of my another (()) (09:11) of this. So basically, you have del F del y into dy dx is same as df dx. So once you know this and you can apply the chain rule. So Phi x takes R2 to and now f is taking you from R2 to R that is it.

So you see the chain rule that we have learnt and apply that chain rule you will just get this one. So what you will do now here is dx dx is 1 so you will get del F del x + but this del Phi what is del F del x because del this is this was equal to 0. So it was 0 F x y was 0. So it will be del F del y dy dx. So what you will finally get is dy dx = del F minus of del F del x / del F del y obviously for this to be meaningful we must have del F del y not equal to 0.

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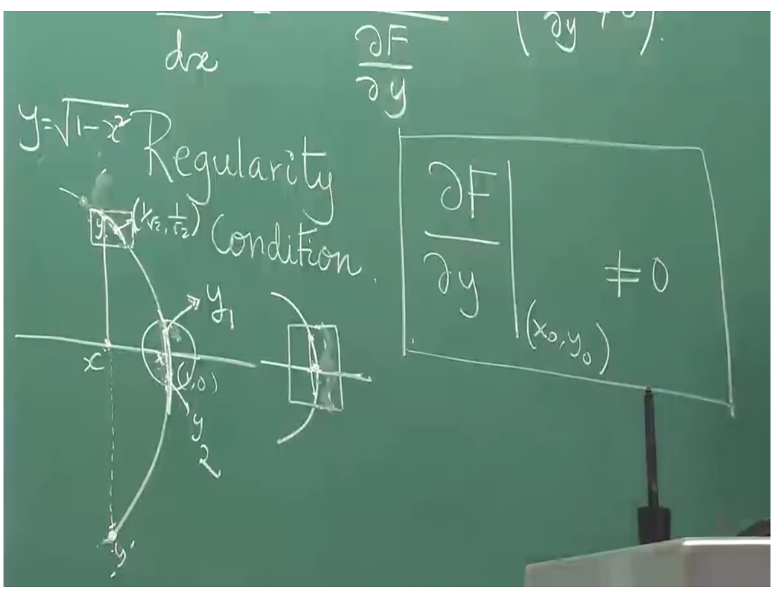

So I have tried to explain you how can you view it and how can you apply the chain rule. These are standard chain rule if you forgotten it let me write it again for you. Suppose you have the thing like z = F x y then for any t you have dz dt you have del F del x dx dt providing x is the function of t + del f del y into dy dt. This is the standard chain rule. In this case putting t = x and that would imply dz dx but which is 0 of course in this case because that F x y was given to be 0.

So dz dx would be equal to del F del x + del F del y dy dt. But rather you can apply the other more general chain rules and put Phi in that form and you can try this out whether you can get this result that the same result what we are writing. So this is the standard chain rule that absolutely the basic chain rule which we started with when we were studying chain rules and I have made a complete explanation of how this comes.

So we will not get into those details right now. But this thing what we have done this procedure of finding dy dx this procedure so when we write that dy dx is minus of del F del x. How is it del F del x is evaluated at that point x y. All are evaluated at the point x y I am not going to write every time that I am evaluating at the point x y. Now I think you are now sufficiently understanding the subject to understand that we are actually computing all these things at a point. Here I am calculating dy dx at a point x so corresponding to that x there is a y. So there at those x y I am computing this del F del x basically.

So del F del y so when I write this way of finding the derivative when I am not really knowing the exact link between y and x such a thing is called implicit differentiation. An example they take from the standard text obviously del F del y = 0 would create problems which will get into at a later stage. But let me just get along with the subject.

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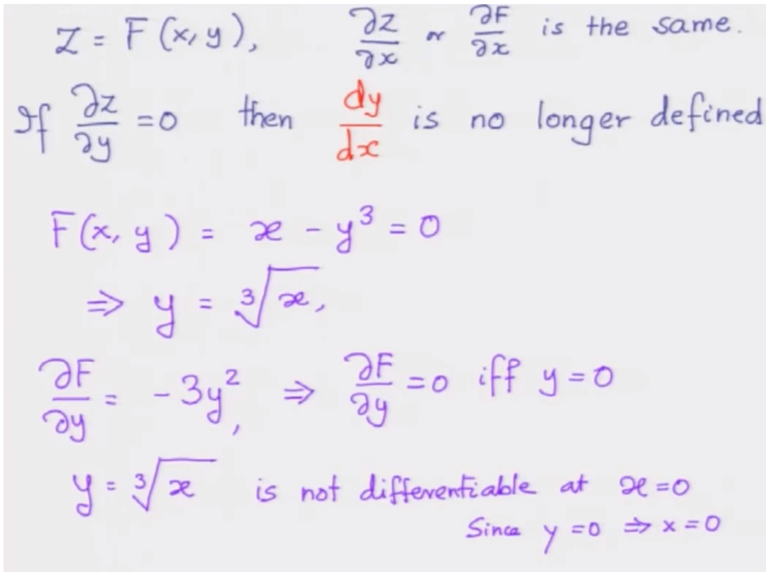

And for example, if I have F x y = 0 where F x y is given as e to the power x - y + x square - y - 1. Now I want to find a derivative of the form of y with respect to x. So what is dy dx? dy dx is minus of del F del x / del F del y. Now let us try to find may be say find dy dx because suppose if I put x and y = 0, 0 suppose I am expecting the function will satisfy when I put 0, 0 here it will be satisfy that is F x y will become being 0, 0.

So dy dx at x = 0 y = 0 which means that when I put 0 and in place of x and 0 in place of y F of x y must become 0. So let me check so F of 0, 0 must be equal to e to the power 0 - 0 + 0 - 0 - 1 to the power 0 is 1 so 1 + 0 - 0 - 1 is 0. So which means that 0, 0 is on a curve. Y is equal to sum f x some the curve whose function whose nature we do not know the exact relationship between y and x.

And now so it is absolutely logical to compute the derivative of y = dy dx at x = 0 y = 0. So what is del F del x in this case? So it is e to the power x - y + 2x. So computed at 0, 0 what do I get I will get I put 0, 0 to the power 0 + 2 into 0. So I will get 1. So del F del y is minus e to the power x - y (()) (16:50) because of the chain rule again here -1. So del F del y evaluated at x = 0 y = 0 is -e to the power 0 - 1 that is - 1 - 1 which is -2.

So dy dx = - 1 / - 2 = half. So this is the example that we have computed and this is an example through which we can show that whatever we have done is actually working. This is the way we can find out. Of course, there is a question what would happen if del F del y del z del y what we because z we have written z = F x y because we have written z = F x y. So you can call instead of del F del y you can call del Z del y means writing this or this is same it is the same.

Now important question is what would happen if del F del y del z del y is suppose del z I am in the position where del F del y or del z del y whatever if del z del y = 0. Then what happens to dy dx? dy dx stands undefine if then dy dx is no longer defined. So suppose I have F xy = x - y cube in this case the relation between y and x is clear. In this case I will have y = cube root of x. Let us see what is del F del y del F del y or del z del y whatever it is the same thing is -2y sorry -3y square.

So **it** it implies that del F del y = 0 if and the only if y = 0 ok so but at y =0 the function y = cube root of x and y cube = x does not have a derivative. You can check it out by drawing the graph it is not differentiable. So y is not differentiable at x = 0 y = cube root of x is not differentiable. So when I put y = 0 of course this is y cube = x. So when y is 0 x is 0 so y this is not differentiable at x y = 0, 0.

Basically, for the single value function at x = 0 this is not differentiable, and you note this that this is the standard form standard. So at it is not differentiable even implicitly at 0 0. So dy dx at 0 0 cannot be computed because in reality dy dx is not you cannot evaluate. It is not differentiable at x = 0 since y = 0 implies x = 0. So this is the kind of demonstration that this is the meaningful only when the lower one is a non-zero stuff dy dx. Now we will come to implicit function theorem. Implicit function theorem tells you ok the following

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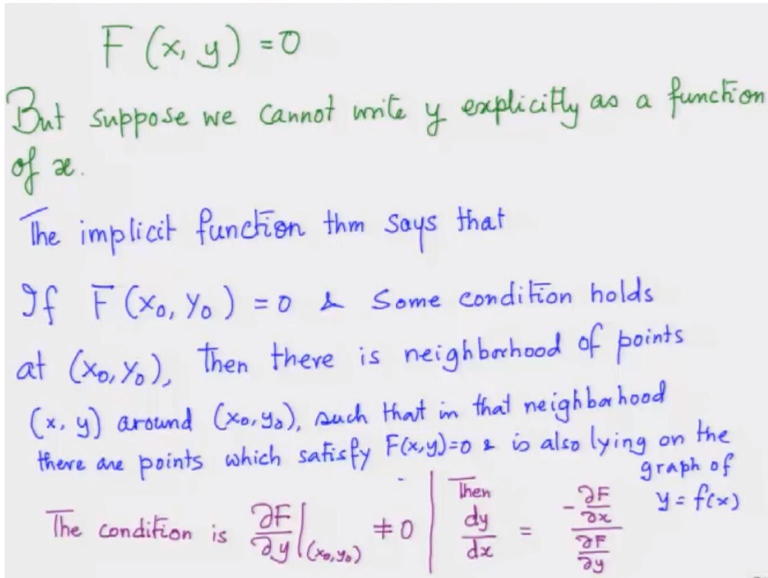

Let suppose F x y = 0 I am missing 2 dimension just for simplicity. But suppose we cannot write y explicitly as a function of x we cannot write. Once we cannot write what can we do what it says is the following. The implicit function theorem says that you under certain conditions if you have a point is if X naught Y naught such as F of X naught Y naught is 0 then under certain conditions in a small neighborhood of that point X naught Y naught that is in a disk containing x naught y naught is a disk in the in 2 dimensional space.

In a disk containing X naught Y naught you will you know you already know what is this ball that disc containing x y X naught Y naught this circular region completely. You will have you will around the around such a point if certain condition is satisfied you will see that y for all those x y within that neighborhood can be represented as a function of x. So locally y can be represented as a function of x.

It so under certain condition you can always write y locally as a function of x and that is very important note. Implicit function theorem says that if F of X naught Y naught is 0 and some condition and we will see what is that condition some condition holds at X naught Y naught. You might think that this whole thing is very dry, but I will show you immediately why it is not. Then there is a neighborhood or a set a points around X naught Y naught hood of points x y around X naught Y naught for which F of x y = 0 and y can be represented as y can be represented as y = f X.

So implicit theorem essentially tells you this simple fact that local representation is possible. But under what is that condition? The condition is that del F del y or del z del y at the point X naught Y naught if we evaluate that that cannot be 0. This is the regularity condition this is sometimes called the regularity condition I am just writing it here regularity condition. Absolute the once you have this condition that this cannot be 0 then so then you will have this kind of representation and not only can be represented as y = f X.

Then you can represent dy dx as minus of del F del x / del F del y. So the condition is del F del y evaluated at a point X naught Y naught cannot be equal to 0. Then dy dx can be written as for every such points as -del F del x / del F del y. At X naught why it does not say around what are the other points it should be at X naught Y naught it cannot be 0. But then because it is not 0 at X naught Y naught in the neighborhood it will not also be 0 basically otherwise this cannot be computed.

That is what it says it will gives you a lot more information. So if this condition is not satisfied then this thing may not be true around the point X naught Y naught which is lying on the curve y when a which is satisfying the equation F x y = 0.

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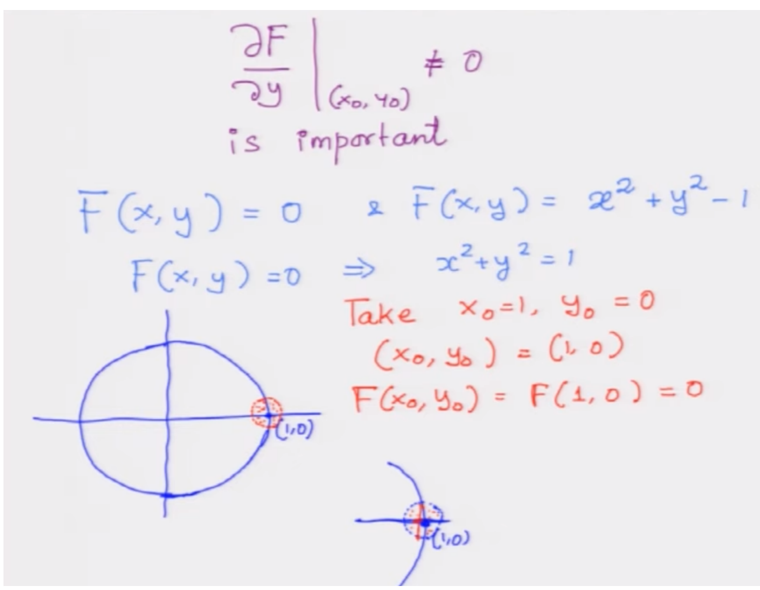

And that is what we are going to show through an example that this condition del F del x y this condition that del F del y evaluated at X naught Y naught = 0 is not equal to 0 is important let us see what happens if it fails. Now consider F of x y considered F of x y = 0 and F of x y = x as square + y square -1. So F x y = 0 implies x as square + y square = 1. So it is circle centered at the origin and having a radius of 1.

Now consider the point 1, 0 or any of the points here which cuts not this point this 2 point 1, 0 and -1, 0 whatever. So take X naught = 1 and y naught = 0. So it means I will now consider x naught y naught as 1, 0. And obviously F of x naught y naught in this case is F of 1, 0 which is 0. Now take a small neighborhood take a disk take all the point here in this disk. Now look at the picture there is magnified.

This is your 1, 0 point and you have taken a disc around 1, 0 point. So some condition holds at X naught then there is a neighborhood of points x y around x naught y naught such that in that neighborhood there are points which satisfy F x y = 0 and is also the graph is also and it is also lying on the graph there is a better way to say it lying on the graph of a function y = f x lying on the graph of y = f x.

So what it says is that so you will also find a neighborhood and in that neighborhood you will always find some points in that neighborhood which will lie on the graph of a function y = f x. But also satisfy F x y = 0. So you will be able to have some points on the curve that is on the curve y = f x and locally you can represent it as y = f x. So a part of the graph part of the curve can be represented as a part of the graph of some function y = f x.

For a but this is for example here if you come so these are lot of let these be the neighborhood points. So only this blue point in the part of the neighborhood the open neighborhood if you want to say only the blue points are the points which are lying on the graph F x y = 0. The blue point satisfies F of x y = 0 in this case. But remember in this case for any x that I take in this neighborhood any x y that you think is lying on the graph corresponding to the same x corresponding to the given x here there is more than one y there is 2 y’s.

Actually, that is what happens that this is the point that you have to understand that however small you make the neighborhood it does not matter. So if this is your 1, 0 and this is small neighborhood you have got does not matter how small you make. Look at this point x here and look at the corresponding y here. This x y is in this neighborhood as well as it is on the it is satisfying F x y = 0 because it is on the circle.

But corresponding to this x drawing is not very good that is any point here for example here for example take this point here any x here you see there is a symmetry actually I unable to draw it properly so here this X there is a corresponding y1 here and there is a corresponding y2 here y2 here and a corresponding y1 here. And this y2 and y1 that is basically at these points x y1 and x y2 both are line on this curve and the both are satisfying F of x y1 is 0 and F of x y2 is 0.

So corresponding to a given x there are two were 2 y's in this part. However small you make the neighborhood it does not matter this will always occur. So if you take a y you can actually have a symmetrical y corresponding to a given x. But if you come here for example say take any point here far off from here on this boundary so you take a point like say point like here 1 / root 2 then take a neighborhood taken square this term does not matter.

You see take any point here y here it corresponds to 1x. So any other y that you have the corresponding symmetric y that you have does not lie in this neighborhood. Here what is happening this if you take y1 here you can symmetrically because you have taken circular neighborhood even if you take a square neighborhood because if 1, 0 is like this if you take a neighborhood like this square neighborhood it does not matter so sorry.

So what happens is that if you take this point here there is another symmetrical point in that neighborhood which is also lying on the graph and corresponding to a single x. But a function for a function by the definition of function given x there is only 1 y given x there cannot be 2y given 2x there can be a single y but given x there cannot be 2y's. Every x corresponds to a unique y and that is a function. Here every x y x here corresponds to a unique y in this neighborhood. This x corresponds to this y technically right. This you have this X y also satisfies the equation if x y = 0.

But this x y point this this coordinate is not inside this neighborhood. So when up at this point 1 root 2 1 / root 2 del F del y is not equal to 0 and hence you can represent this y the function of x here y will be root over 1 - x as square. So that is the idea of implicit function theorem. That locally there is always a chance if del F del y is not equal to 0 local representation is possible you can.

But del y = del F del y = 0 local representation is not possible and that is the idea of the implicit function theorem. This is a difficult theorem it takes some time to understand you can go through. So the idea is that you take a neighborhood and in that neighborhood you will find points which are lying on that curve and also you can represent that piece of curve as a graph of a function given as y = sum f x.

Please observe this point corresponding to this y here there is a symmetrical y here. But this x y point this coordinate corresponding to this x of course this x and there is a y here another y dash here. But x xy dash this coordinate is not in this neighborhood around the point x0 y0 which is 1 / root 2 in this case just for an example. So with this i end my talk this is a slightly difficult thing but from the next class we will go in to much more mechanical thing which you can understand much more simply.

And hoping that you take this course little seriously we will get into more detail things. But this will be extremely useful without an understanding of function of 2 variables or function of three variables. You know doing difficult tasks in engineering solving engineering problems it is it becomes really difficult without mathematics things done do not advance there is a society of industrial and applied mathematics in the US and there is Siams and their motto is also Siam science and industry advances with mathematics.

So I will just request you that whatever been done here you can look up other books you can look up the net look up other sources just do not depend on single soul. I can be mistaken I can have a little maybe mark there some issues where I would have conceptual failures maybe that I have to admit here you to its human to or so you can correct me if you want on the on the portal and what it is very it is important to go through sources importance to see applications because in this little course of 20 hours is not possible to do a complete justice.

This is actually a course of whole semester in each class would be of an hour. Ok so good night and actually it is night here thank you very much.