Curves And Surfaces.

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Module-II.

Surfaces-1: Smooth Surfaces.

Lecture-09.

Tangent, Normal.

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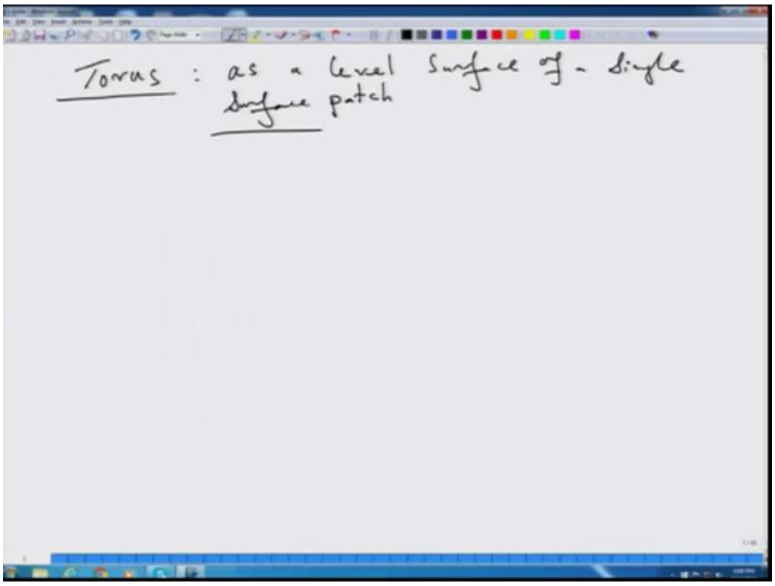

Okay, so as I said in the last lecture, construct a surface, we actually construct the surface patches smoothly and then add them together in a way that transition maps are smooth. So, as a example, I will do a very simple example but this is very important if you will carry on your study on different Geometry or topology, that is we have done this thing before also, Torus. So, we will realise this as a level surface of a single surface patch. Now, remember what was the restriction in this Torus that you take a circle in a plane and then axis which does not intersect the circle and you rotate that circle around that axis.

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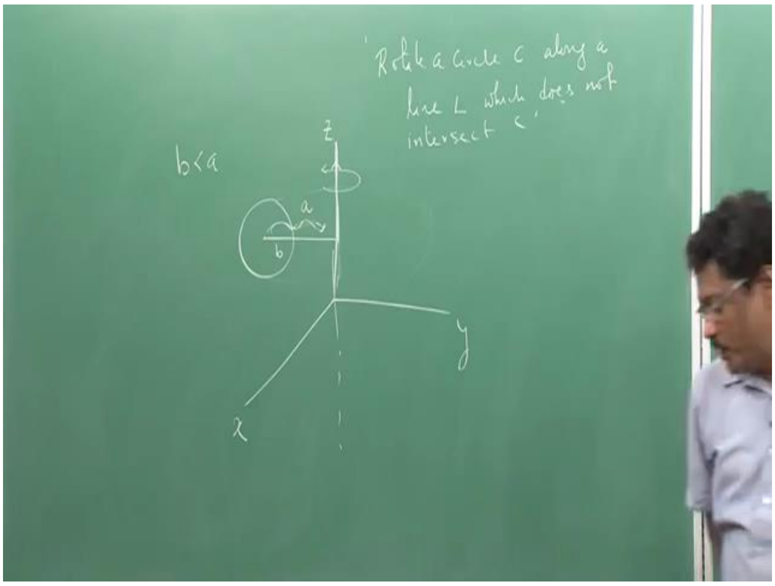

So, for convenience it does not matter where we take, so let us let me take for that convenience circle in the XZ plane. This is the radius B, this is Centre and the distance of the Centre to on that axis to be Z axis and this distance I take to be A. And of course I must have B less than A, so let the axis does not intersect the interior of the circle. So, idea is in general rotate Circle C C along a line L which does not intersect C. So, in this particular case, so we have taken this circle in the XZ plane and L to be the line Z axis.

So, if you rotate it along the Z axis, what will happen, every point will rotate I will have picture like I mean you can do, you can try to very fancy picture of a Torus through your, I mean you can try or you can try to generate it through Math lab or Mathematica, but usually if you see in any lecture, people usually do denote the Torus by a donut, that is…

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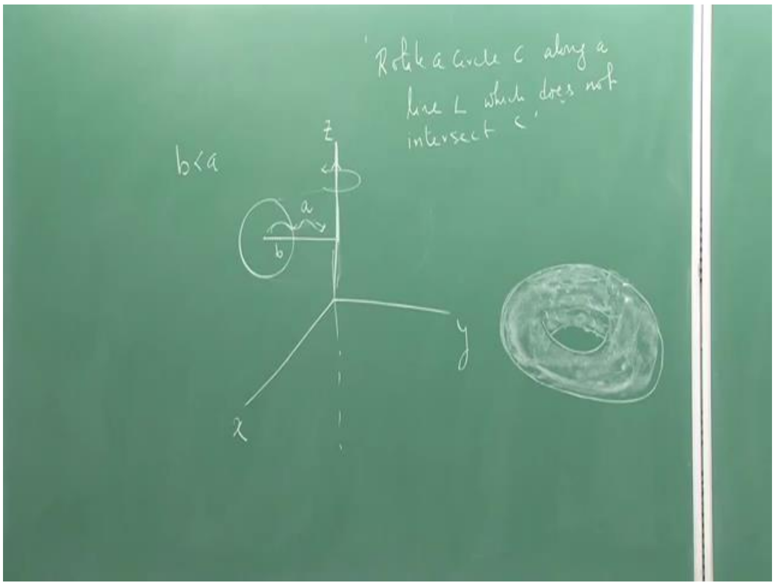

This is a typical picture of a Torus. I think I drawn it earlier also, so you have to see it from above and if you see it from above, it will really look like a donut.

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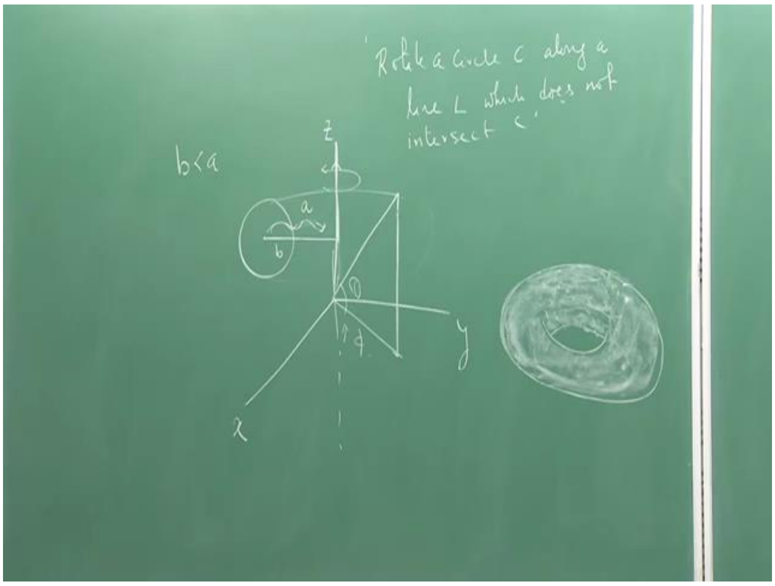

This is called Torus with janus1 anyways, this is a language which you are not concerned with.

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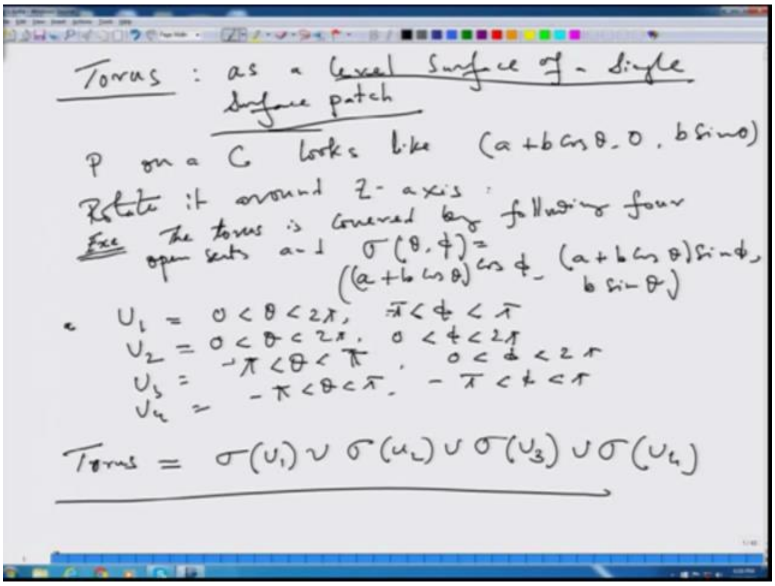

Now, a typical point, so as we have done before, any point here will take its position on XY plane then this angle to be theta and this angle to be Phi, then any typical point P on C, the circle when we start looks like A - B cos theta, you are in the XZ plane, Y coordinate is 0 B sin theta. When you rotate it along the Z axis, then you will, this is a simple exercise, all of you will be able to do it, otherwise ask in the forum that how it is done. Be sure about this example.

The entire Torus is covered by following 4 open sets and Sigma Theta Phi equal to, let me write it here A - B cos theta cos Phi, it is very easy actually if you think it geometrically. A - B cos theta sin Phi B sin theta. And what are the open sets? U1 theta between 0 to 2 pie and pie between - pie to pie again theta between 0 to pie, this is between 0 to pie, theta between - I mean Symmetric… And the Torus. Okay, this is the parameterisation of Torus, there will be many points which are common, try to identify those common points which will be common to the surface patches and between Sigma 1, I mean Sigma 1 and Sigma 2, both are covered by certain points but these 4 patches will cover the Torus.

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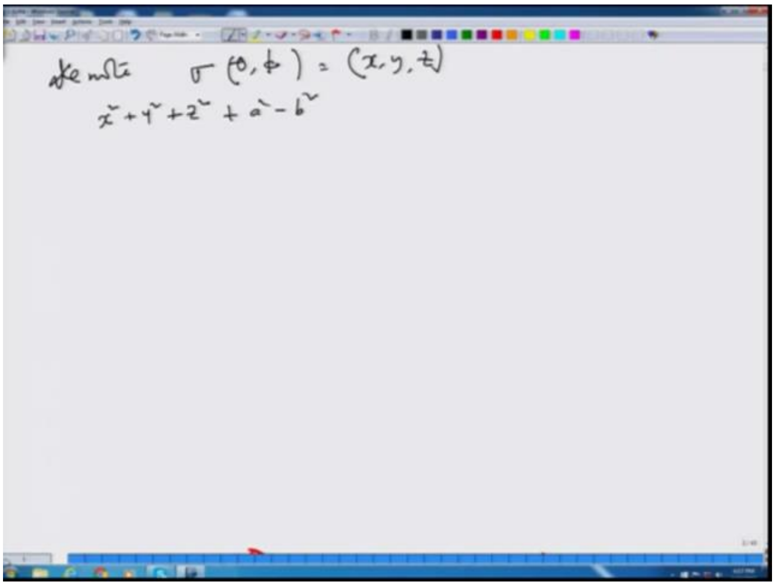

And we want to realise it as a level surface of a single surface patch. How? That is done very easily. Suppose I want to denote Sigma Theta Phi as XYZ, right. Where I recall, what was Sigma Theta Phi, this is the definition of, this is the parameterisation of the Torus. Now, if I do a simple calculation, X square - Y square - Z Square - A square - B square.

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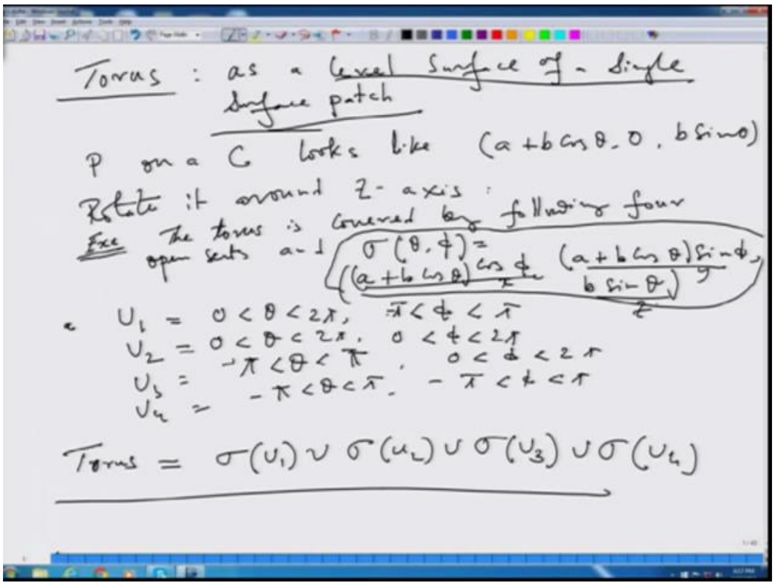

Comeback here, so this is the X coordinate, this is Y coordinate, this is Z coordinate. So, do X square, this simple exercise X square - Y square - Z Square - A square - B square.

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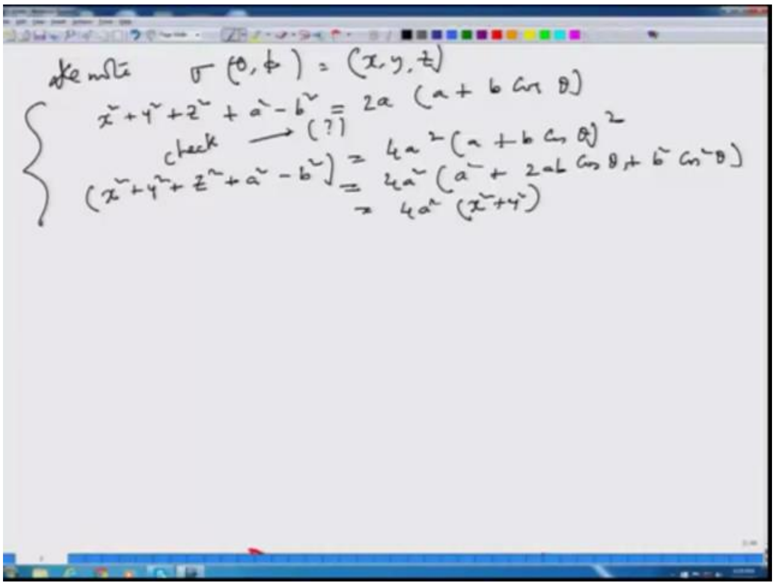

You can see very easily, you have to check it, this will come out to be 2 A A - B cos theta. So, I have to eliminate theta, so what I do, I take a square again, this will be 4A square A - B cos theta square, but now this will be 4A square - A square - 2 AB cos theta - B square cos square theta which will be you can easily checkout is X square - Y square. Again, this is X, this is Y, okay. Take some time to do this calculation, this part, this is very simple, I am just eliminating theta and Phi. So, this is way, we realise Torus as a single surface patch or as a level surface of a single surface patch.

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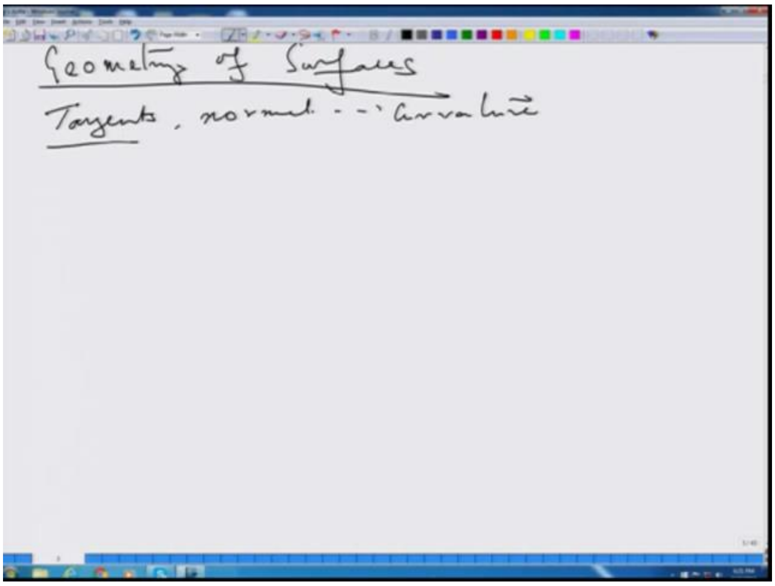

Next we move onto geometry of surfaces. So, by surface again throughout this course as I said, it means regular smooth surface. Now, as per the curve what is Geometry, we define, which I have to define tangent, which I have to identify a surface through its tangent, for recall the curve tangent, then the unit normal, binormal and then torsion and then finally a curve in R3 is uniquely determined by Frenet Serret formula that we did last time last week.

So, similarly for surfaces you have to define tangents, normal, and more co of course curvature, so there are many quantities related to that and as the course progresses you will see that actually surface will be determined by its fundamental forms, 1st and 2nd. We will come to that eventually but let me for time being start with tangent today. So, how do I define a tangent to a surface? Now, surface is a three-dimensional thing, unless a curve, unlike a curve which is, for a curve it was easy, right.

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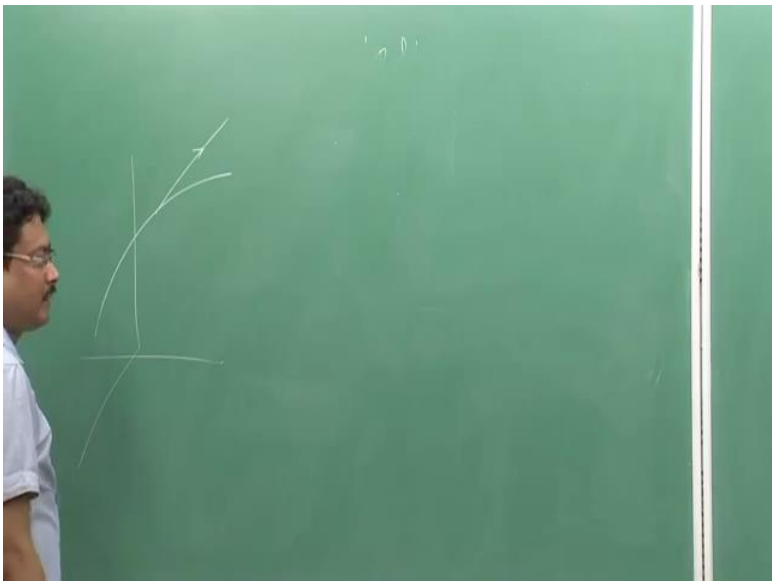

For a curve it was easy, this curve or whatever planar curve, you take a, take a point and we define a tangent, which is anyways clear.

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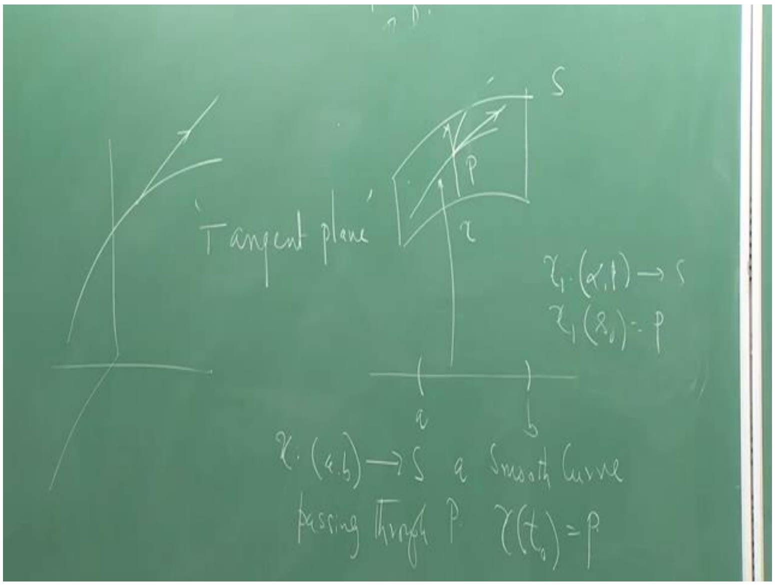

But suppose I have a surface and I have a point P here, how do you define tangent to a curve at this point P? Well, I know after all this surface consist of patches. I can consider below it an interval, let us say A, B and a curve passing through P, smooth curve. So, let us say gamma of some T nought is equal to P. For that curve I can define a tangent. No problem. But immediately we will say, okay, there will be another curve, there will be another curve, say gamma 1 defined on same interval or some other interval and I will have tangent, another tangent.

So, how I, so what we should call it tangent at the point. You understand that I cannot just take one single vector to say something is a tangent. But after all, a surface is consist of patches from R2. So, though it is in R3, it is two-dimensional thing. Right. So, locally it is two-dimensional. So, what we define here is not tangent but so-called tangent plane.

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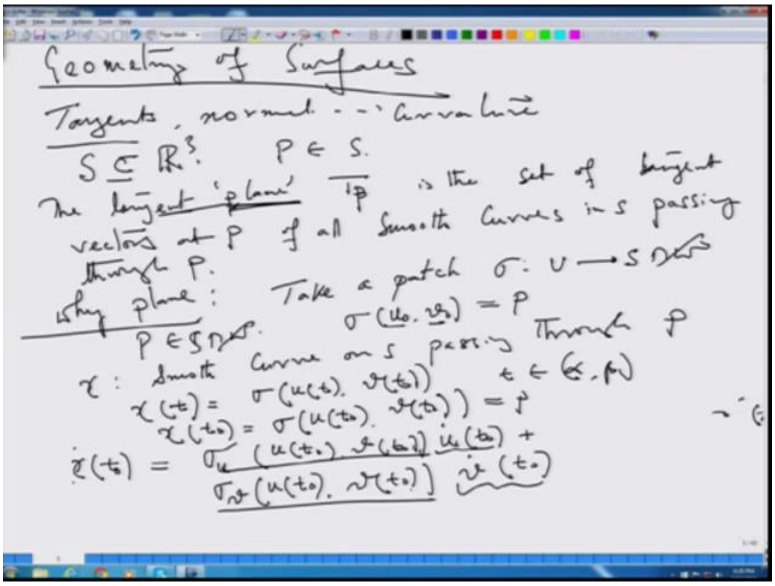

What I mean by that that suppose S is a smooth regular surface in R3, P is a point on S, the tangent plane, T P is the set of tangent vectors at P of all curve in S, all smooth curves passing through P. But then immediately comes to the point that why I am calling it a plane. So, why plane? A plane is two-dimensional, right? I mean intuitively to be plane but I must justify that.

Okay, let us take any take a surface patch, take a patch Sigma S Intersection W P in S Intersection W and I can actually forget this W part, I can, since I am doing everything locally and let us take Sigma U nought V nought equal to P. So, gamma is a smooth curve on S passing through P. So, let us so, gamma T will be Sigma UT VT, where T belong to some interval A, B or Alpha, beta and they have written on the board. With some gamma T nought equal to Sigma U T nought VT nought which is P. And I denote that point by U nought V nought.

Now I am looking a tangent at that curve gamma, so that is gamma dot T, right.You have done it for the curves, what will be that? I apply the chain rule Sigma U at UT nought VT nought U dot - partial derivative of Sigma with respect to V UT nought VT nought V dot T nought, correct. These quantities are scalars. So, this gamma dot T nought is a combination of scalar, combination of these 2 vectors.

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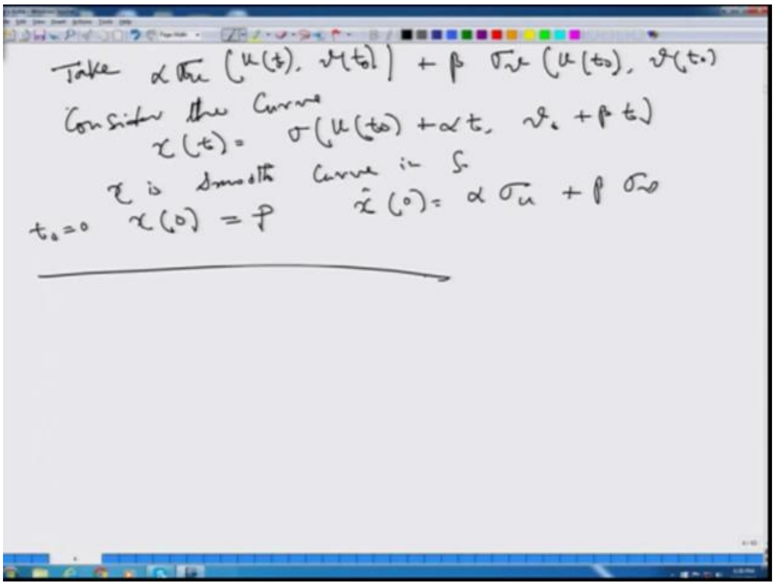

Now, take Alpha Sigma U, I am not writing, okay, let me write no problem. Sigma U UT VT and consider the curve and consider the curve gamma T equal to… then gamma is smooth because it is a smooth surface smooth, U and V are smooth, so gamma is a smooth curve in S. Gamma is 0, that is T0 equal to 0 is equal to P and what is gamma dot F0? This is Alpha Sigma U, I am not writing the other part Sigma V. So, what it says, that whatever curve I take, this along with the previous page, this one,

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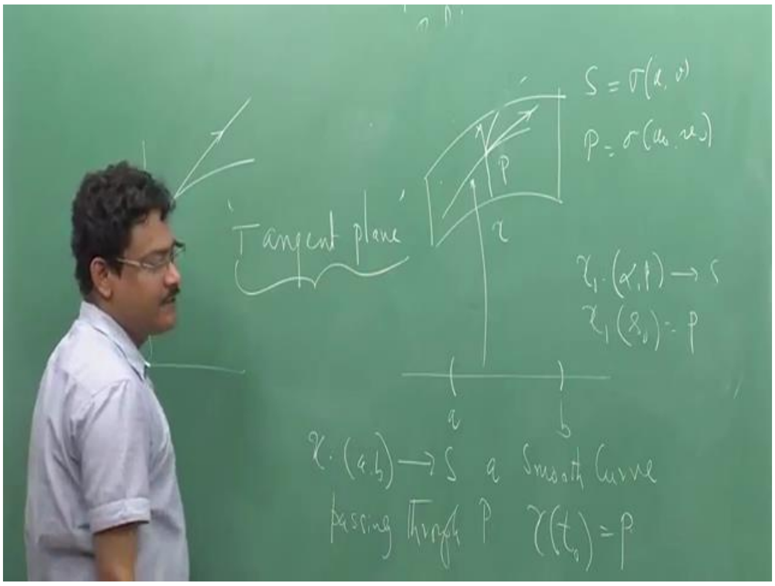

What it says that whatever curve I take, suppose this surface patch is Sigma U V and my P is Sigma U nought V nought, then this set of all tangents, what we have defined as tangent plane, set of old tangent vectors at P, all smooth curves passing through P, that is in this case…

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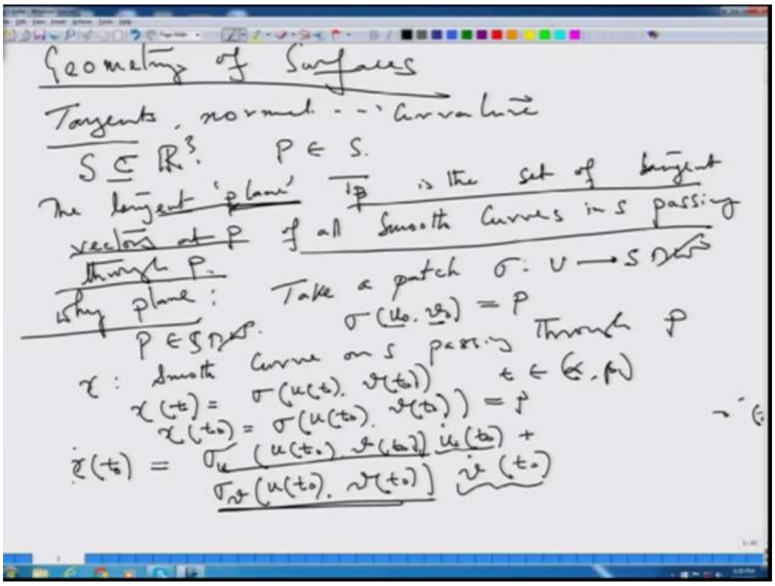

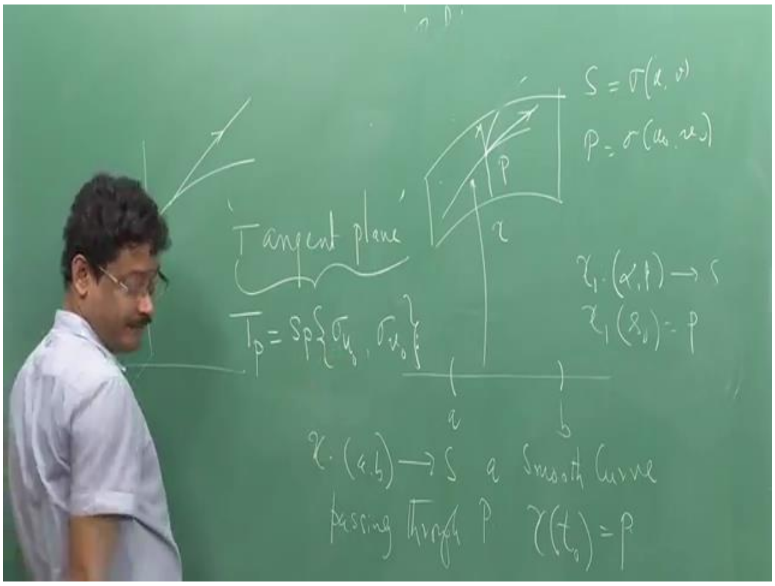

So, it is generated by 2 vectors and anything generated by 2 vectors we usually call it a plane, right. So, that is a tangent plane and that justifies a language plane here. Okay. But, as I said that everything you define for a surface should be invariant under parameterisation.

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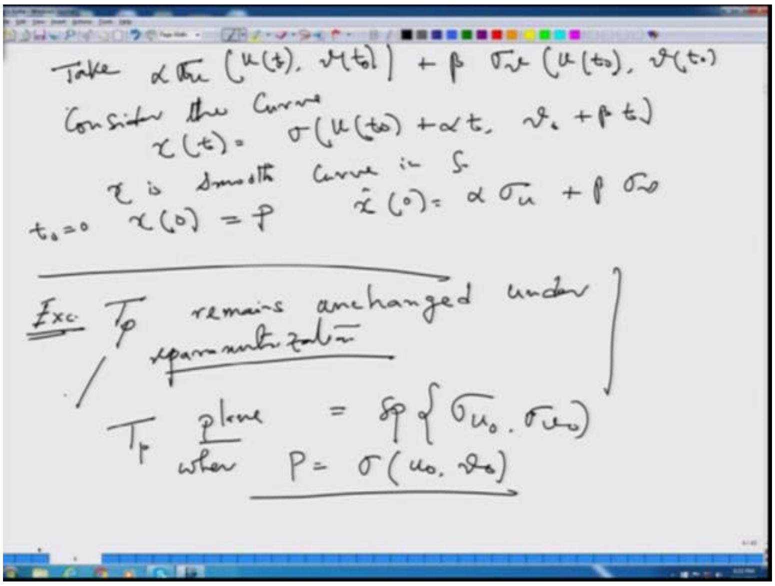

So, one must verify, this is a part of 2nd assignment also that tangent plane remains unchanged, it is very easy though to do under reparameterisation. What will happen is actually if I take reparameterisation, then instead of Alpha, beta, so you will have some scalar Alpha, beta appearing depending on the reparameterisation. So… Sigma inverse composed Sigma tilde at U nought V nought. But check it anyways, it is always better to check it.

Now, so we are justifying to call this TP as a plane, so this is generated by span of Sigma, U nought Sigma V nought where P is Sigma U nought V nought.

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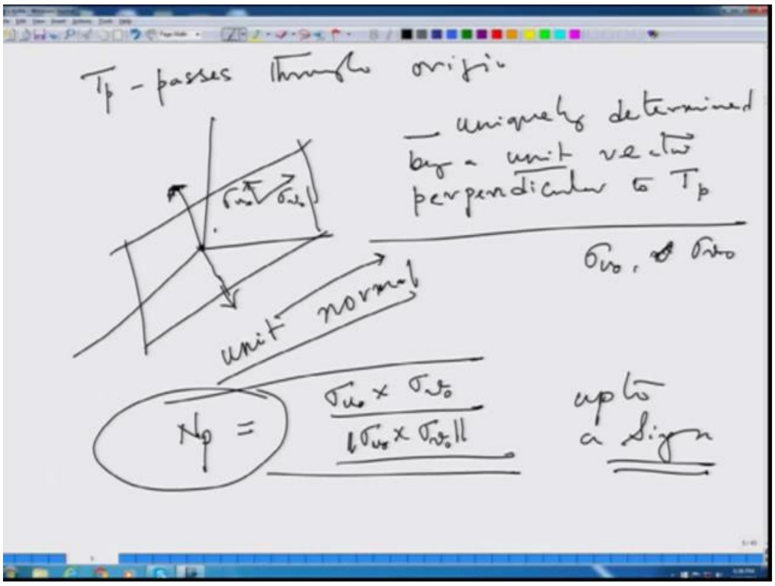

Now, any plane that passes through origin, right… So, plane will pass through origin, it is generated by 2 vectors. But whatever passes through origin, something is, somewhere the plane is passing through origin, how to determine the plane? Then just determine, this you need to determine by unit vector perpendicular to it.

So, if I take the point here, let us take a 0 point, so if I look at this vector P, this vector, this direction will completely determine the plane. And something perpendicular to tangent plane, I should be calling it normal and I am taking unit vector, so I should call it unit normal. But there is a little problem here, you can, I may located in this way that this is a vector but somebody standing on the other hand of the sphere may say, okay, not this one. The vector is actually the other one, the opposite direction.

So, unit normal at point P, either I can define, so this is a vector perpendicular to Sigma U and Sigma V and then. So, if I have 2 vectors Sigma U and Sigma V, U nought V nought. Then the vector perpendicular to it is the cross product. And I normalise it through norm, then what I get is a unit vector. But as I said, direction maybe of our choice, I can either choose this direction, you may choose other direction. So, this fellow is up to a sign. So, unit normal, I cannot just define it without mentioning what sign I should put here and that leaves us to the concept of orientable surface, orientable surfaces, so next lecture I will talk about orientable surfaces. Thank you.