Curves And Surfaces.
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Module-II.
Surfaces-1: Smooth Surfaces.
Lecture-08.
Reparameterisation.

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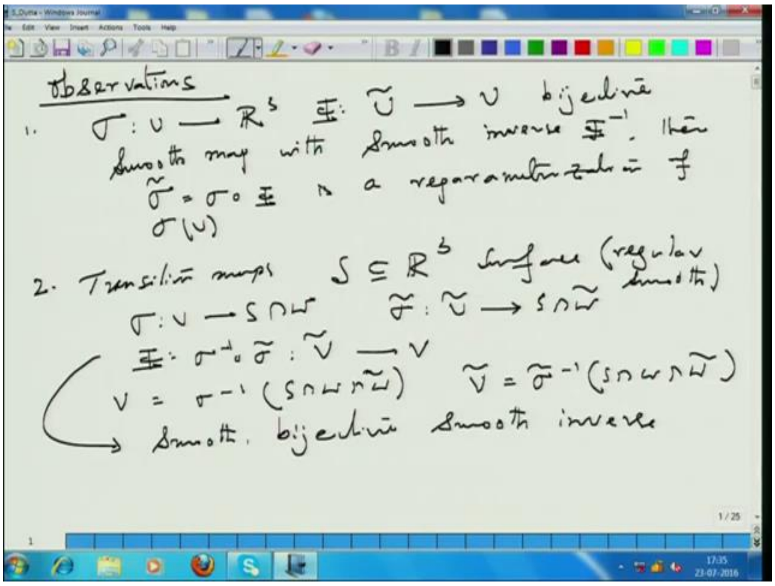

Welcome to the 3rd lecture, so let us recall what we have observed in the last lecture. 1st of all if I have surface patches Sigma from U to R3 and I have another open set U tilde and we have a bijective smooth map with smooth inverse, then Phi tilde equal to Phi compose Phi Sigma composed Phi is a reparameterisation of Sigma U. This was the 1st observation we made, we did not prove it, as I said the proof will be posted and the 2nd observation was that about the transition maps. That I have surface S in R3 which is regular and smooth, we will all be, all we will be dealing with smooth surfaces.

I have Sigma from U to S Intersection W, you can look back at the definition in the 1st lecture. And Sigma tilde from let us say U tilde to S Intersection W tilde, then I have this map Phi which is from Sigma inverse compose Sigma tilde from B tilde to V where V is this Sigma inverse of S Intersection W Intersection W tilde, V tilde is Sigma Sigma tilde inverse of S Intersection W Intersection W tilde, so this map is transition maps which is smooth, bijective and smooth inverse. By these 2 observations, we have decided that properties of smooth surfaces are modulo reparameterisation.

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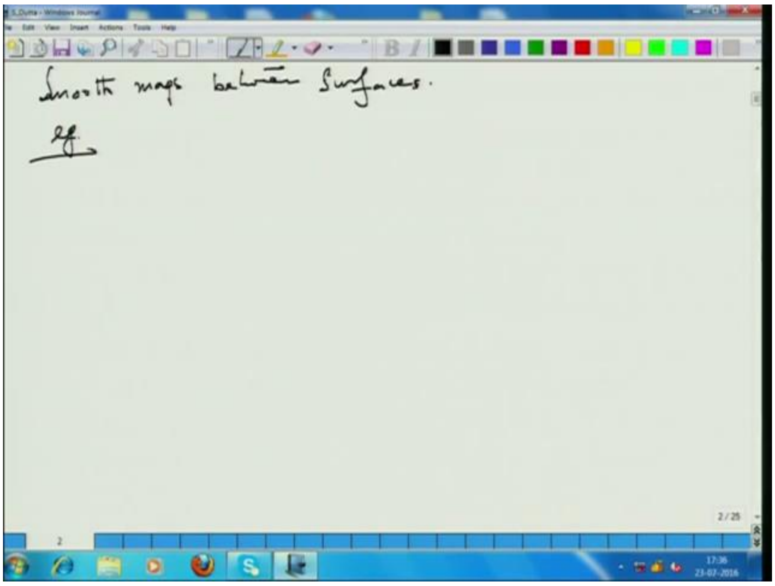

So, what about properties we define for the surfaces in R3? It is up to reparameterisation, so it should be true on the reparameterisation. So, with that idea, let us define smooth maps between surfaces. As usual, I will start with an example.

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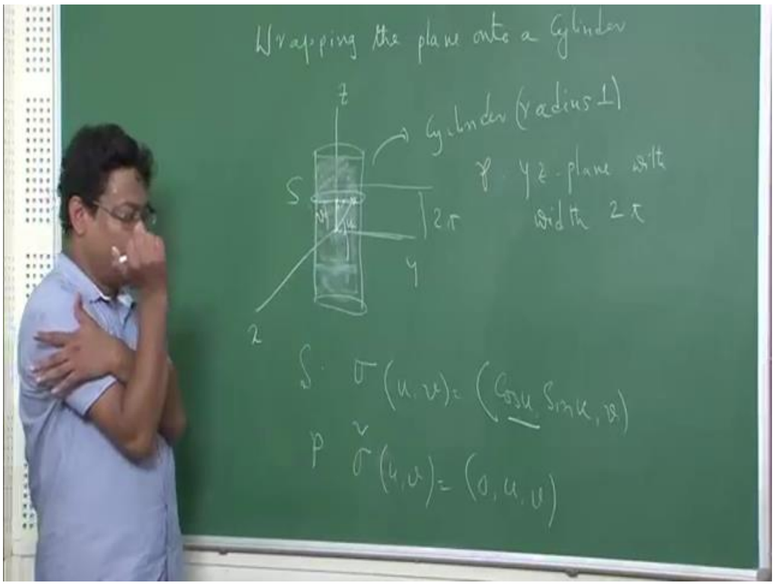

My example is, 1st example is wrapping the plane onto a cylinder smoothly. So, I take this cylinder, this is my z-axis, this is the cylinder, radius 1, okay. And I take the plane, so this is S, this is the surface of the cylinder. I take the plane P, it is a YZ plane with, not the entire YZ plane but 2 pie.

Now, what I am doing, here I have the cylinder, I have the plane out here, I am trying to wrap this plane onto the cylinder smoothly. Now, as you know, as you can see very easily, this can be parameterised by single surface patch that is cos U sin U V, a typical point on cylinder is, this is the height V and this is the angle U. We will draw it little bit carefully, so, a point here, this is the height V and the angle U, so this is, this point can be uniquely determined by cos U sin U and the height V.

Now how is the YZ plane is parameterised? YZ plane is parameterised by Sigma tilde U V is 0 U V. Now if I try to wrap it, what will happen?

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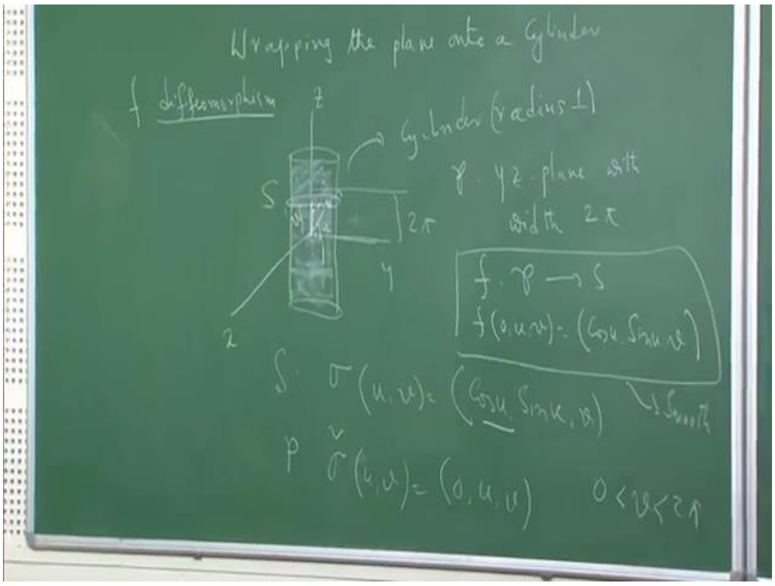

So, you immediately understand that this point will go to, so this particular point which is at angle U with YZ plane, that will be mapped into this point and according to, so V is between 0 to 2 pie and that means the V coordinate will map to the height of the point. So, I can define the map if… Of, maybe I will write here or, okay let me write it on the board itself.

F from F of 0 in V is cos U sin U V. You can check this F is a smooth map in the sense that all these partial derivative exist in a continuous, all partial derivatives all of them exist and continuous. Actually F inverse also exist and this F is actually a so-called diffeomorphism, what is a diffeomorphism, it is a homeomorphism with which is smooth, that means all its partial directives exist, partial derivative all exist. So, we will try to remember this example that such a way you can generate smooth map between surfaces.

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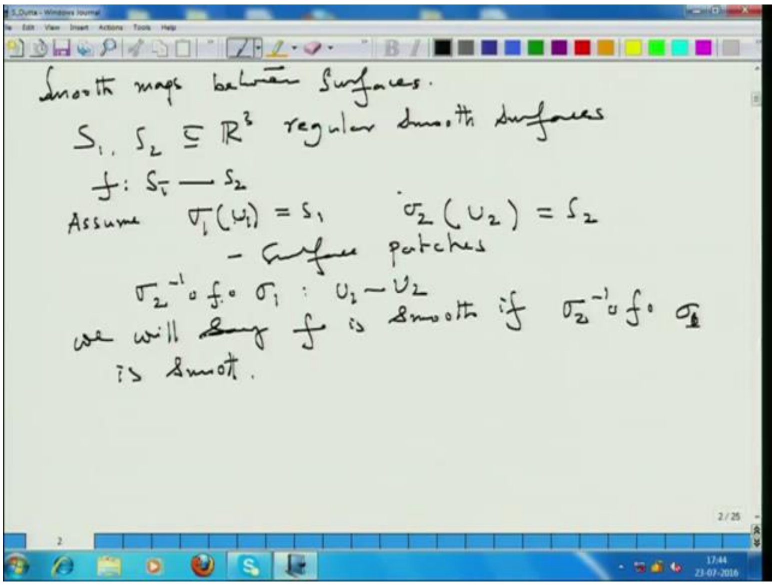

So, in general while defining, let us make things little formal that suppose I have surfaces S1 and S2 regular smooth surfaces, so I have map from F from S1 to S2. Assume S1 is covered by, okay let us say Sigma 1, single surface patch does not matter, otherwise you can always go back locally. And S2 is covered by and Sigma 2, so these are surface patches. Then I have a map, Sigma 2 inverse compose F compose Sigma 1, Sigma 1 takes U1 to S1, F takes S1 to S2, Sigma 2 inverse takes S2 to U2. So, this is from U1 to U2.

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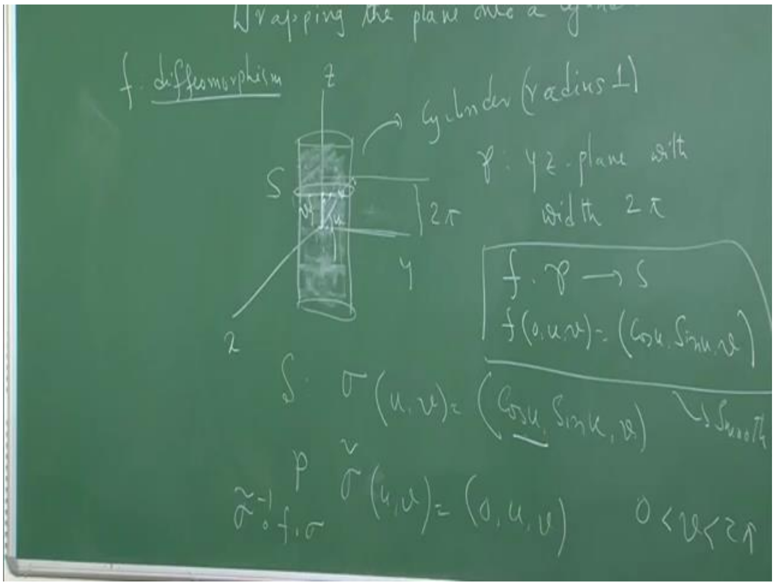

So, we will say F is smooth if this map is smooth For example here I have a plane at P S1, my other surface S2 is the surface of the cylinder, this F is smooth, then provided Sigma tilde inverse F compose Sigma is smooth but here it is trivial map, F itself. So, this is an example of smooth map.

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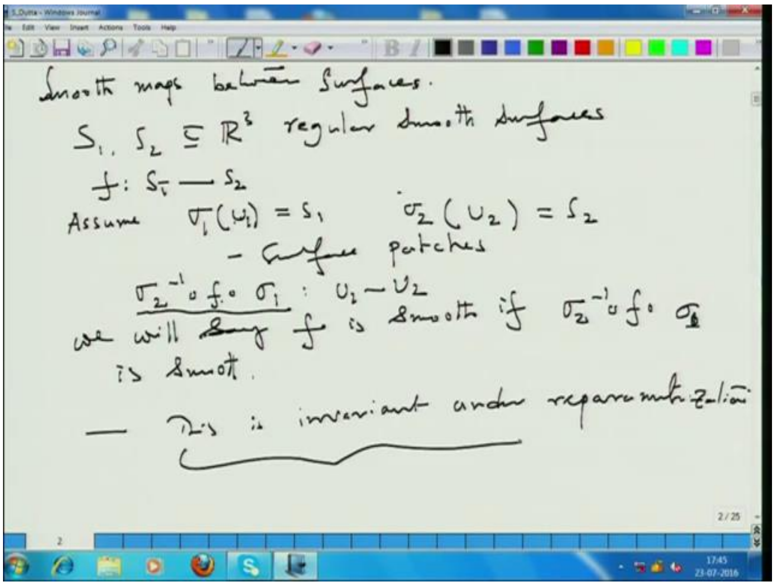

So, this is, this definition is invariant under reparameterisation, that is the way we have done it, right. And looking about it, if I have this S1 and S2 are same, then under reparameterisation map will be smooth if for every reparameterisation Sigma 1 and Sigma 2 it is this mode.

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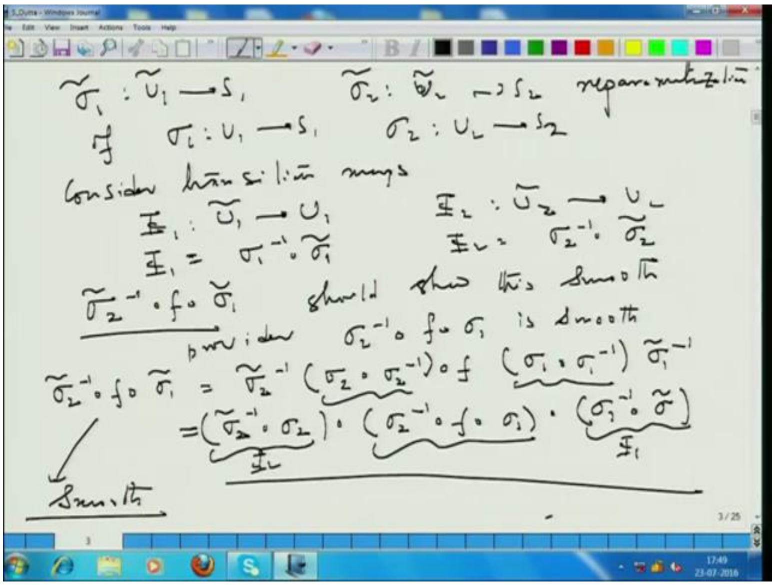

I mean, to illustrate this point, maybe we will look at Sigma 1 from U1 to S1, Sigma 2 from U2 to S2, Sigma 1 tilde Sigma to tilde U1 tilde U 2 tilde reparameterisation of whatever we have done in the earlier page. Sigma 1 U1 is S1, Sigma 2 U2 is S2, suppose if I have 2 reparameterisation, Sigma 1 tilde of Sigma 1 and Sigma 2 tilde of Sigma 2. Then consider transition maps Phi 1 from U1 tilde to U2 and Phi 2 from U2 tilde to U2 brand Phi 1 as usual is given by Sigma which one, Sigma 1 U1 tilde to Sigma 1 inverse compose Sigma 1 tilde, Phi 2 is Sigma 2 inverse Sigma 2 tilde. Then you consider Sigma 2 inverse compose F compose Sigma 1 tilde.

What I should show? I should show this map is smooth provided, so should show this is smooth provided Sigma 2 tilde inverse compose F compose Sigma 1 is smooth. That was an previous page, right. Okay, what the big deal here? After all Sigma 2 tilde inverse compose F compose Sigma 1 tilde, I can write it as Sigma 2 tilde inverse, I can put in Sigma to compose Sigma 2 inverse which is identity compose F and before Sigma 1 I can put Sigma 1 Sigma 1 inverse Sigma 1 tilde inverse, these 2 maps identity, right because they are homeomorphism. So, what will happen? This is Sigma 2 tilde inverse compose Sigma 2 compose with Sigma 2 inverse compose F compose Sigma 1 compose with Sigma 1 inverse com compose with Sigma 1 inverse tilde.

Now, the transition map, we have done a theorem, transition map of smooth surfaces are smooth. So, these are transition maps, this is smooth, these are smooth, this is Phi 1 and Phi 2, this is Phi 2 and this is Phi 1, they are smooth, this map is smooth and composition of smooth maps are smooth, so this map is smooth.

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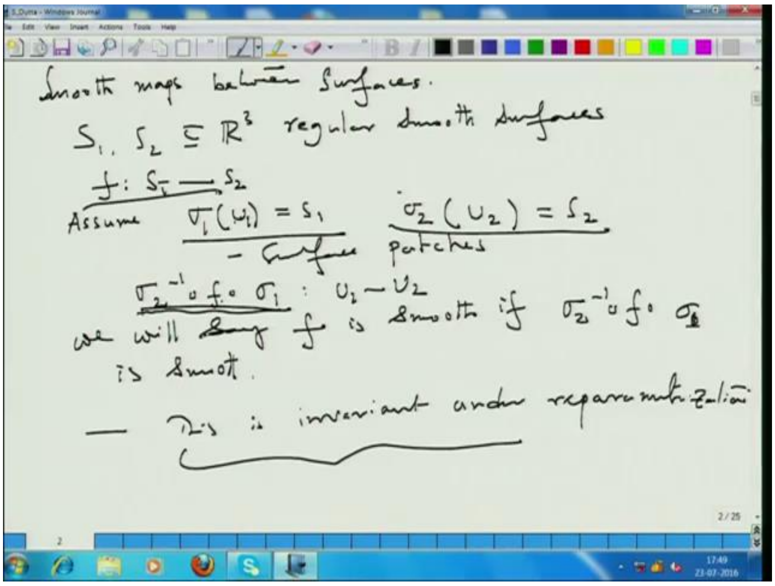

So, whatever you have defined the concept of a smooth function between 2 surfaces is invariant under reparameterisation. Is it clear? Just look at it once again, I will have one-minute pause, look at the 2 pages together and understand what they are trying to say, that any, for idea is whatever we define on surface, property of a surface, it has to be, the property has to be invariant under reparameterisation because I change, property changes in reparameterisation, then you cannot delete that property.

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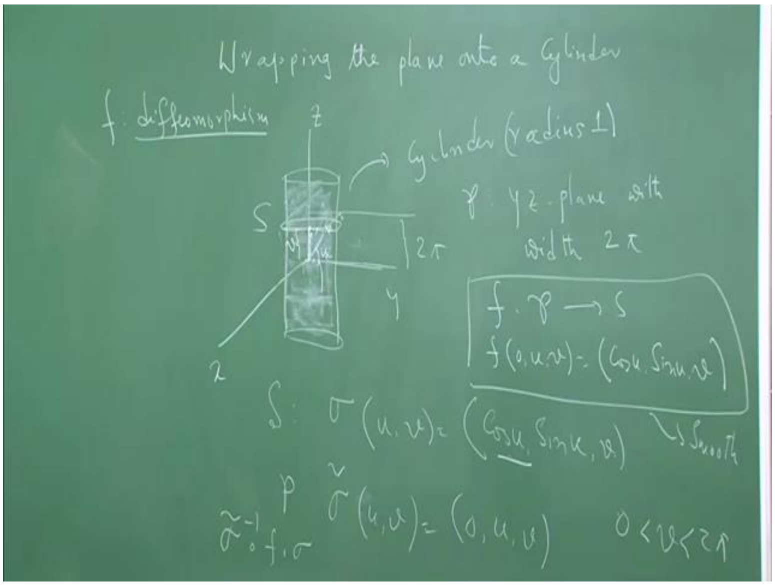

So, to illustrate that, we had this example of wrapping the plane on the cylinder smoothly and to make it formalised, we have done what is called as the smooth map between surfaces in general, this example in mind. And here we have shown that actually what we are seeing that these maps are, the property of the smooth maps between 2 surfaces, the definition is invariant under reparameterisation.

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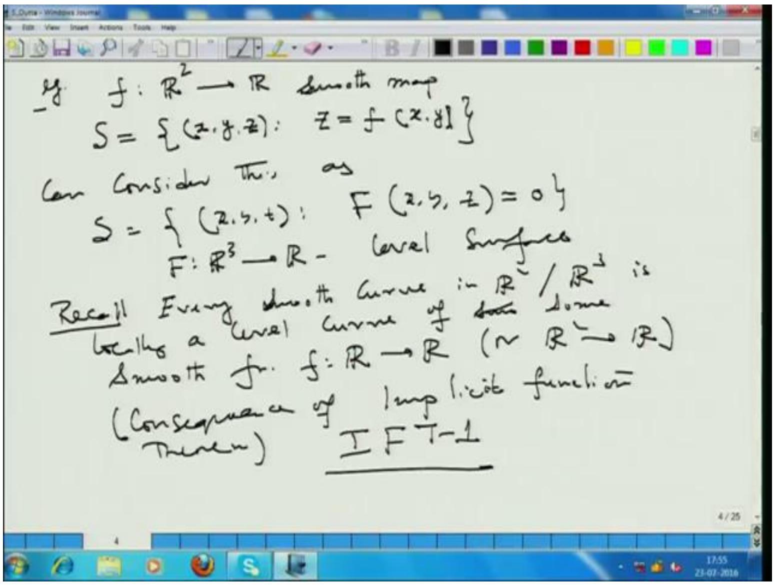

Now, this example we have done in the beginning that if I have, if from R2 to R a smooth map and if I look at the level surface of this graph, I think I denoted it by GF last time, level surface Z equal to FXY. You can also consider this as changing F is equal to 0, now F from R3 to R. So, this is a level surface. Now, if you recall for curves, a lecture in our last week lecture, every curve we realise as locally as level curve. So, recall every smooth curve in R2 or R3 is locally a level curve of some function R2 R or R2 to R as the case may be. And this was consequence of implicit function theorem. Right.

Which I usually denote as IFT 1, in my previous NPTEL note, if you go to the portal, you will find that there is a post on IFT 1 and IFT 2, implicit function theorem and inverse function theorem.

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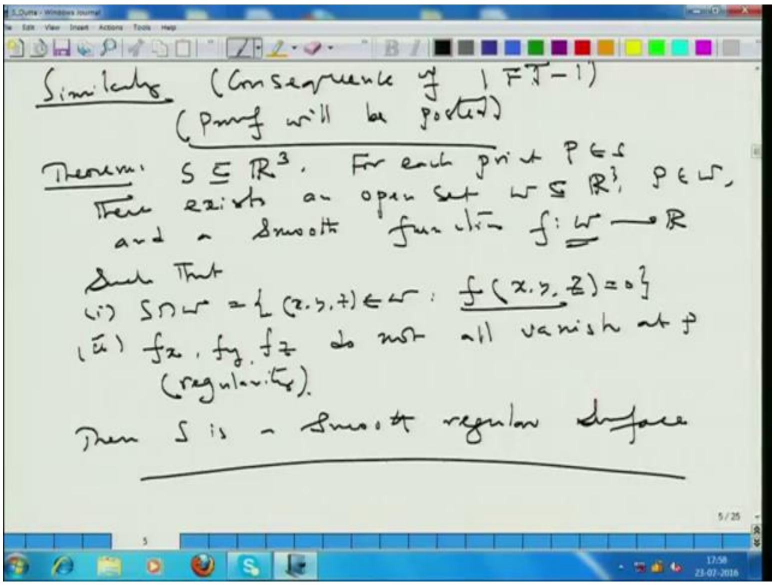

On a similar line, similarly, this is again consequence of IFT 1, implicit function theorem, proof will be posted, do not worry too much about the proof because that is not going to be what you need for the course. And this course is unto learn certain basics about the curves and surfaces. So, this proof is not important for the course.

The theorem we have here, it says that, okay, locally again, every surface is a level surface. So, S in R3, let for each point P in S, there exists an open set an R3 containing that point. And a smooth function such that S Intersection W is actually the level surface for this and for regularity purpose, so this is the regularity part, then F is a smooth or regular smooth or smooth regular surface.

Okay. So, this is the direct consequence of implicit function theorem. And this is also a way how you construct a surface. That is, okay, I choose open sets around W certain (())(24:01), I construct the patches of the surface by this, taking a smooth function F and look at, taking a smooth function and look at the living set of that function F and I put the patches together, then I will get a smooth regular surface. So, next time in the next lecture I will show you how torus, we have discussed about torus, the example of torus is there in the assignment also, how torus can be realised as a level surface of some smooth function F. So, that is in the next lecture, thank you.