Curves And Surfaces.
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Module-I.
Curves In R^2 And R^3.
Lecture-05.
Simple Closed Curve And Isoperimetric Inequality.

Okay, welcome to the last lecture of this module where we aim to prove the so-called and very famous iso-perimetric inequality for planar curves. But before we start that, we need to gather some information on simple closed curves in R2.

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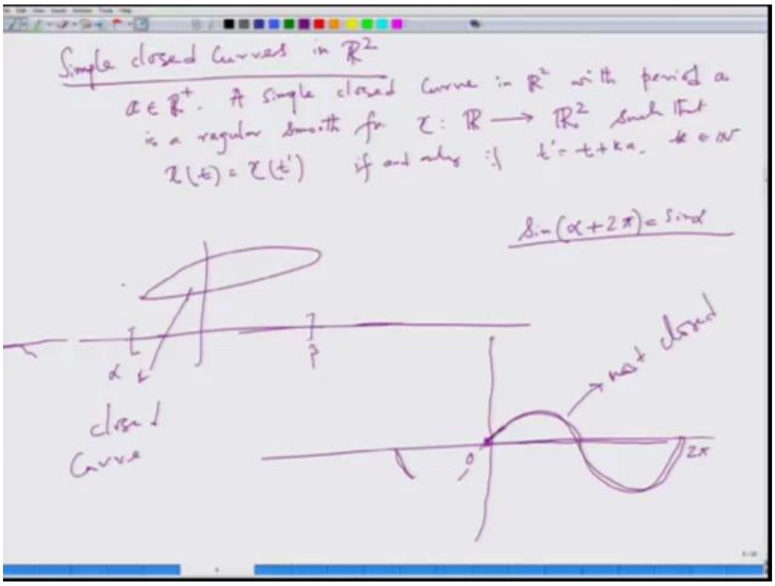

As the name suggests, suppose A is a positive number. A simple closed curve in R2 with period A is a regular, we know what is regular, smooth, thus we have tangent at every point function gamma defined on entire R to R2. So, instead of defining on an interval, I define it on the, I can extend it R to R2 such that gamma T equal to gamma T Prime if and only if T Prime equal to T + some KA. KA is some positive integer.

So, what it means here? If I have a defined on some interval say Alpha, beta, now and I also have that, so I have a curve Alpha, beta which is like this that it intersects itself at the same point, at one point only and everything is smooth. Then actually I can extend it to entire R with a period beta - Alpha, that is what I mean. For instance, you can check the sine curve. Now, consider a sine curve with this portion.

So, from 0 to 2 pie, I have this curve, now I extend it periodically to entire R using the formula that sin Alpha + 2 pie is sin Alpha. But this is not a closed curve, this is not closed, whereas we are concentrating on closed curve.

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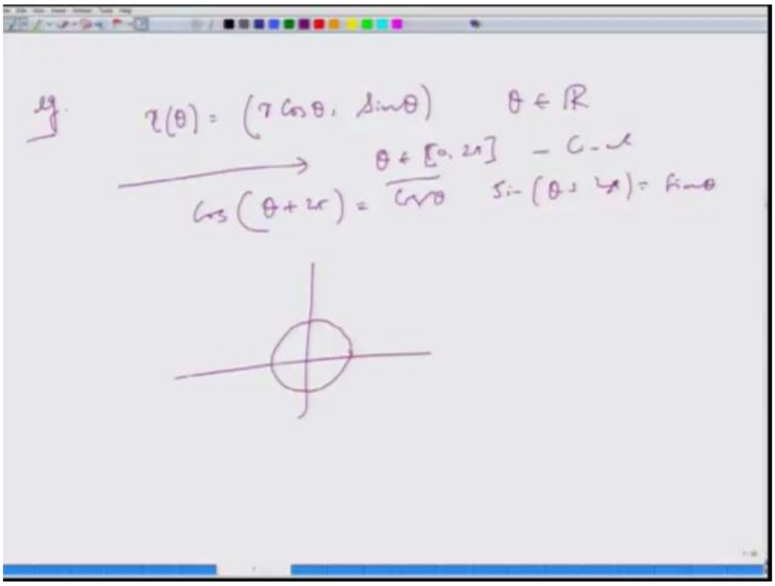

For instance if you say for example, if you look at this curve gamma theta equal to say some R cos theta sin theta. Theta in R, what this will give us? Okay, for theta in 0 to 2 pie, this will give us circle. But now I go for any theta, theta +2 pie, since cos and sin are 2 pie periodic. So, if I extend it, I remain on this circle, I do not move anywhere other than that.

So, this is a simple closed curve on R2. Now there is a great theorem which we cannot prove with our tool, it needs some kind of more advanced techniques and complex analysis.

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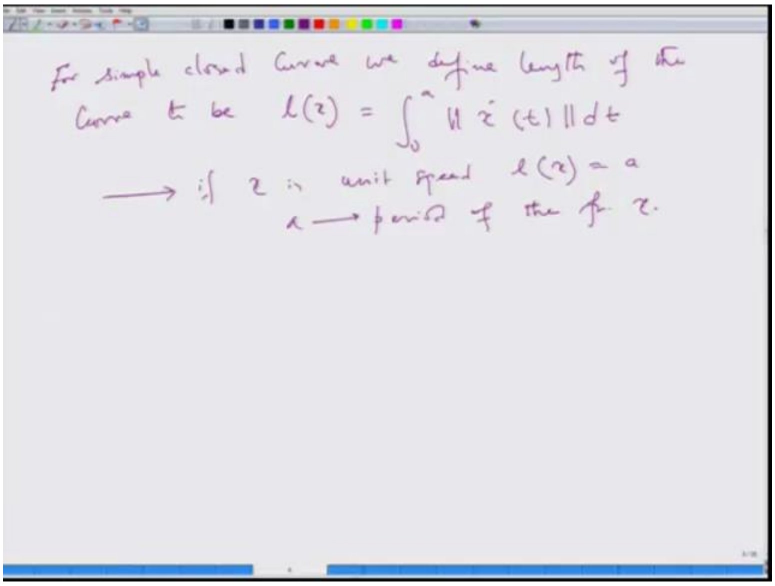

And, okay before that. Here is a, for a simple closed curve, we define length of the curve to be L gamma 0 to A gamma dot T norm dt and you can easily see that if gamma is unit speed, length gamma is equal to A where A is the period of the curve. Okay. So, this is just a definition and definition or meets every kind of a demand we may make.

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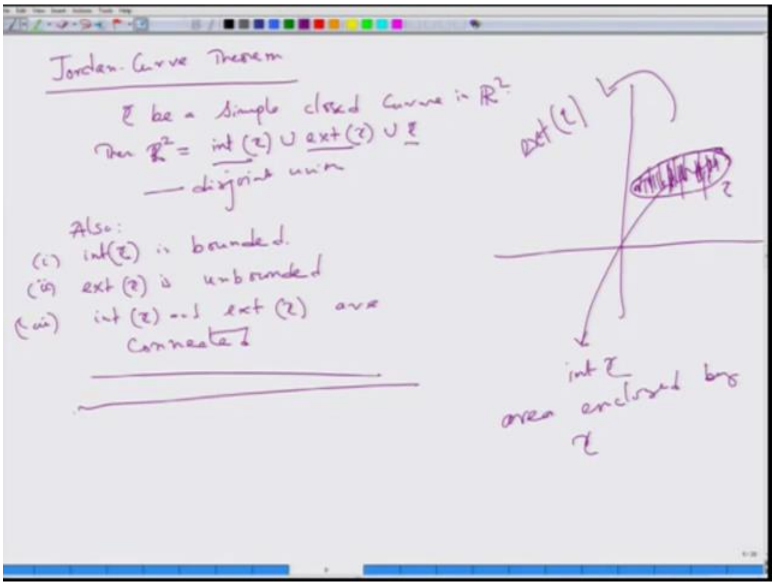

Now, there is a very famous theorem as I was telling is called Jordan curve theorem and which we cannot prove with our existing tools we have so far developed, but we will use a statement. If you look at the statement, the statement looks like a very trivial but to prove it mathematically, it needs a lot of effort. Just so, because we will satisfy with the statement only, so statement is gamma be a simple closed curve in R2. Then, this entire plane R2 can be divided into disjoint union of 3 regions, of 3 sets, not region, 3 sets.

Interior gamma union exterior gamma union points of gamma. What is Interior gamma? The shaded portion, this part is called Interior gamma that is the area enclosed by gamma. The portion outside, this is called exterior of gamma, that is the area which is outside gamma. And of course there are some points on the curve gamma itself that I have to take care and this is, the union is disjoint union. So, in the area include by this gamma, we do not consider the points on the curve. This looks like very simple but it needs the proof. And it says also something more.

That is Interior of gamma is a bounded region, okay. All of us understand what is bounded. Exterior of gamma is unbounded and 3rd both regions Interior gamma and exterior gamma are connected, okay. So, this is the famous Jordan curve theorem and we use it in every part of mathematics kind of, unfortunately nowhere it is proved. But it has a proof and you can find the proof in a standard book, but its proof needs some more terminology, some more tools of mathematics, well we will use this as usual and with that theorem, we define what is area of interior gamma. Its notation is A gamma.

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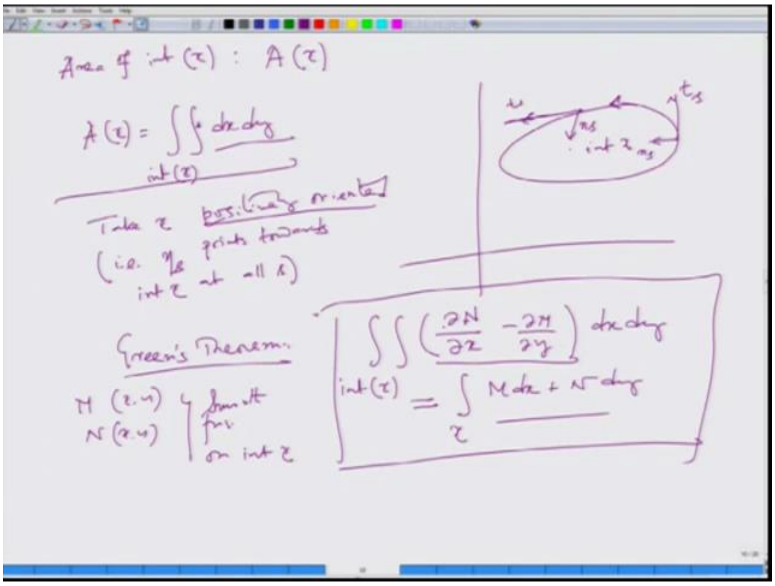

Well as you know if I have a curve like this, closed curve, A gamma should be by simple calculus Interior gamma dx dy. Now, how do I express it in terms of the curve? Okay. Take gamma positively oriented, what are you mean by that? That is eta S points towards Interior gamma at S. So, if I have unit normal here, that should be pointing towards the interiors gamma. If you see that this is the direction I take… Then this will be a tangent vector, so eta S will be pointing towards Interior gamma.

Similarly here, here is a tangent vector, so eta S will be pointing towards Interior gamma. So, positively oriented simply means if you take anticlockwise direction, so, you are moving anticlockwise direction, if you move the other direction, you will see that it will go outside, I mean usually point to exterior gamma. Now, there is this theorem in calculus which says that that if I have such a region and I have 2 functions, smooth functions, this is a statement of so Mxy Nxy, these are smooth functions on interior of gamma.

Then this theorem states you that this relation holds. The integral, this integral, area integral is a line integral over gamma. Now, look back here. I have Interior gamma dx dy, right. So, I need to find 2 functions N and M here such that which is equal to 1 because this is just one here. Any 2 functions N and M will do such that Dell N tell X - DM Dell Y is 1. So, then I can express it as line integral over Mdx or Ndy.

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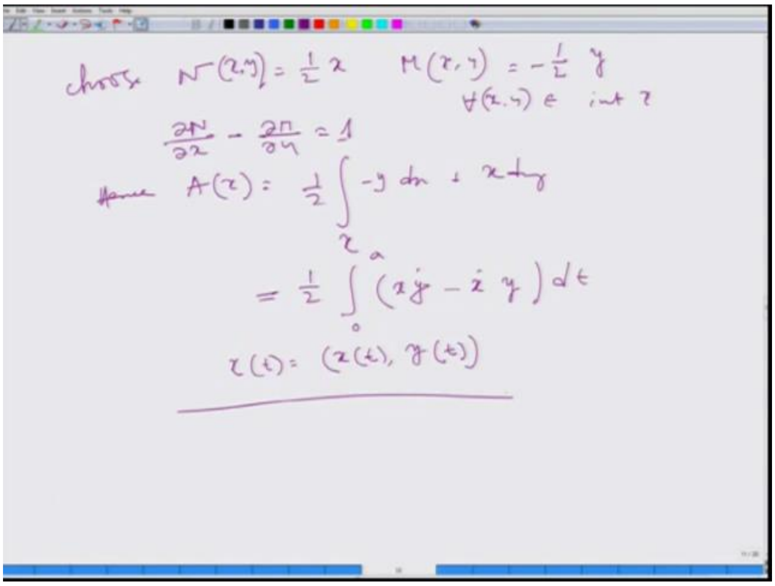

So, choose N of XY equal to half of X. M of XY equal to - half of Y for all XY in interior gamma. Then you see Dell N Dell X - Dell M Dell Y is just 1. So, hence A gamma will be given by gamma half of - Y dx + X dy and if you put it in a parametric form, this will be 0 to A, XY dot - X dot Y dt. Where gamma T I take XT YT. Okay. So, this is the area.

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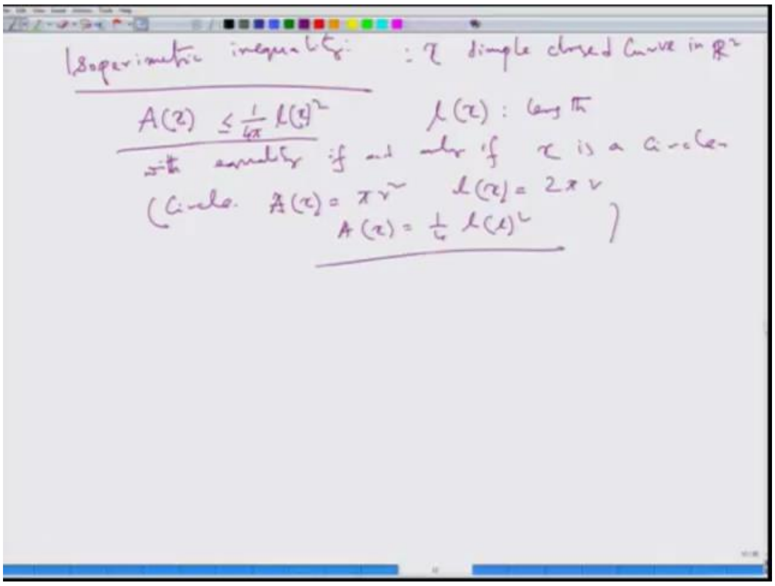

Now, the famous iso-perimetric inequality states that area of gamma, so gamma is a simple closed curve is less than 1 by 4 pie L gamma square, where L gamma, recall this is the length. With the equality, if and only if gamma is a circle.

See, for a circle what is A gamma? This is pie R square, right. And what is L gamma? This is 2 pie R. So, then A gamma equal to 1 by 4 L gamma square. This is iso-perimetric a new quality. It is read in some layman's way that given all simple closed curves with a given with a fixed perimeter, so among all the simple closed curves with fixed perimeter, circle has the largest area because then it will be equality, otherwise all the curves will have less than equal to. Okay.

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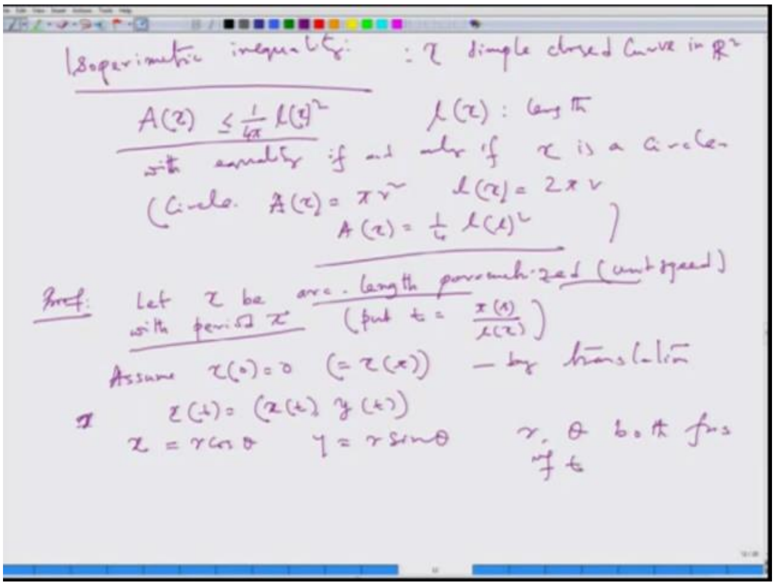

So, for the rest of the lecture I will prove this iso-perimetric inequality. So, without loss of generality, let gamma be unit speed, that is arc length parameterisation, arc length parameterised unit speed parameterisation which has been referred to with period pie. How can I do that? I have to change, just put T equal to pie S by L gamma. That will give you arc length parameterisation with period pie. Whatever the period is, you make that period pie by this translation.

Assume gamma 0 is 0 which is also equal to gamma pie because it has period pie, this I can achieve by translation, right. Whatever the curve is, at 0 I can bring it to the origin. Now, consider gamma T has 2 coordinates, XT and YT and I put X equal to R cos theta and Y equal to R sin theta in polar coordinate. So, R and T, R and theta, both functions of T. Here is the picture.

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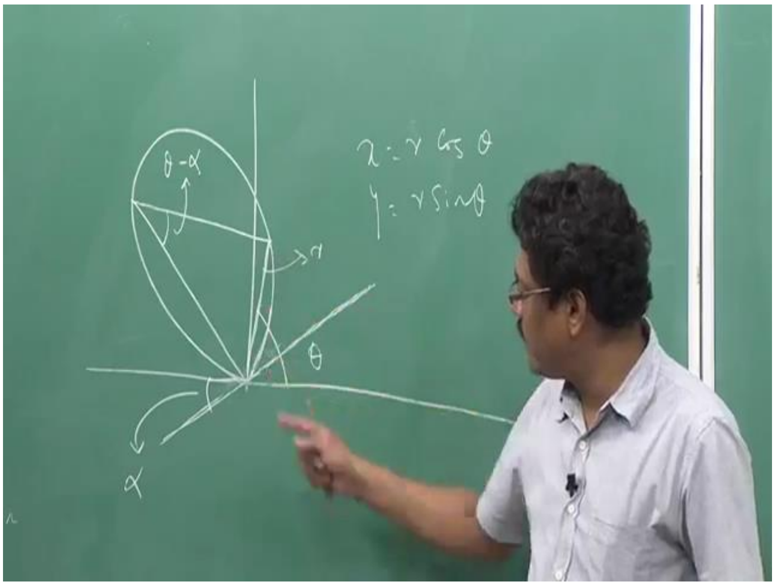

So, this is my gamma, this is the angle theta, it makes tangent, and this particular point XT YT I put in the polar coordinate.

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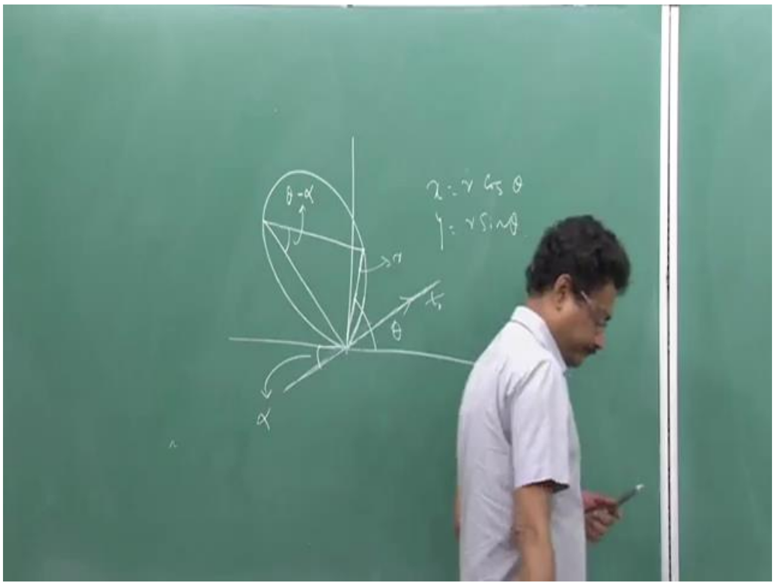

And what I do here, this is at R, I draw a perpendicular at this point, it intersects the curve and I join these 2 points in this angle I call theta which is the angle between x-axis and the tangent vector. So, this is the tangent vector at T0. Okay. So, what do I get from these 2?

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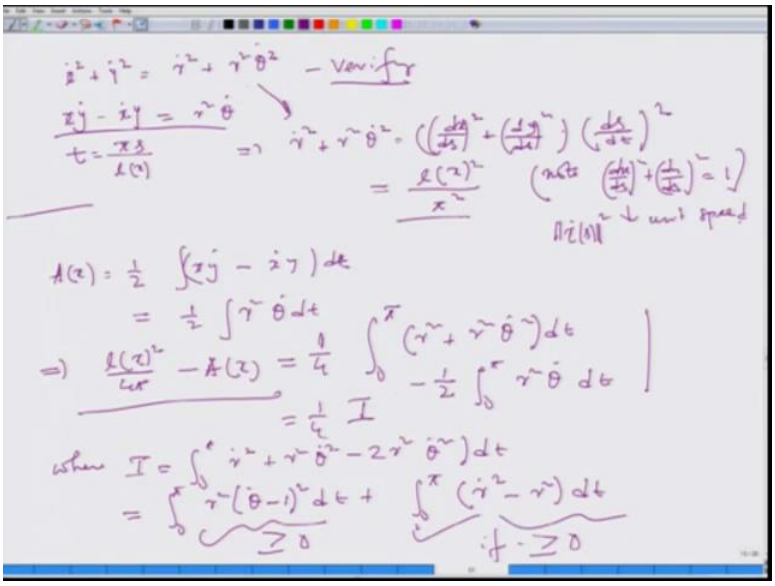

I get X dot, dot is with respect to T, X dot square + Y dot square equal to R dot square, you must to verify this part, theta dot square, verify. And we also get XY dot - X dot Y equal to R square theta dot. Now, what is T, T equal to pie S, this is the translation we have made, L gamma. So, that will give us this quantity, R dot square + R square theta dot square equal to dx ds square dy ds square ds dt square which is equal to, you can easily calculate, L gamma square divided by pie square. Correct.

Why is this so? Because, gamma is unit speed, we have taken unit speed. Note, this is the norm of gamma dot S norm square. So, therefore I get this is equal to A gamma square by pie square. Now, what is A gamma? A gamma is half XY dot, just now we proved H dot Y dt. This was done in here. So, which is equal to, by this equation half of our square theta dot dt. So, L gamma square divided by 4 pie - A gamma, so compare these 2 equations, is equal to 1 by 4, 0 to pie R square + R square theta dot square dt - half 0 to pie R square theta dot dt, check. This follows from whatever I have written down, from this, this follows.

Let us call this equal to 1 by 4 I where I equal to 0 to pie R square R dot square + R dot theta dot square - 2 R square theta dot square dt which I write it as 2 integral 0 to pie R square theta dot -1 square dt + 0 to pie R dot square - R square dt. Now look at this integral. This is square of something square of something integral. So, this is always greater than equal to 0, because positive quantity.

And what about this quantity? If I show this is greater than equal to 0, so if this is greater than equal to 0, I will get this fellow is greater than equal to 0 and I will get my iso-perimetric inequality, correct. If I show this is greater than equal to 0, then I will get this is greater than equal to this always.

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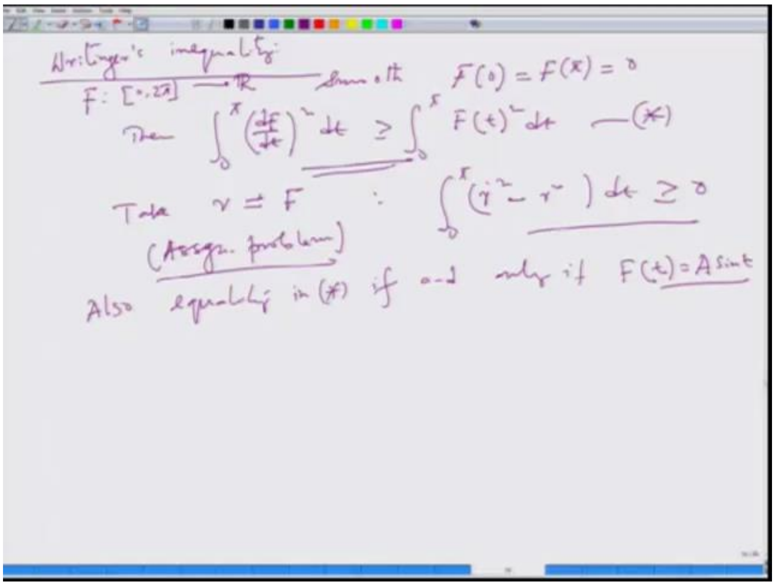

Now, this is a consequence of so-called Writinger's inequality, which says suppose I have a function from 0 to 2 pie to R which is smooth with F0 equal to F pie equal to 0, then 0 to pie df dt square dt is greater than equal to 0 to pie ft square dt. So, take R equal to F, this will give you the 0 to pie R dot square - R square dt is greater than equal to 0.

Writinger's inequality, I put it as assignment problem, this is just doing some manipulation. So, that, and it has also come also the part of inequality, Writinger's inequality, equality, there is equality here in star, if and only if ft equal to A sin T.

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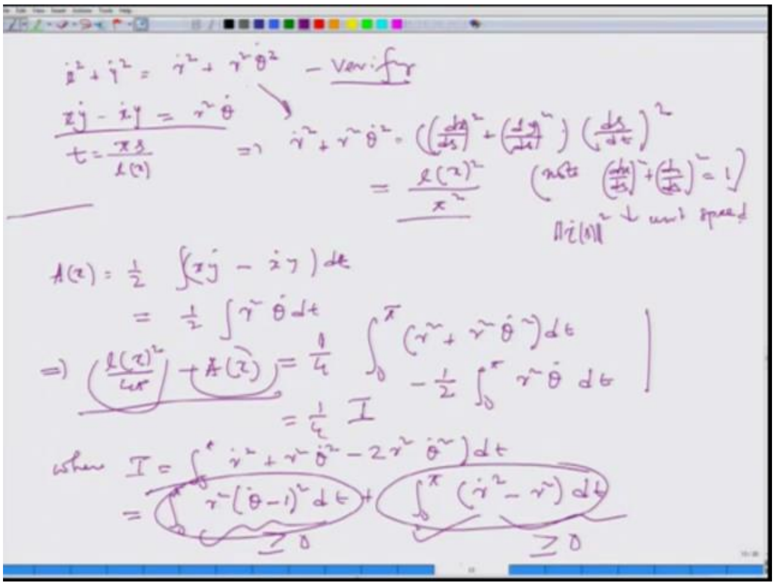

Now, go back here, Writinger's inequality will tell you this is, now if I go move, by Writinger's inequality; this is greater than equal to 0. This is greater than equal to 0 because of the positive quantity. So, I get iso-perimetric inequality that this fellow is always greater than equal to this fellow. Then what is the equality? Equality is both, so I will have equality in iso-perimetric inequality if this is 0 and this is 0.

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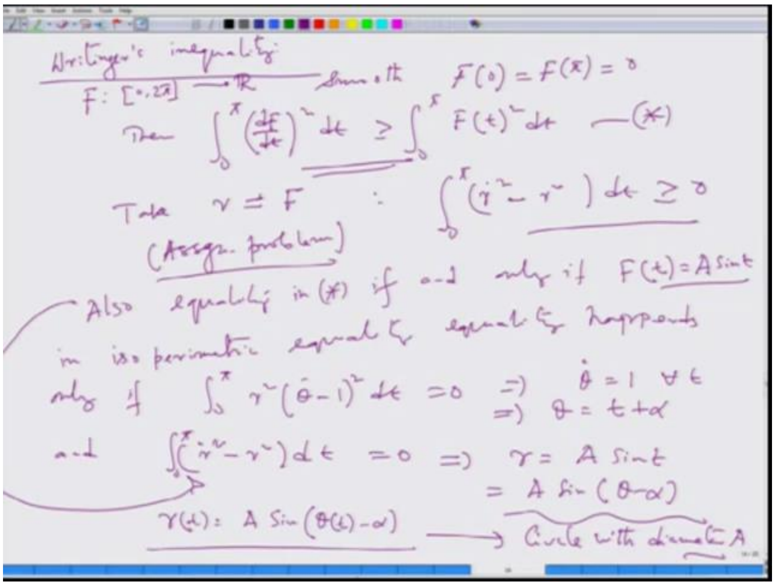

So, that means in iso-perimetric inequality, equality happens only if 0 to pie R square theta dot -1 square dt is equal to 0 which implies Theta dot equal to 1 for all T which implies theta equal to some T + Alpha, some constant. And so 0 to pie R dot square - R square dt equal to

0, but now from this part, this follows that we must have R equal to some A sin T which is sin T which is equal to A sin theta - Alpha.

Now, what is this? RT equal to A sin theta T - Alpha. What is equation of this? This is a circle with diameter A. So, that is a proof of iso-perimetric inequality which as I said in layman’s terms is that I am the all close curves with the given length, given iso-perimetricer, circle has the maximum area. That ends this module for curves and surfaces, next time we will start with surfaces.