Curves and Surfaces.
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Module-I.
Curves in R^2 and R^3.
Lecture-04.
Frenet-Serret Formula.

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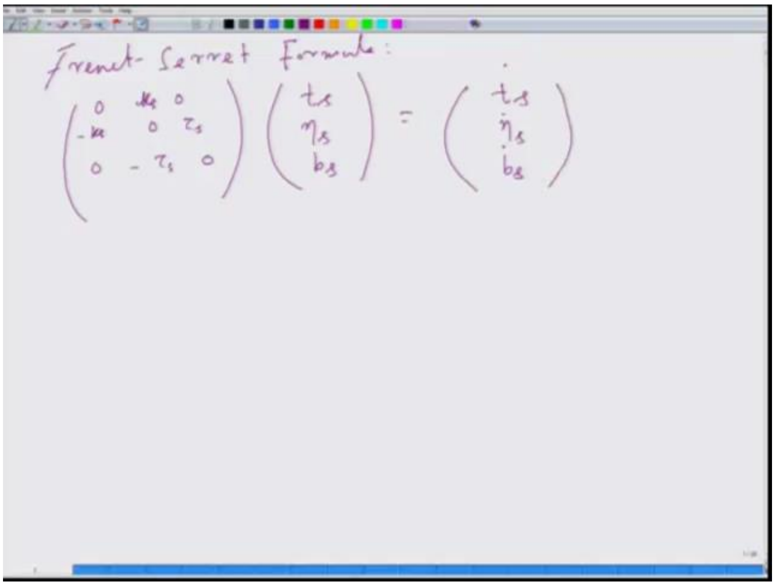

Welcome to the 4th lecture of this module. So, we start by recalling Frenet Serret formula which we wrote down yesterday. Which actually says that, okay, I can write it in a matrix form. That KS0 - KS 0 tau S 0 - tau S 0 TS eta S BS equal to… So, it gives the relation between these quantities we defined that is unit tangent, unit normal and unit binormal.

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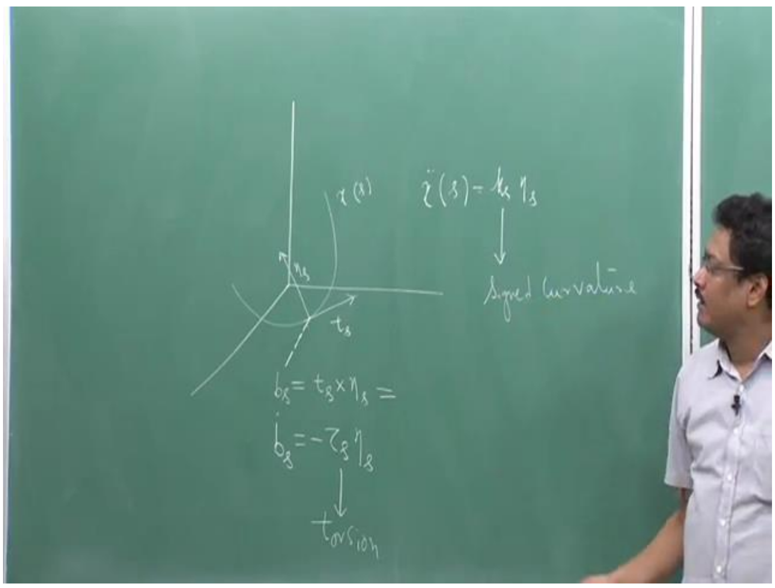

Recall that we have, how we defined these quantities. So, this is a special curve gamma S. I have a tangent at a point S nought S tau S, and you rotate it by pie by 2 and you get the unit normal. And binormal is tau S cos theta is the vector, unit vector perpendicular to both. And the torsion is B dot S, multiple of eta S, so we usually use the - sign. And the signed curvature is given by this one. So, just to recall. This Frenet Serret gives actually, I mean it is a tool to prove what is called invariance theorem for regular smooth curve in R3 and from that you can also deduce the invariance of smooth curves in R2.

What it, what I mean by that? I say that for planar curves, the signed curvature determines the curves of 2 rigid motions. For R3, the spatial curve, the signed curvature and torsion determine the curve of 2 rigid motions of R3. And now for planar curve, as you know the torsion is 0, we have proved it, so the previous theorem follows from this one. So, let me write this as a theorem.

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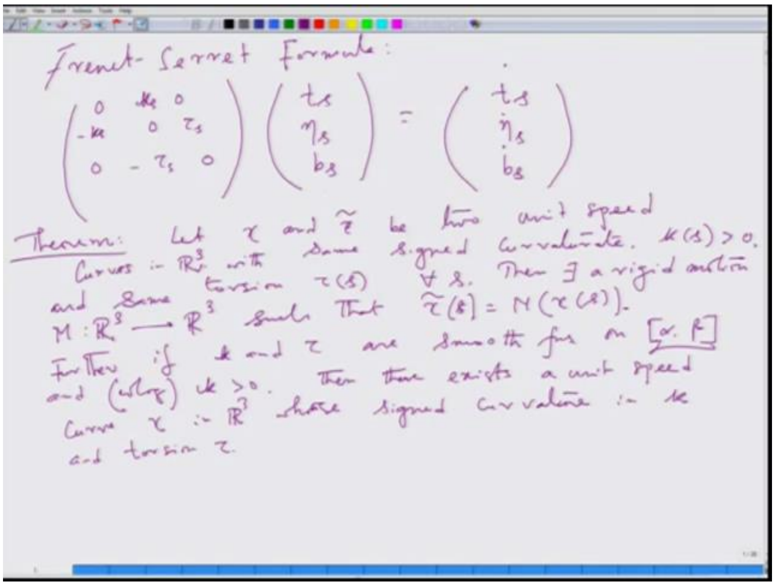

Now, let gamma and gamma tilde be 2 unit speed curves in R3 with same signed curvatures, let us assume KS greater than 0, no problem in Assuming KS greater than 0, otherwise you would look at it in 2 parts, KS greater than 0 part and KS lesser than 0 part. And same torsion tau S for S. So, unit speed curve, so we are considering arc length parameterisation.

The conclusion says then there exists a rigid motion, so recall the rigid motion is always composition rotation followed by translation or translation followed by rotation M such that gamma tilde for each S is M, S. So, to curves with unit speed, when you put it in unit speed, every curve, every regular curve can be, can have unit speed parameterisation. And if they have same signed curvature and same torsion, then they are basically same in the sense that I can rotate and translate to match them to each other.

Actually this has a converse also, in the sense that further, if K and tau are smooth functions on some interval say Alpha, beta. And let us assume without loss of generality K greater than 0, then there exists unit speed curve gamma in R3 whose signed curvature is K and torsion tau. That means, is also defined on the same interval Alpha, beta and this solution is satisfied.

So, for today's lecture, I want to prove this theorem in detail. So, follow the proof carefully, if you have any question which you cannot follow during the proof, please raise it in the discussion forum.

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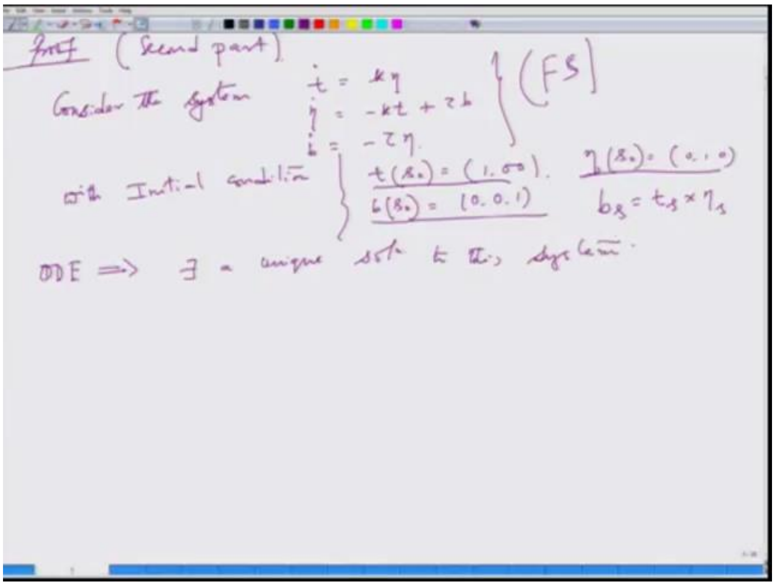

So, let us prove it very slowly so that everybody can follow. I will prove the 2nd part 1st, that is, what was it? That if K and eta are smooth functions and, okay without loss of generality KS greater than 0, then there exists a unit speed curve gamma in R3 whose signed curvature is K and torsion is tau.

So, what I suggest… Consider, this is from, I am following from Frenet Serret equation. Consider the system T dot equal to K eta eta dot equal to K tau, so S is suppressed everywhere, so S is there everywhere, + tau beta and beta dot equal to - tau eta. So, this is the FS equation, right. So, this is a system of linear equation, 3 variable T, Eta and B and K and tau are known quantities. And consider system with IC, initial condition, that your TS nought is 100, eta S nought is 010 and 001. So, what I am trying to do? I am try to find a curve which passes through 0 and at 0 the tangent vector is this one, so passes through some point S nought, S nought can be assumed to be 0 as well.

The number should be perpendicular, so I take this perpendicular, unit channel vector as perpendicular and binormal to this one. Because recall that I should must have BS equal to TS dot eta S. And so in this system, this is satisfied. Now if I have, look at this linear system FS with this initial condition, then the general theory of ODE will tell you, so ODE will tell you that there exists a unique solution to this system.

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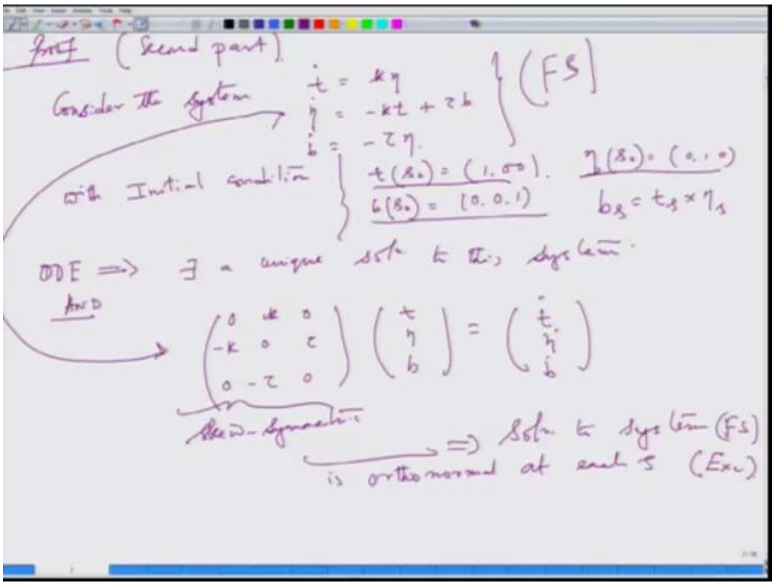

And importantly if I look at here, this is in matrix form what? Recall, this is 0 K 0, then - K 0 tau 0 - tau 0 T eta B equal to T dot eta dot beta dot. And this matrix is skew symmetric. That will imply us that the solution to system FS is orthonormal at each S. I put it as a simple exercise. If you do have, if you have done your vector calculus in +2, this is very easy to prove. So, what I got? The system has a solution, so I can find T, eta and B. So, now I have to define the curve.

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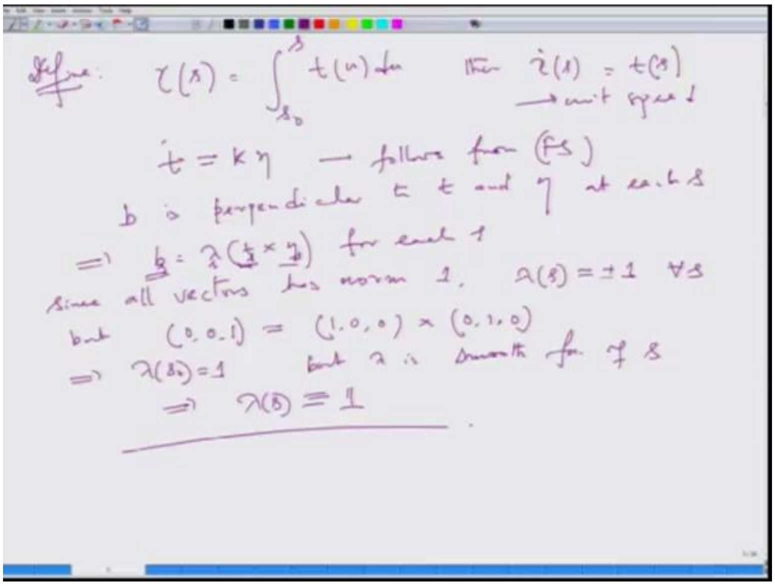

So now define gamma S TU dU, then gamma dot S is TS. So, this is unit speed because T is a unit speed vector. You can verify because this is a solution of FS, this follows from FS and now B fellow is perpendicular to T and eta at each S hence B should be some lambda times T cross eta. BS lambda S T S eta S for each S. Since all vectors are unit norm 1 lambda S must be equal to + or -1 for all S. But look at this point.

This implies lambda S equal to1, it is not equal to 1. But lambda is a smooth function, we have discussed this part because B is smooth, B is smooth function, T is smooth function, eta S is smooth function, this is a smooth function of S, this will imply lambda S is identically 1. So, this is simple vector calculus, nothing great in it. Okay. So, this is the proof of the 2^{nd} part. Let me prove. 2^{nd} part proof is okay, I hope everybody?

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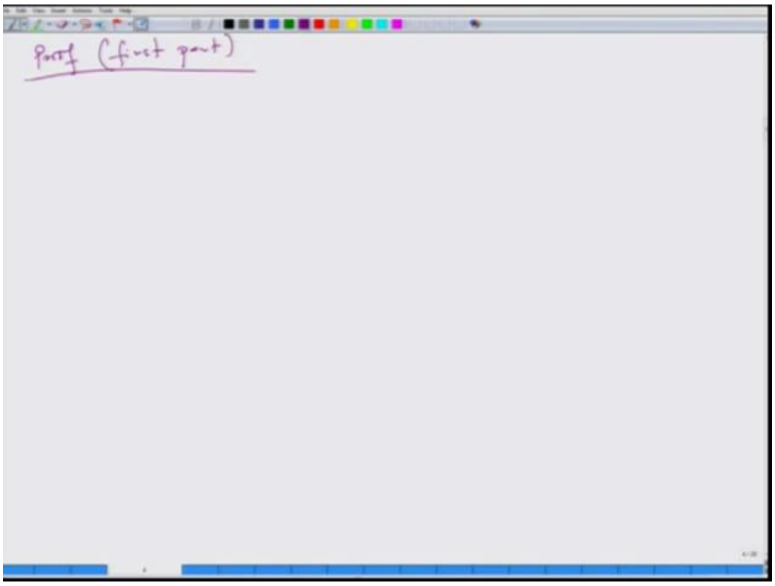

Then let us go to the proof of the 1st part. Okay. So, what I have here? I have a situation like this.

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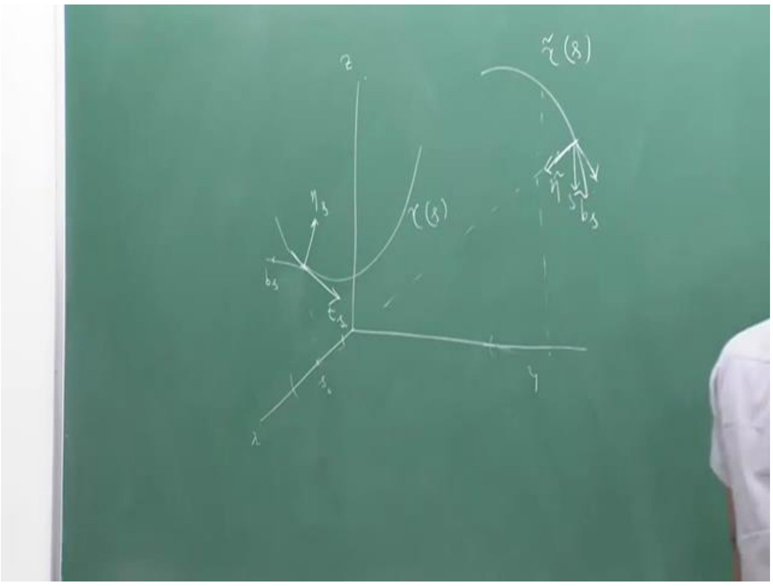

So, I have some spatial curve sitting here and I have another spatial curve sitting there. So, this is the point, here is x-axis, S nought. And here I have some other interval Alpha, some other interval I should write this way. Okay. So, now 2 curves are there which are both are unit speed. What are same? We should extend it little bit. Yeah. What is given to be same?

Tangent vector is here and binormal, these are binormals and here I have tangent vector this way. Let us say T tilde this is gamma tilde S. This is gamma S, this is a point S nought this is eta tilde S and binormal will be which way, this way, right. So, this this beta tilde S, B tilde S. So, this is the situation here and what I am given?

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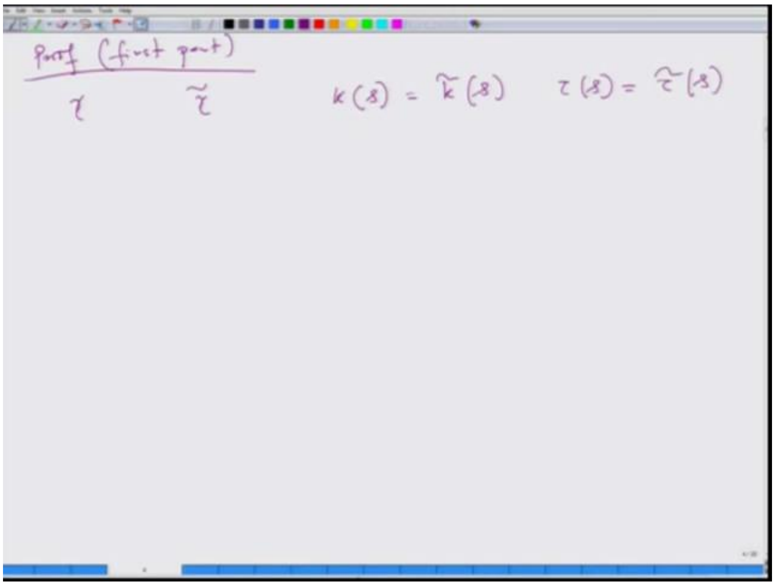

So, I have 2 curves gamma and gamma tilde for which I have given for each point KS equal to K tilde S and tau S equal to tau tilde S.

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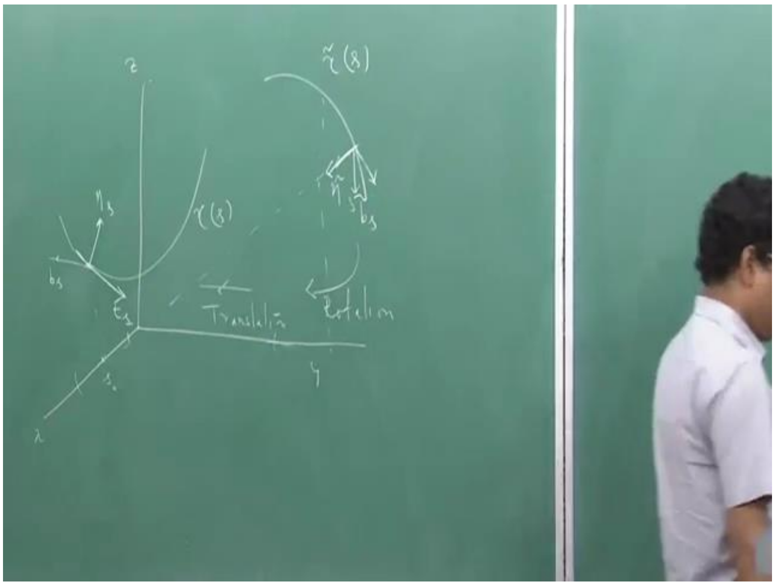

Now what I can do, I can always put a rotation followed by a translation to matchup by rotation followed by a translation,

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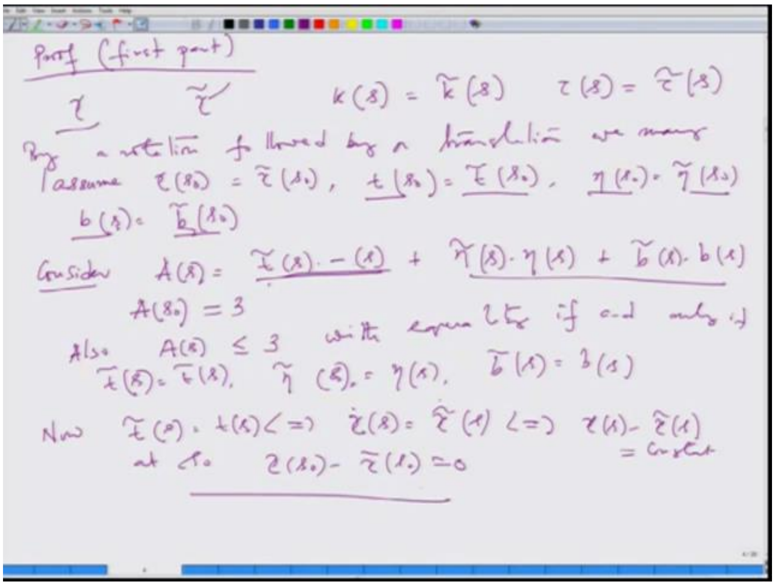

we may assume that the point, the point is not I am looking at, gamma till, gamma S nought is gamma tilde S nought. Anyway T S T T S nought is given to be T tilde S nought. And eta S nought is eta tilde S nought and BS nought equal to beta tilde S nought. So, beta tilde, tilde is a quantity corresponding to gamma and without tilde, this is a quantity which is corresponding to, okay without tilde it is gamma, with tilde it is gamma tilde.

Consider this particular fellow AS, this I defined a function. T tilde S dot TS + eta tilde S dot eta S + B tilde S dot BS. What is AS nought, how much is that? This is unit vector, this is unit vector, so this is 1 at S nought, unit vector unit vector unit vector unit vector, so this is equal to 3. And also AS is less or equal to 3. Why? Because this fellow is of course less or equal to its norm, this norm is 1, this norm is 1, this norm is 1, so AS is less or equal to 3. With equality, this is by Cauchy Schwartz, if and only if T tilde S equal to TS eta tilde S equal to eta S B tilde S equal to BS. Right.

Now, T tilde S equal to TS will give us, what does it mean, this is if and only if by definition gamma dot S equal to gamma tilde S. So, this will happen if and only if gamma S - gamma tilde S is constant. And the constant and S nought, gamma S nought is 0. So, this is an observation, with that let us calculate what is A dot S.

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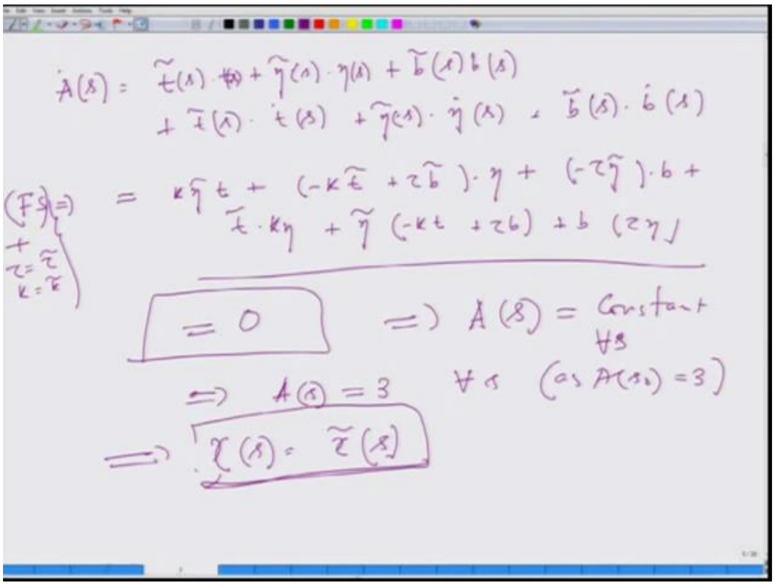

A dot S is T tilde S dot T eta tilde S dot eta beta tilde S TS eta S BS now I have to take care of the other quantities. T tilde S T dot S + eta tilde S dot eta dot as + B tilde S dot beta dot S.

Use the formula, this will give us K eta tilde T and suppressing S there + - K T tilde + tau B tilde dot eta + - tau Eta dot B + T tilde K eta + eta tilde - KT + tau B + B tau eta. FS + using tau equal to tau tilde K equal to K tilde. So, these 3 you used together, you will get the formula. And you from, just open up the bracket and to see this is 0. So, this will imply AS is constant for all S. And, so this implies AS equal to 3 for all S as we have seen AS nought is 3. So, that will give us gamma S equal to gamma tilde S. Okay.

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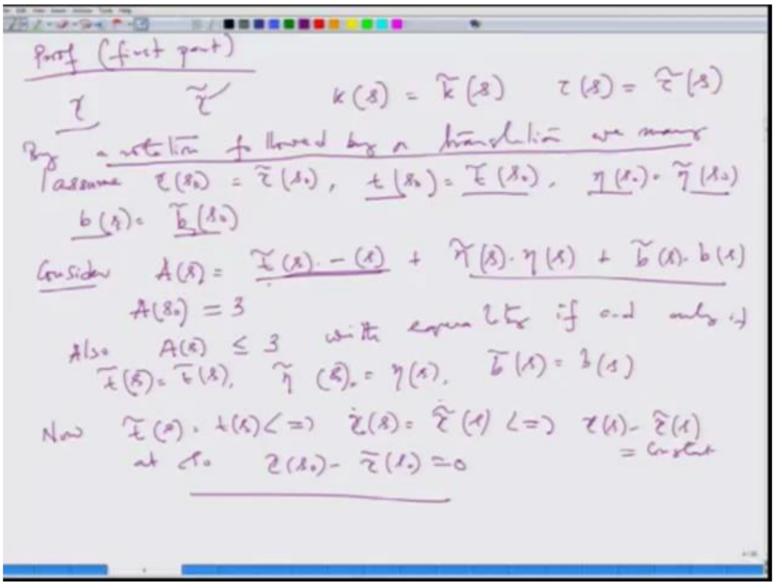

This is up to, I have written this but this is up to which one, rotation followed by a translation.

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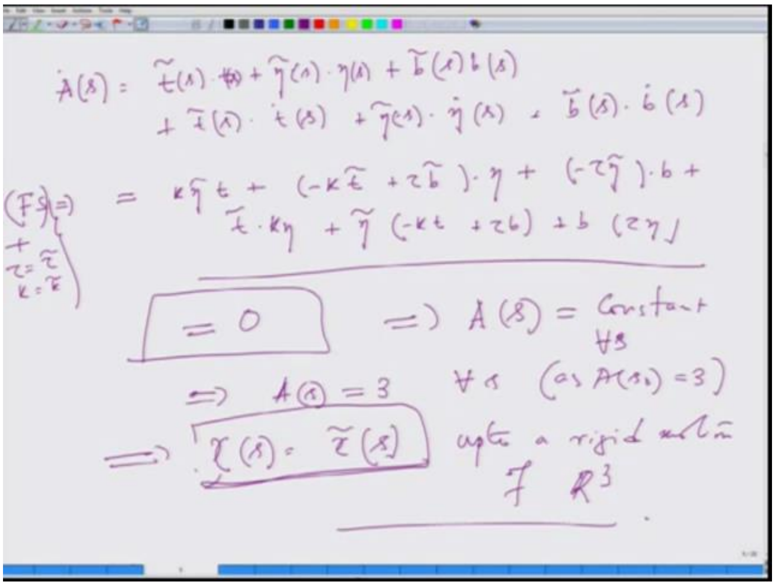

That means, up to a rigid motion of R3. So, this is the proof for the fact that if 2 planar curves smooth regular planar curves have same portion and same signed curvature, they are same up to rigid motion of R3. Again, simple vector calculus. Next lecture we will see some beautiful application of all these ideas and that will be end of curves. I covered this curve part very quickly because I want to concentrate more on surfaces from module 2 onwards. So, next lecture we will do what is known as, an application, what is known as iso-parametric inequality. So, till then goodbye.