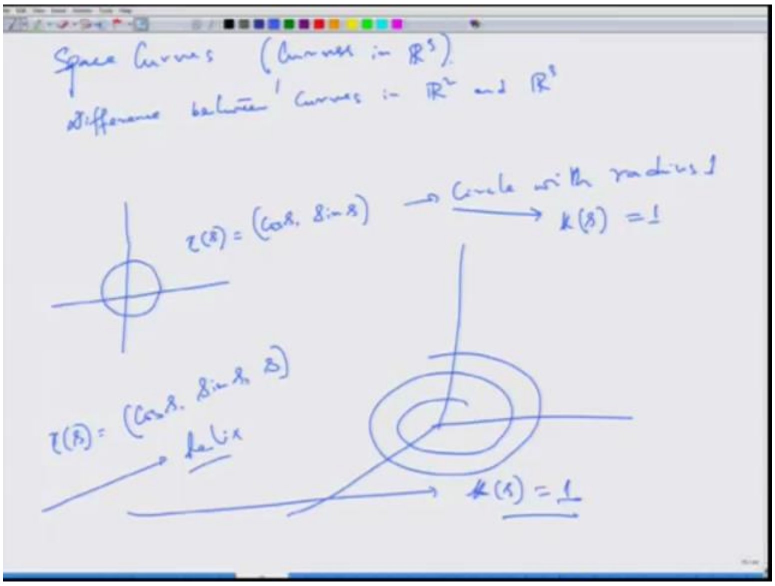
Curves and Surfaces.

Proffessor Sudipta Dutta.
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur.

Module-I.
Curves in R^2 and R^3.
Lecture-03.
Curves in R^3, Principal Normal and Bi-normal, Torsion.

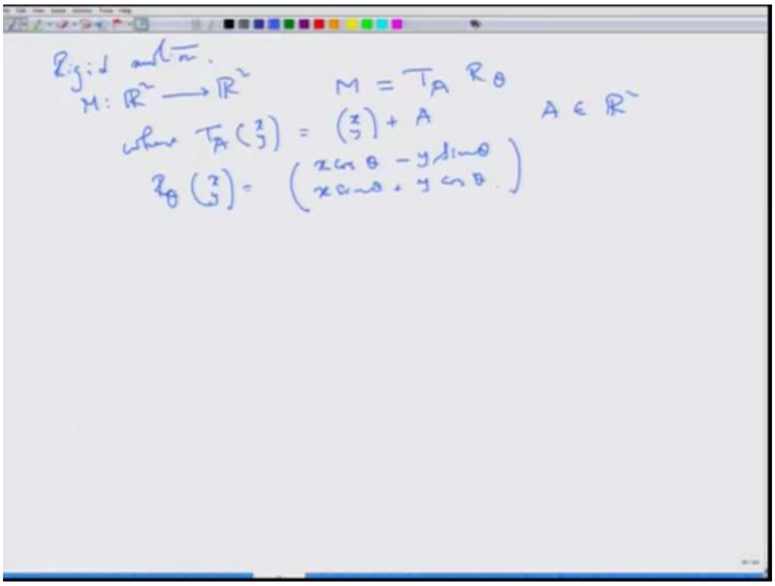
(Refer Slide Time: 02:00



Okay. So, in the third lecture we will talk about space curves, that is curves in R3. And in the process we will also complete something we have left undone for the planar curves. So, first see, let us see those fundamental difference between curves in R2 and R3. I have in the first lecture, suppose, in the first lecture, I had given this example, circle. Let us say, S equal to cos S sin, circle with radius 1. Unit speed parameterisation, right. Also, we have talked about this fellow.

Which is gamma S equal to cos S sin S S which is a Helix. Now if you calculate the sign curvature of circle, this will be 1. And similarly, for the sign curvature of Helix, this will also be 1. So, Helix what happens, Helix is a circle on x, y plane and it spirals on Z axis. And so a circle can never be mapped to Helix by rigid motion. What do we mean by rigid motion? Okay?

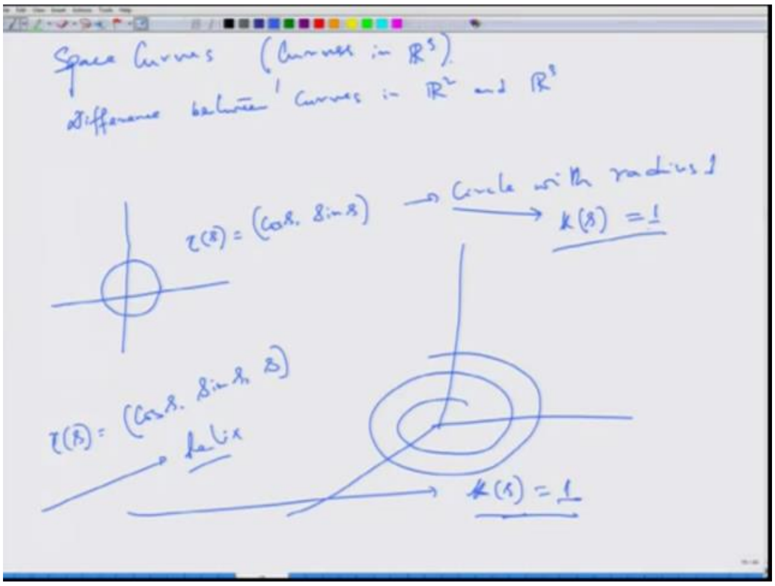
(Refer Slide Time: 03:10)



Rigid motion, let us say in plane is a map from R2 to R2 which is in, which is given by a translation by a vector composed with, so first rotation and translation or first translation and then by rotation, so what is translation? Translation is, suppose I were better x, y is vector x, y + A, A in R2.

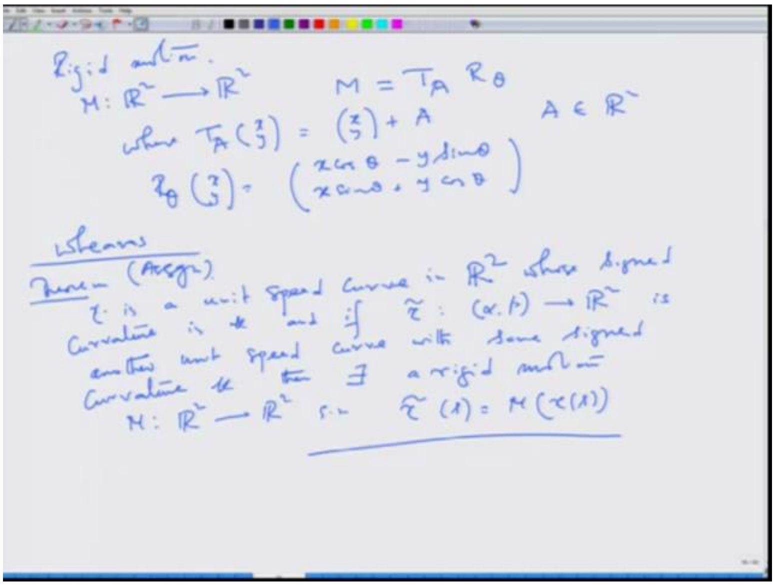
And what is rotation? This is simply rotating by an angle theta. X sin theta by cos theta. So, such a thing is called a rigid motion. So, you can imagine that if I have a planar circle and if I translate it or rotate it or whatever I do, I cannot make it a Helix.

(Refer Slide Time: 3:25)



So, both of them but has sign curvature, same sign curvature but no rigid motion can map a circle to Helix.

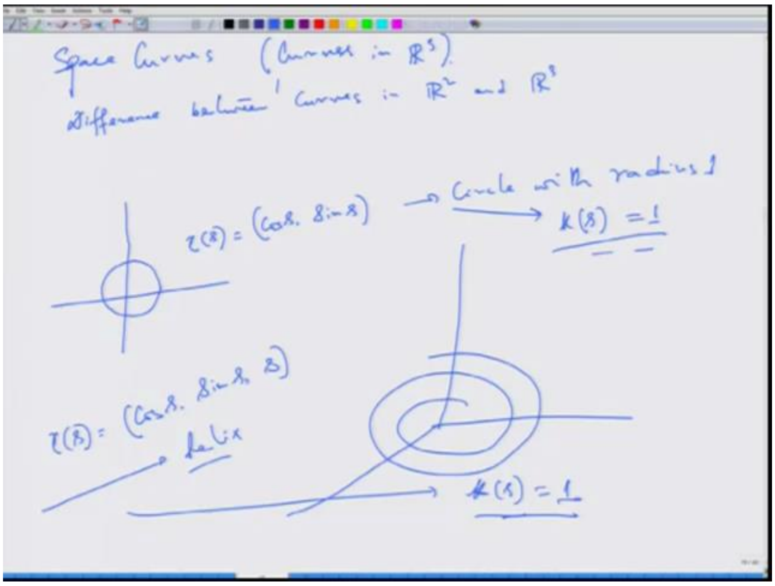
(Refer Slide Time: 05:05)



But whereas, I have this theorem which is a part of the assignment, that suppose gamma is a unit speed curve in R2 in the plane, whose sign curvature is K and if gamma tilde for some of the interval Alpha, beta to R2 is another curve, another unit speed curve with same sign curvature K, then there exists a rigid motion M from R2 to R2 such that gamma tilde S equal to N gamma S.

This is what I meant by in the last lecture, that any planar curve is determined by its sign curvature of 2 translation or rotation. That if 2 curves in a plane has same sign curvature, then one mapped into another by translation and rotation.

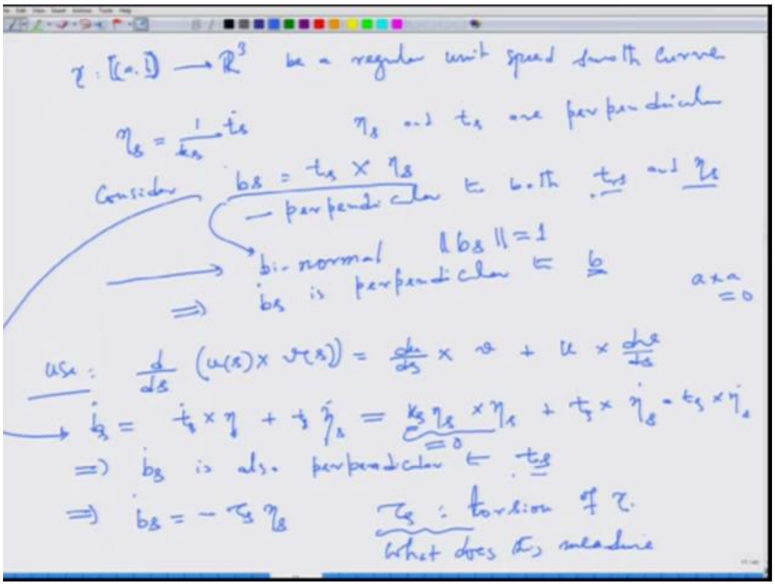
(Refer Slide Time: 05:35)



But as you see just in the previous example, if in R3, if I have same sign curvature of 2 curves, I can never map through any rigid motion of R2 or R3 a circle into a Helix. A circle cannot be made to a Helix by any transformation, any translation or rotation.

So, to study curves in R3, we need to have some more quantities to define. So more invariance of the curves.

(Refer Slide Time: 6:00)



So, let gamma from a,b to R3, let us make it a,b this way, be a regular unit speed smooth curve. Okay. What we have done before? That its unit normal is equal to KS by T dot S, this we have defined before, right? KS is, we have defined it this way. And you know Eta S and

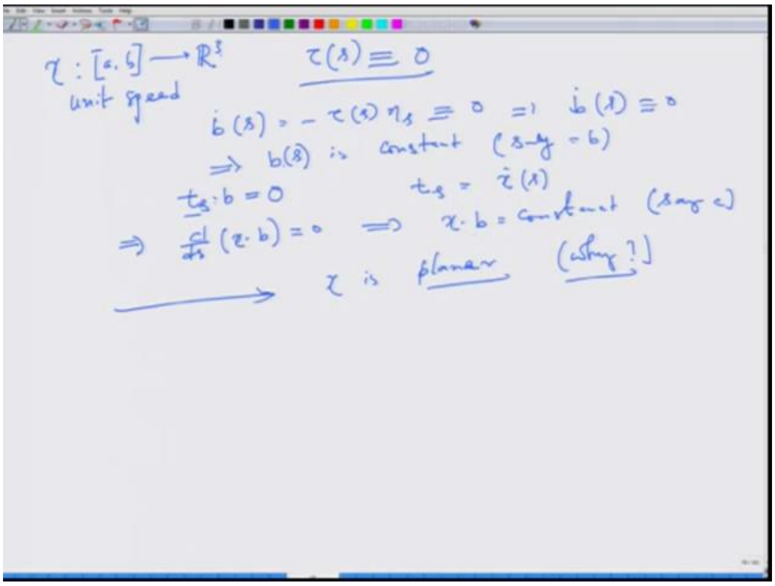
TS are perpendicular. Now, consider this vector BS equal to T as cost product with Eta S. So, this is perpendicular to both TS and Eta S. At any point, this vector is perpendicular to both unit normal and unit tangent vector.

This fellow is referred as, so this has a name, it is called binormal. Of course this is a unit vector and any unit vector, so B dot S is perpendicular to B. Now, B is perpendicular to both TS and Eta S. B dot S is perpendicular to B. Okay. Now you see, if you use this thing that I have used ds of 2 functions I have u and v, and if I look at their crossproducts and if I take the derivative, this is du ds crossproduct V + U crossproduct dv ds. So, use this, what I get B dot is equal to from this part equal to T dot, I am suppressing the S part + T Eta dot, I will use the previous formula.

This is K Eta, sign curvature, okay, let me put the S. Look at the formula for sign curvature cross Eta + T cross TS cross Eta dot S, let us repeat. So, this would tell as that B dot S is also perpendicular, now this fellow is 0, right, because KS is a quantity, scalar quantity and I am taking crossproduct of Eta S with Eta S. Crossproduct of A with A is 0. So, this is equal to TS cross Eta dot S. So, what it says, B dot S is also perpendicular to TS. So, it is perpendicular to B, it is perpendicular to TS. And B is perpendicular to TS and Eta S. That implies BS dot is equal to some constant at point S. For some reason I have put a negative mark here, you will see why.

This fellow tau S is referred as torsion of gamma. See, this makes only, makes sense only in R3 because if I take in R2, this crossproduct does not make sense. If I have 2 vectors in R2, the crossproduct does not make sense. So, the torsion only makes sense for curves in R3. So, what is this, so what is this quantity? What does this mean?

(Refer Slide Time: 11:14)

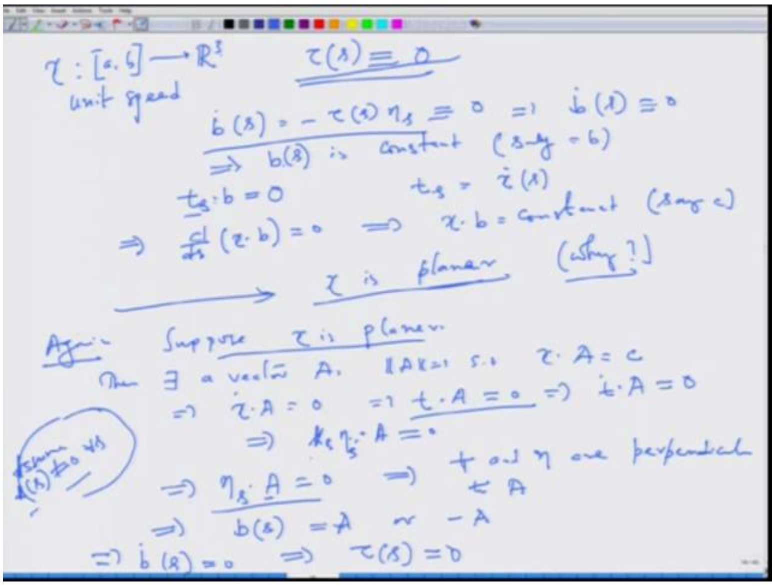


Okay. So, to see this let us say, I have a curve gamma and suppose unit speed regular everything everything, smooth and this is identically 0. So, I want to see what torsion is there.

So, let us see if I have a curve whose torsion is 0, what does it say? What does it mean? B dot S which is - tau S Eta S, this is 0 identically. That means B dot S is 0 identically. What does it say? BS is a constant, say equal to B. Now TS dot B, B is binormal is perpendicular to both T S and Eta S. Binormal is perpendicular to TS and Eta S both. And BS is identically B, so it TS dot beta is 0. What does it say? B is a constant. What is TS, recall TS is gamma dot S. So, therefore B BS was gamma dot B, this is 0. Implies gamma dot B is constant. Say C. Gamma dot B, so gamma is a curve in R3 listen to me carefully again gamma is a curve in R3 whose dot product with a fixed vector is constant.

That gives gamma is planar. Why? If I have a curve in R3, I cannot have a fixed vector, unit vector B whose dot product with A is always constant. Why is so? So, why, just think about it… If I do not have a, I have a proper curve in R3, so all the 3 coordinates are, it cannot be all the 3 coordinates are 0, any one coordinate cannot be 0 all the time, then it cannot have a fixed vector whose dot product with it is 0, because dot product is an angle geometrically. So, B is a vector in R3. I am explaining this why part, you think about it yourself also, this is a vector in R3, gamma is a curve whose angle with the fixed vector remains same.

(Refer Slide Time: 14:30)

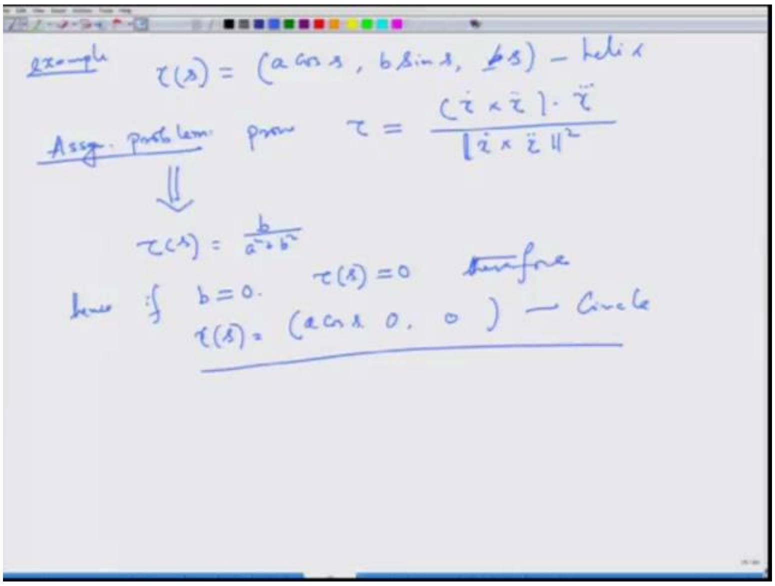


So, gamma S to be planar, planar in the sense it may not be the plane (())(14:34). Now, also again, suppose gamma is planar, okay. So, then there exists a vector, unit vector A such that gamma dot A is constant. You just take the vector which is perpendicular to the plane of gamma. So, that means gamma dot A, this is also 0, that gives us T dot A T is 0. And if I do another derivative, T dot dot A is also 0. But T dot is sign curvature Eta S, so sign curvature Eta S A dot equals to 0. Assume KS not equal to 0 for all S. That will give you Eta S dot A is equal to 0.

That will give you T and Eta both are perpendicular from this and from this perpendicular to A. Therefore binormal by definition, this is equal to A. Or - A because binormal is a unit vector which is perpendicular to Eta S and TS and A is a unit vector which is perpendicular to T T TS and Eta S. So, binormal is either A or -A. Therefore B dot S is 0, which will imply torsion is 0. So, what it says, suppose I have a planar curve which moves and has vanishing curvature 0 torsion, then, sorry. I have a planar curve which moves and has vanishing curvature, then torsion is 0 and on the other hand, if I have torsion as 0, then the curve is planar.

So, what does torsion measure now? Torsion precisely measure how much it is away from being a planar curve. Okay? So, that is the geometrical interpretation of torsion.

(Refer Slide Time: 17:37)

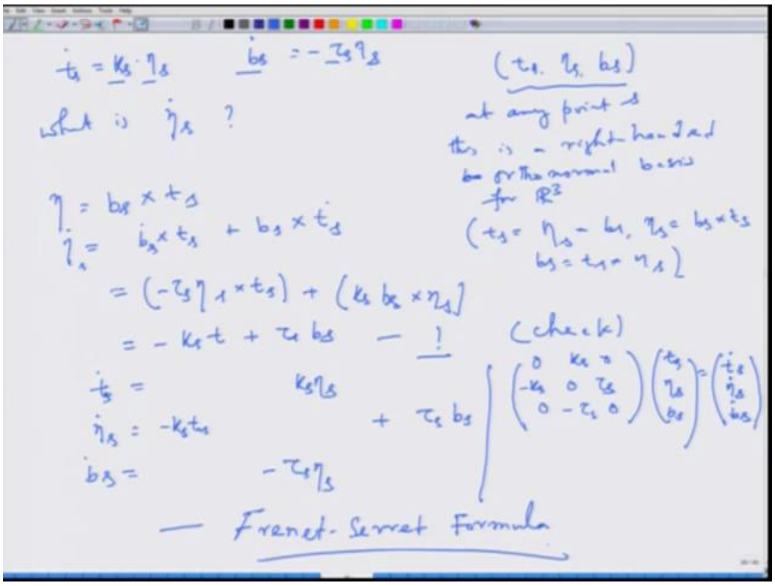


So, let me put an example here. Again the helix, let us say BS. There is a problem for assignment also. Torsion is equal to gamma dot gamma double dot dot gamma triple dot, count the number of dots, important. Gamma dot gamma double dot square. Here is the formula for torsion. So, this formula will give us for this particular curve, this helix, tau S is B by S square + B square. Hence if B equal to 0, then torsion is 0 and therefore gamma S is A cos 0, this is a circle.

So, this example was to illustrate that a planar curve, a curve is planar in R3 curve in R3 is planar if and only if the torsion is 0. This torsion measures that how much it leaves from the plane. Now you see, I can to make the variance of a curve in plane, sign curvature is a complete invariant. So, sign convention of 2 curves unit speed has same sign curvature means, by some rigid motion I can map them, map one into another. What do I do for curves in R3?

To do that there is a very important tool in this subject, I will write it down. So, what we have seen?

(Refer Slide Time: 20:15)



We have seen that T dot is sign curvature dot Eta S. We have also seen B dot, the binormal S is - tau S into Eta S. So, I got T dot S B dot S. Now we have TS eta S BS. At any point S, any point S, this is a right-handed orthonormal basis for R3, in the sense that TS is Eta S cross BS, Eta S is BS cross TS, BS is TS cross Eta S, correct. And they are all orthonormal unit vector. But I know T dot S in terms of, I know T dot S in terms of sign curvature Eta S. I know B dot S in terms of sign curvature Eta S. So, what is Eta dot S?

Well, I know eta equal to, just now I wrote, BS cross TS. So, again using the previous formula, ETA dot S is equal to B dot cross TS + BS cross TS dot which is equal to, if I use formula, I have a formula, TS cross TS + KS BS Eta S, that is simply - KS T + TS BS. How do I get that? Check. So what do I get? I got T dot equal to KS TS is Eta, K dot S, KS eta S, eta dot S equal to - KS TS + BS and B dot S equal to - tau S into Eta S. I am writing this here for a purpose because I have to write it as a matrix. Okay. Or in matrix form, what I have? 0 KS 0 - KS 0 tau S 0 - tau S 0, TS eta S for each point BS equal to T dot S eta dot S B dot S.

So, this is the relation between derivatives of the quantity TS Eta S Beta S. Relation between vector TS eta S BS at any point to their derivatives. And this is a famous formula called Frenet Serret formula. And this will be used to prove the invariance, so we will prove in the next lecture that this quantity sign curvature and the torsion determine a planar curve, sorry, spatial curves up to rigid motion of R3. So, let us do it for the fourth lecture.