Curves And Surfaces.
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Module-IV.
Surfaces-3: Curvature and Geodesics.
Lecture-22.
Surface Area And Equiareal Map.

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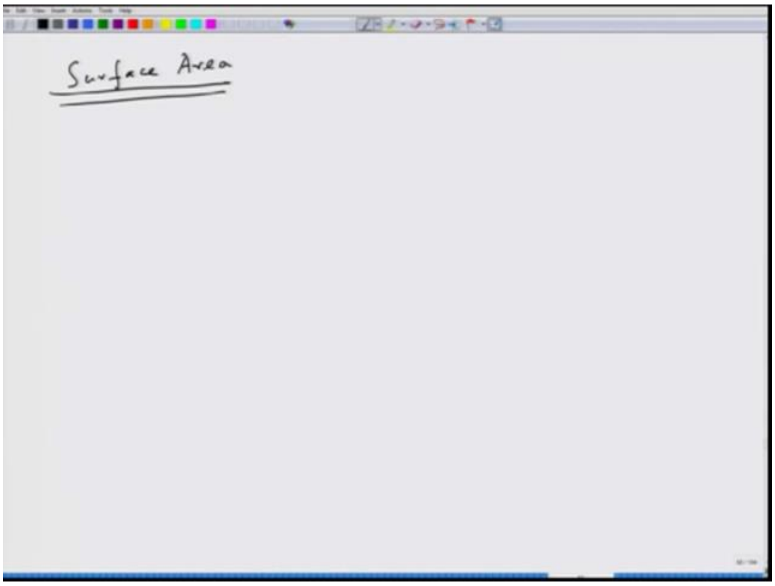

Okay, in the last lecture we started with surface area. As in the curve we have one parameter is curve length, this is for surface, we should be concerned about surface area. Now, how do you measure a surface area?

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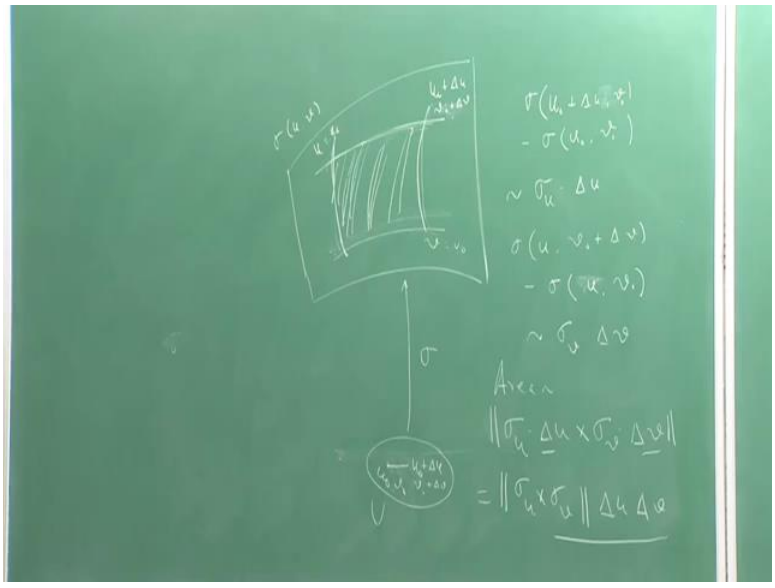

That you see, suppose I have a surface patch over some U so, this is my surface patch Sigma UV. So, the idea is as you measure length by approximation, here also you measure by approximation. So, you choose some fixed point U0 here U0 V0, so that will give us 2 curves, let us say like this U equal to U0 and let us say V equal to V0. Now, if I move on U to U0V0 in surround direction, so U0 + delta U and V0 + delta V, I get 2 more curves nearby, this is U0 + delta U, this is let us say V0 + delta V. Now, the change in the U direction, so Sigma, so this area, this patch which looks like parallelogram, you can as see really well that area of this, so change, how much change from U to U0 and U0 + delta, how much change in Sigma for any V actually?

This can be measured, approximated by Sigma U delta U. So, this length can be measured Sigma U by the which length, yah, this one… Similarly for this length, change in V coordinate, sign may differ but it is not going to, this apply mean value theorem. So, area of this patch, this little patch under covered by these curves U equal to U0, U equal to U0 + delta U, V equal to V0 + V0 + delta U, that area is approximately how much? Sigma… You know the area of the paragraph is Sigma U delta U cross Sigma V delta V. A parallelogram with sides A and B, the area is A crossproduct B norm. But delta U and delta V are scalars, so this is basically Sigma U cross Sigma V delta U delta V. Then if you get this patch, what do you do? You integrate.

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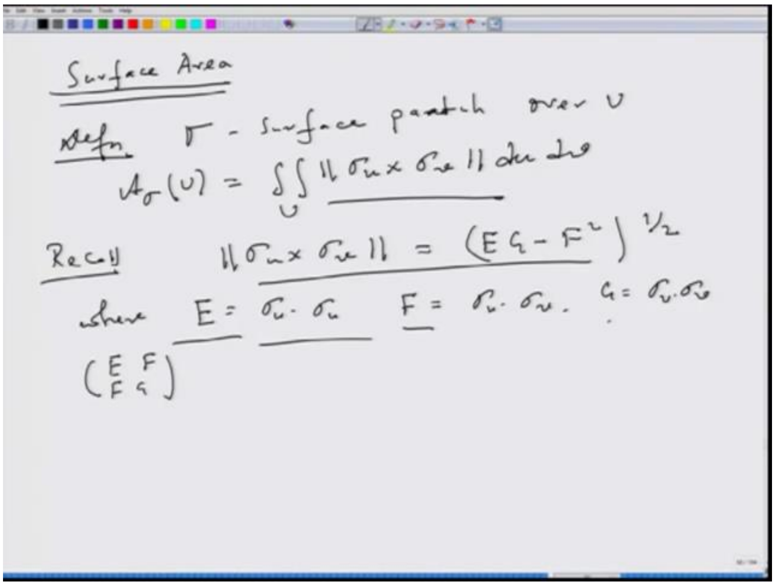

So, after you integrate, your surface area definition so, suppose I have a patch Sigma, surface patch over some U to area A Sigma U, this is integral over U Sigma U cross Sigma V du dv.

But one of the exercise I left which is very easy to prove actually, why not, recall what are the exercises. That this Sigma U cross Sigma V is actually EG - F square. Where E is Sigma U dot Sigma U, F is Sigma U dot Sigma V and G is Sigma V Sigma V. The entries in the… So, this is the determinant of the matrix, we have used it while discussing 1st fundamental form. So, entries in the 1st fundamental form that also enters in the area. How do you prove this thing? This proof is very easy, it is just, let me prove it, why not.

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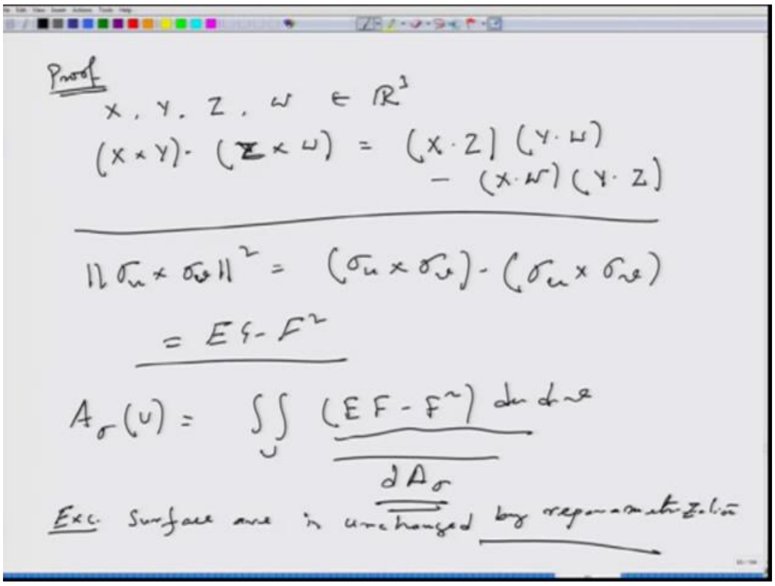

You know, if I have any vector, let us say X, Y, Z, W vector in R3, then I know X cross Y dot Z cross W, this is X dot Z Y dot W - X dot W Y dot Z. This is from vector algebra. So, you just calculate this in this form, Sigma U cross Sigma V square which is Sigma U cross Sigma V dot Sigma U cross Sigma V, calculate, you will get this is equal to EG - F square. So, surface area over patch can also written as, okay. So, this quantity sometime written as, okay, forget it. This is a surface element, sometimes written as dA Sigma. Okay. Of course one has to show that surface area is unchanged by reparameterisation. This is expected, Jacobean determinant will come up but then it will cancel each other in the crossproduct, so try to do it. Okay. So, this is about surface area.

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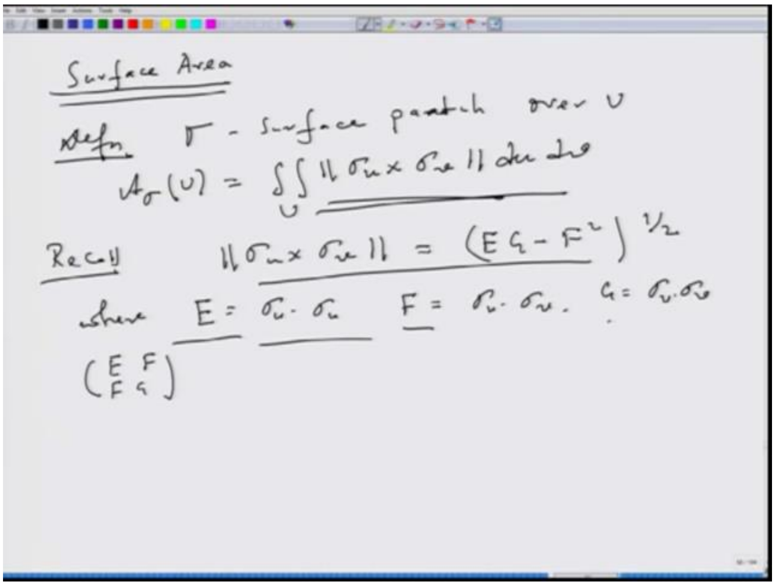

And with surface area, okay, this formula or this formula, whatever you want to use.

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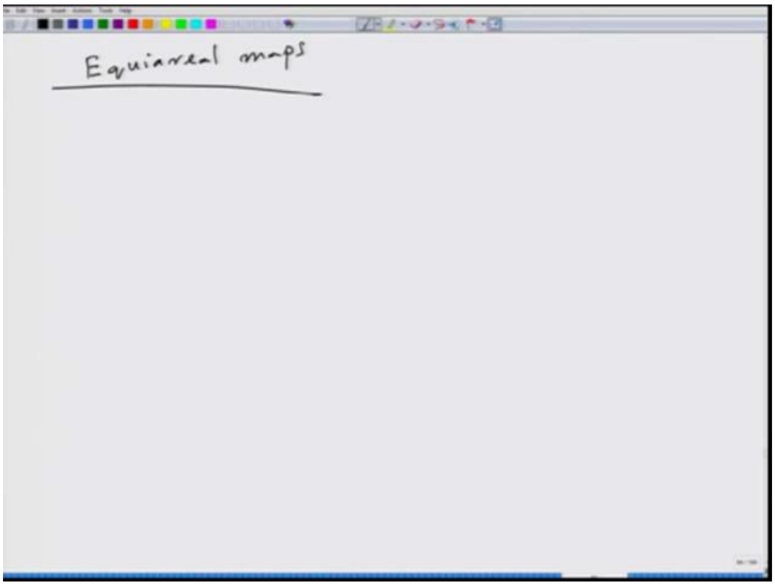

With this I want to define so-called equiareal maps. What is equiareal map, okay, what is isometry? Isometry preserves the length of every curve on the surface. Conformal maps, they preserve angle between the intersection of 2 curves, so equiareal maps which are the maps which preserves the surface area.

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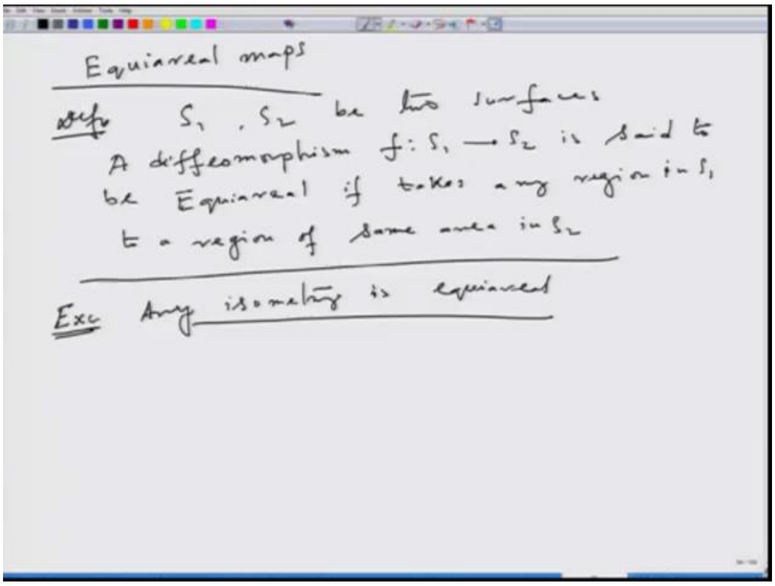

That means, suppose I have 2 surfaces S1 and S2 be 2 surfaces, a diffeomorphism f from S1 to S2 is said to be equiareal equiareal if it takes any region, any patch, any region in S1 to a region of same area in S2. Okay. Simple. This preserves the area. We will see one example of equiareal maps. 1st of all one must convince yourself, any isometry is equiareal. Isometry is conformal, isometry is equiareal, so isometry preserves everything, converse must not not true, one must find some example for that. But, before I do example, before I do one example of equiareal maps, very classical one,

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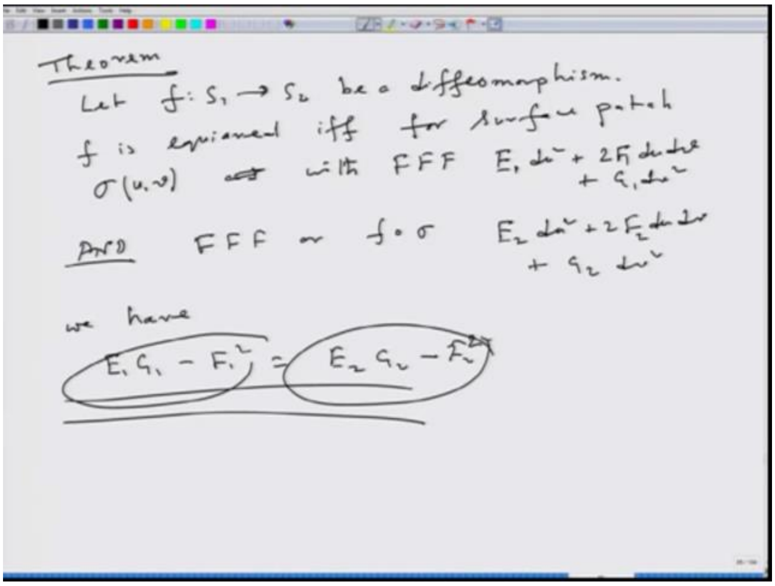

I want to write down this theorem of which I do not prove because proof is straightforward, very easy. it is like the proof of conformal maps. So, let f from S1 to S2 be a diffeomorphism. Okay. Then f is equiareal if and only if for every surface patch Sigma UV and okay, with FFF E1 du square 2F du dv + G dv square. If and only if surface patch Sigma UV with FFF and FFF of f compose Sigma, f compose Sigma, let us say E2 du square + 2F du dv + 2 F2 F1 G1 G2 dv square. We have E1 G1 - F1 square equal to E2 G2 - F2 square. Remember this is the quantity that appears in the integral of calculation of surface area. Okay. So, this is expected, this should be true, so try to prove it. So, equiareal if and only if1 G1 - F1 square equal to E2 G2 - F2 square, okay. So, let us assume this theorem and try to give an example which is due to Archimedes, it has a nice story.

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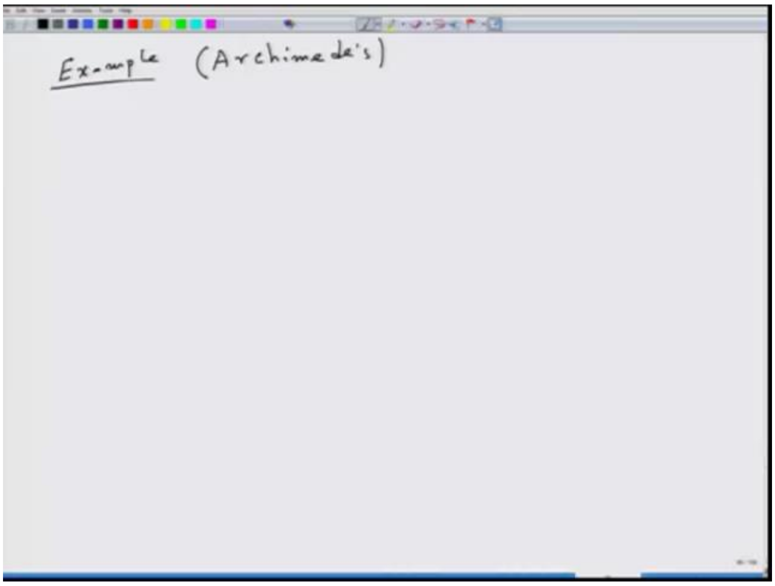

Let us forget the story part. so do one example due to Archimedes. But Archimedes of course did not used 1st fundamental form, he has to do some basic methods. Okay. This is like this, so you consider the sphere inside the cylinder.

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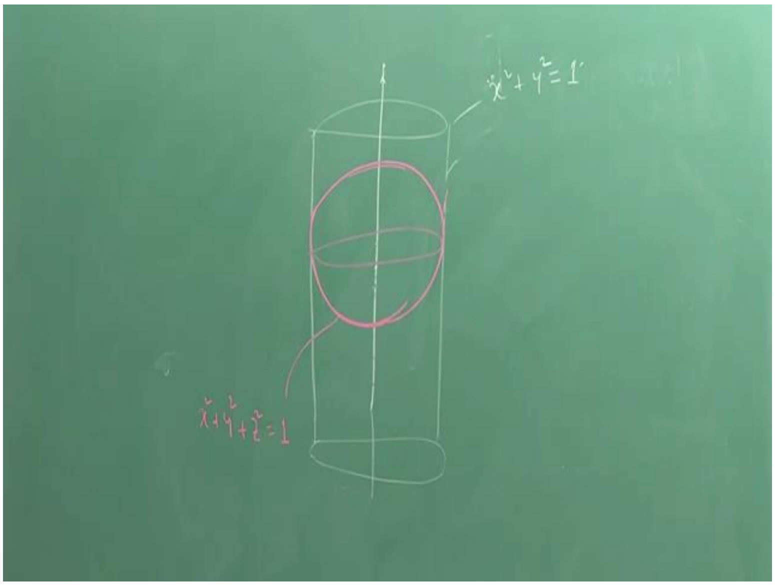

Let us take the unit cylinder and you consider the unit sphere inside it, I should have drawn the sphere 1st, then the cylinder, anyways… This is the cylinder, this is the sphere, okay. I am going to construct an example of equiareal maps.

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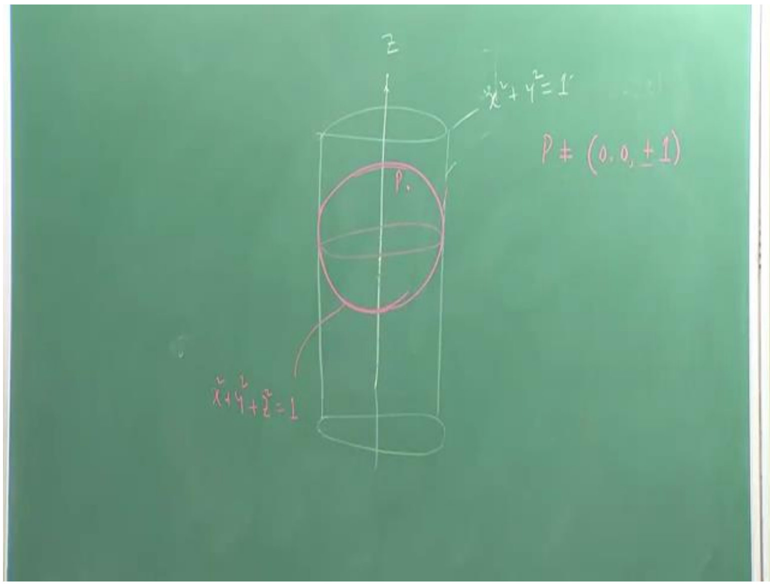

Okay. So, let us take a point P on the sphere. Okay, other than the poles, not equal to poles, not the poles, so you can remove the poles from the sphere to construct this map. So, this is my Z axis, right, so through this sphere, there will become if I draw a line which is parallel to XY plane and passing through P and the Z axis, okay.

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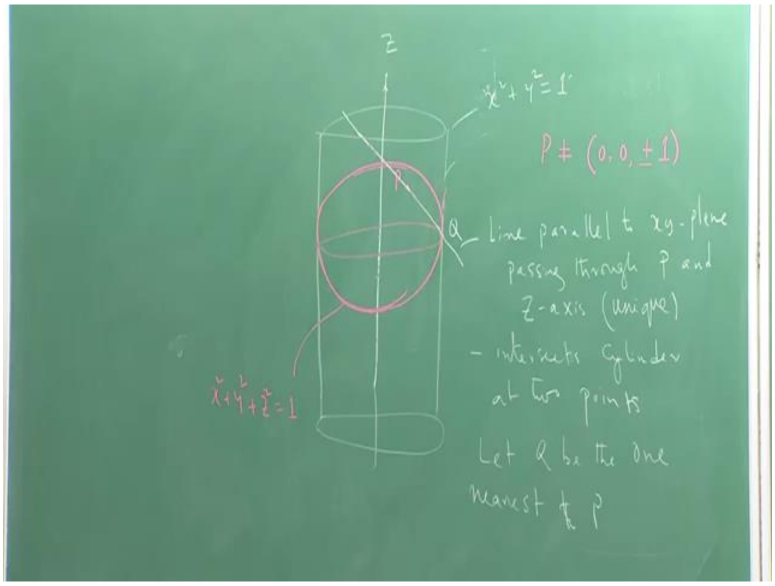

I draw a line parallel to XY plane passing through P and Z axis, that line is unique, such a line is unique. So, this will intersect the sphere, this will intersect the cylinder in 2 points. So, this line will intersect the cylinder in 2 points 1 in this side one on that side, the surface of the cylinder, this will intersect cylinder at 2 points and one of them will be nearest to Q, P. Let Q be the one nearest to P, so Q maybe this point. Okay.

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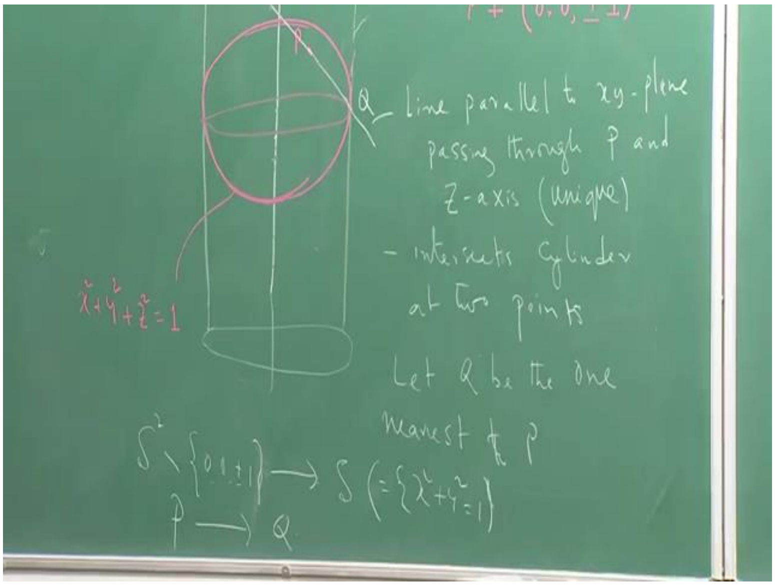

And I take my map, the map is the sphere delete the poles to cylinder, precisely this map, P going to Q. And according to Archimedes, this is one example of equiareal maps. How? We will see.

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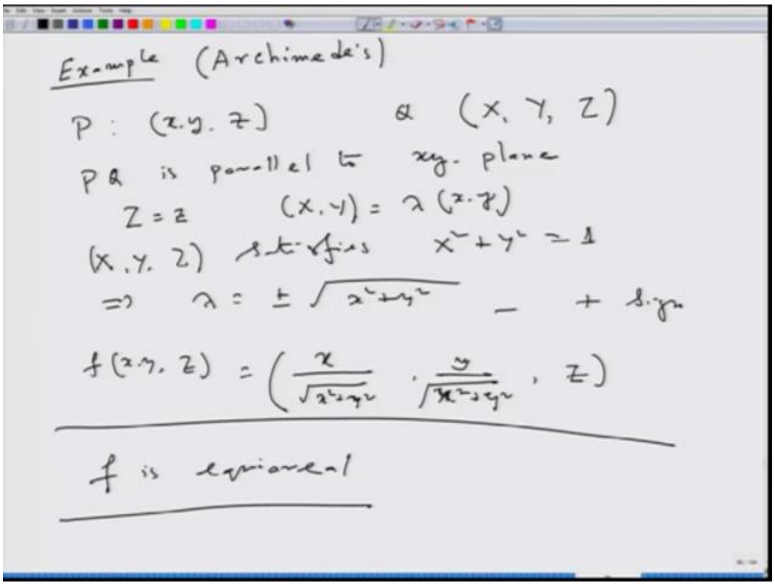

So, let us write down the formula for F. So, let P be x, y, z and Q be X, Y, Z, theek hai. So, this PQ line is parallel to XY plane, so Z will be same and xy will be proportional to XY, but X, Y, Z satisfies x square + y square equal to 1. That will give me lambda equal to + - root over x square + y square. So, take the + sign and here I want the nearest point, so f of x, y, z will be x by root over x square + y square, y by root over x square + y square and Z. This f is equiareal. How do you prove it?

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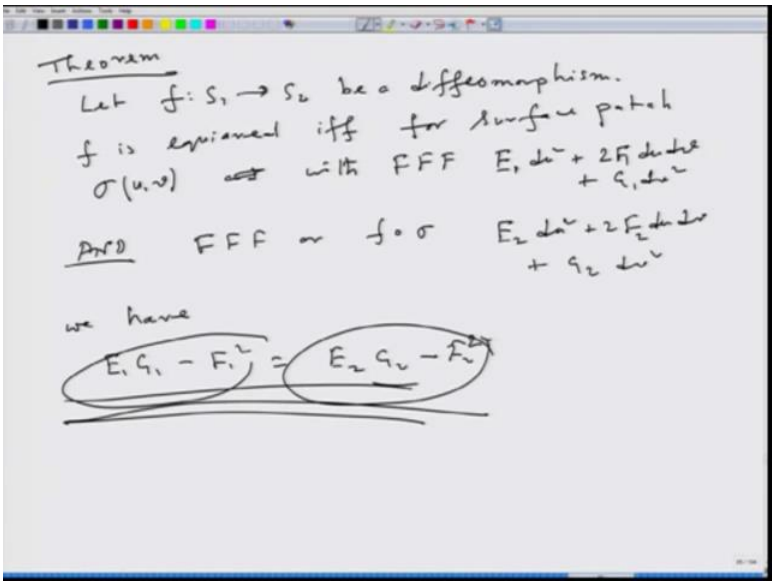

Well, to prove we actually come back here. We want to show that E1 G1 for the sphere E1 G1 - F1 square and similarly further is equal to E2 G2 - F2 square for the cylinder.

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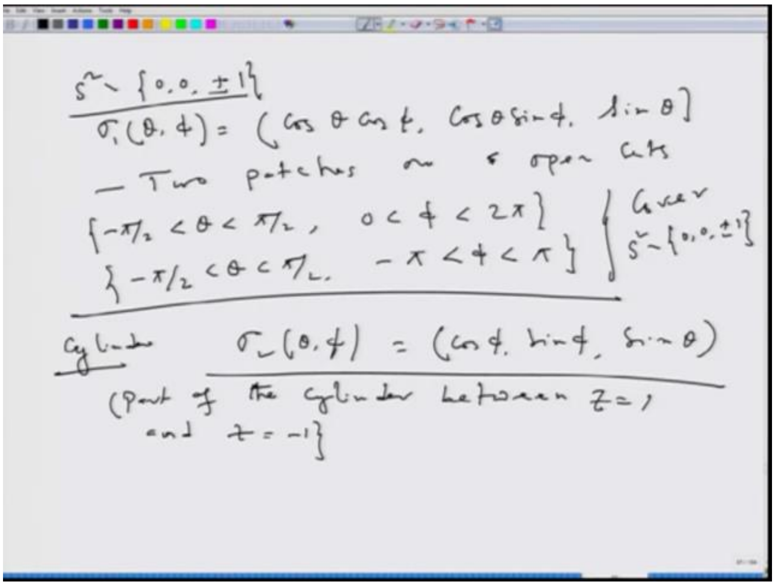

So, you take the parameterisation, so you take the parameterisation Sigma 1 Theta, phi is one, we have been taking throughout the course cos phi cos cos Theta sin phi sin Theta but I have to take 2 patches on open sets. What are the open sets? One this, another is a rotation by angle pie, okay, this will cover, this will cover S2 - the pole.

For cylinder I need only single patch. Sigma 2 Theta, phi is equal to cos Phi sin Phi sin Theta, right. So, this is not for the entire cylinder, this gives the part of the cylinder, this is part of the cylinder between Z equal to 1 and Z equal to - 1.

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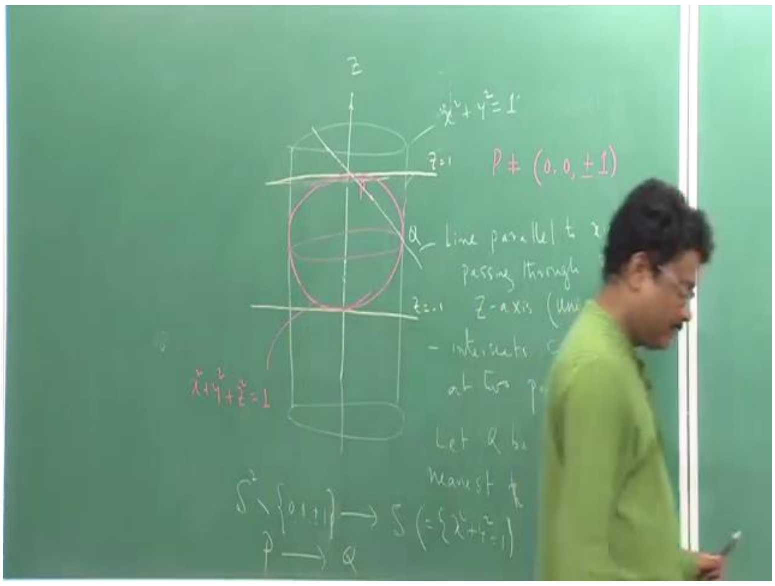

But that is a part where our sphere lies, rights, that is exactly the part where our sphere life, sphere this North pole is Z equal to1 and South pole is Z equal to -1 is 2 plane.

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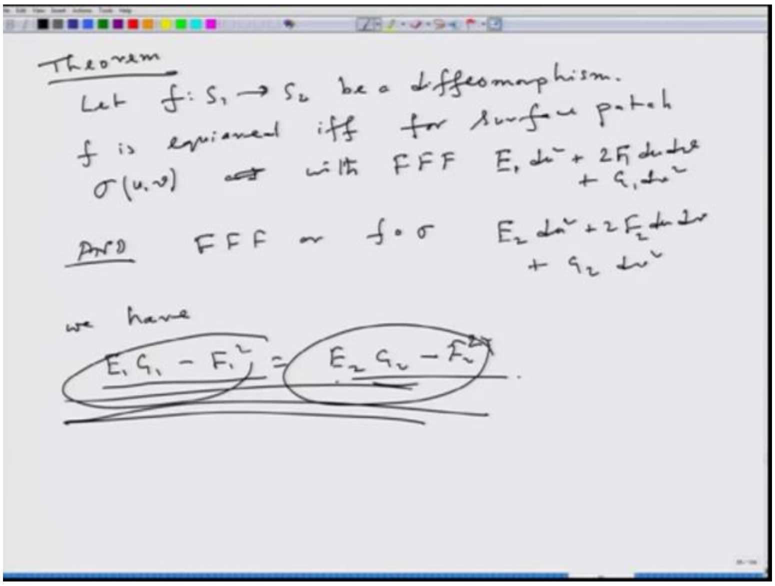

Now you do what? Simply calculate these quantities E1 G1 - F1 square, E2 G2 F2 square and you see they are same.

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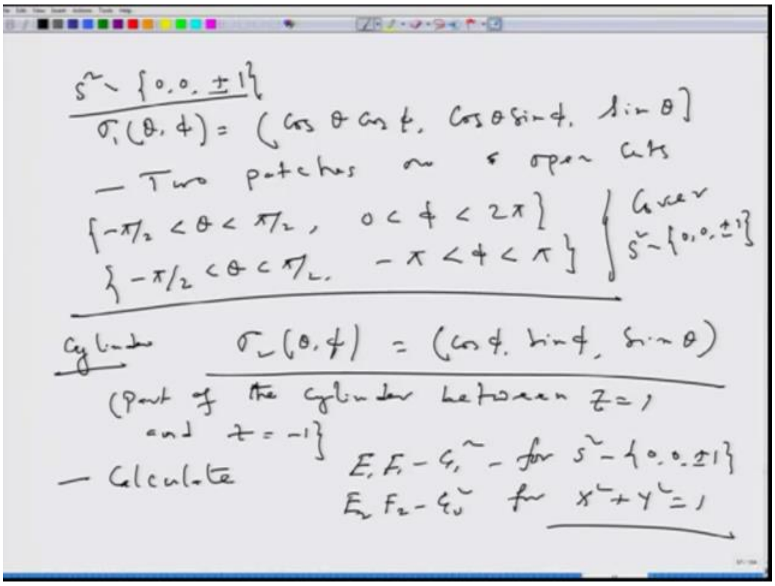

You have the parameterisation, you know how to calculate now, okay, so calculate from here, calculate E1 G1 - F1 square for S2 - 0 0 + -1 and E2 F2 - G2 square for x square + y square equal to1 and you see they are same. And that is it, so from the theorem we have map is equiareal. So this is one example of equiareal map and that will be the end of this lecture. Thank you.