Curves And Surfaces.
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Module-Iv.
Surfaces-3: Curvature And Geodesics.
Lecture-19.
Geodesics On Curvature Of Revolution; Clairaut's Theorem.

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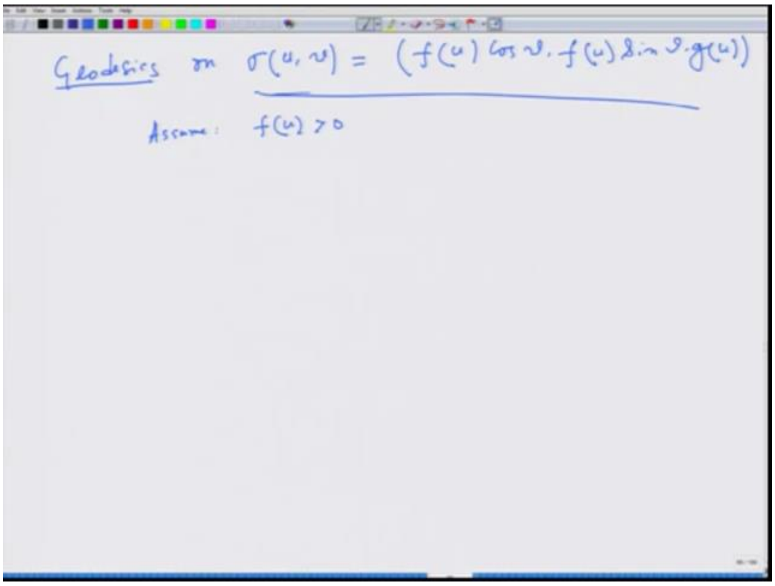

Okay, let us continue, so our aim is to find geodesics on a surface of revolution which is parameterised by Sigma UV FU cos V FU sin V GU. And what are the assumptions?

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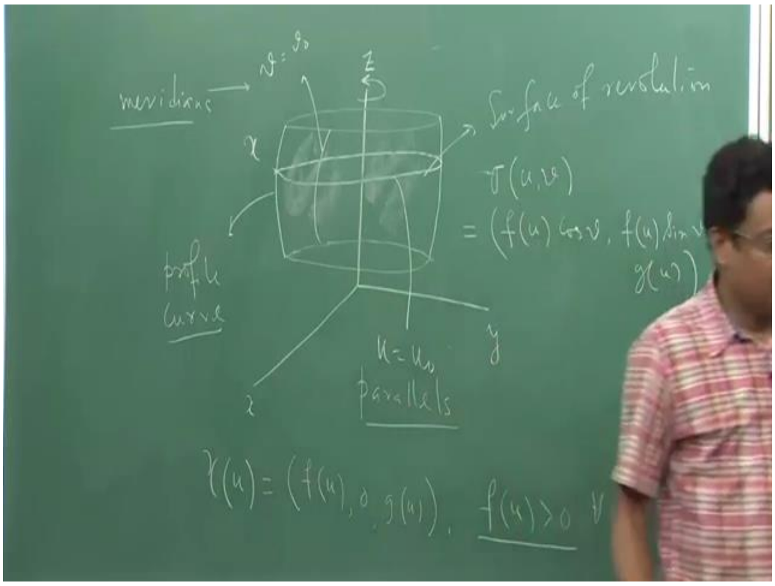

We assume that FU is greater than 0, so that the curve does not come here on the board, curve does not intersect the Z axis and we take, so my curve gamma is FU0 GU,

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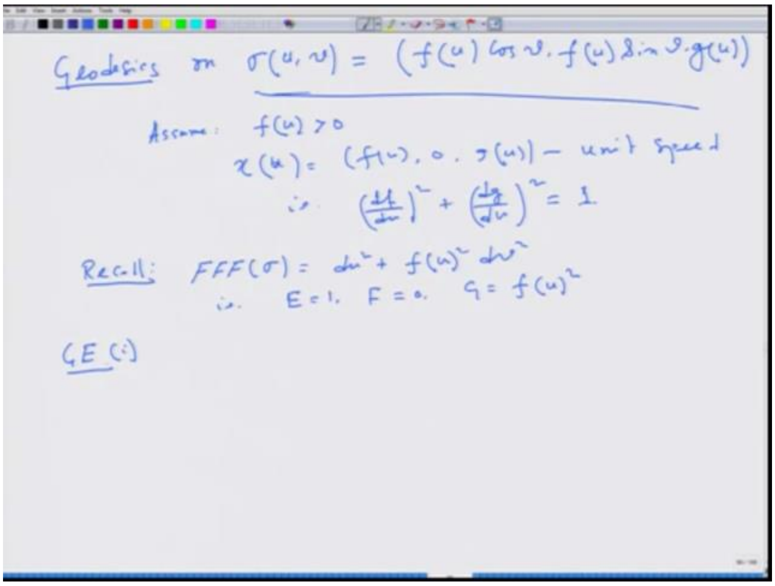

and we will take it as usual a unit speed, that is df du square + dg du square equal to 1. Okay. We have, we will just recall, it was exercise problem, so the assignment problem or I did it myself, I did not remember. The 1st fundamental form of Sigma is simply du square FU square dv square. That is E equal to1 F is 0 and G is FU square. Okay. So, what happens to geodesics equations?

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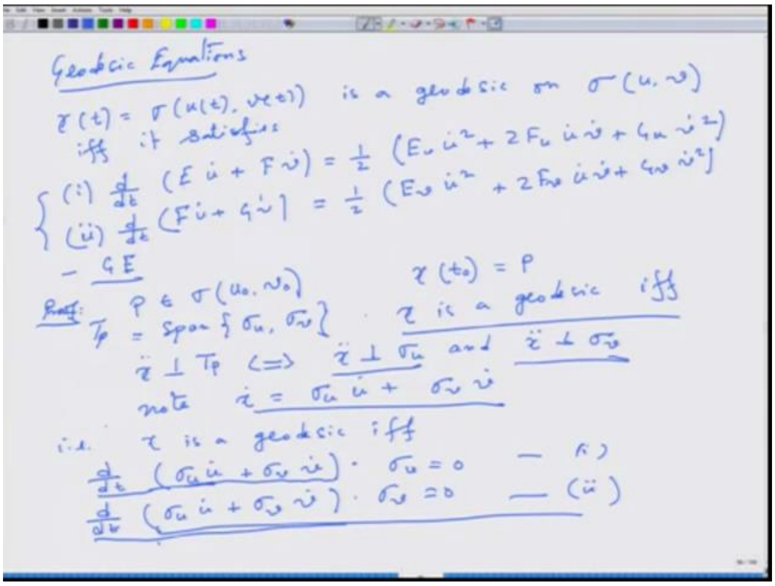

E is one F is 0. So, this part EU U is 0 FU is 0 and GU is, G is FU, so GU is FU square.

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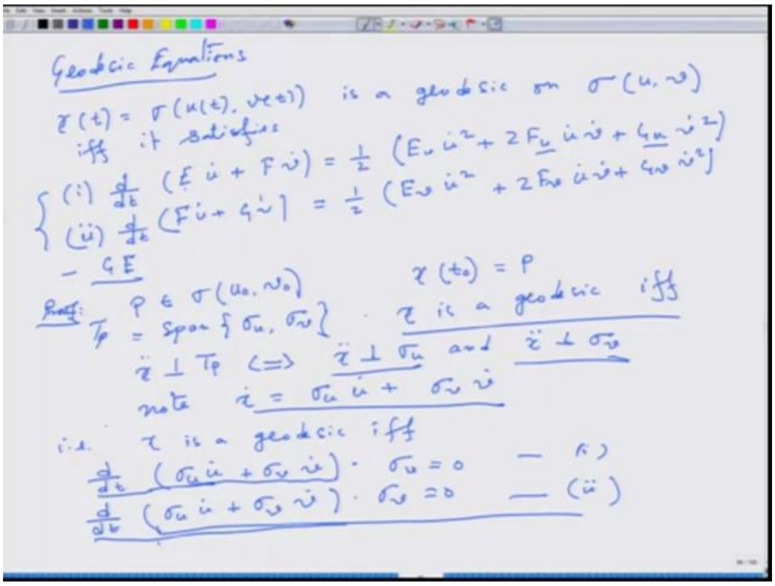

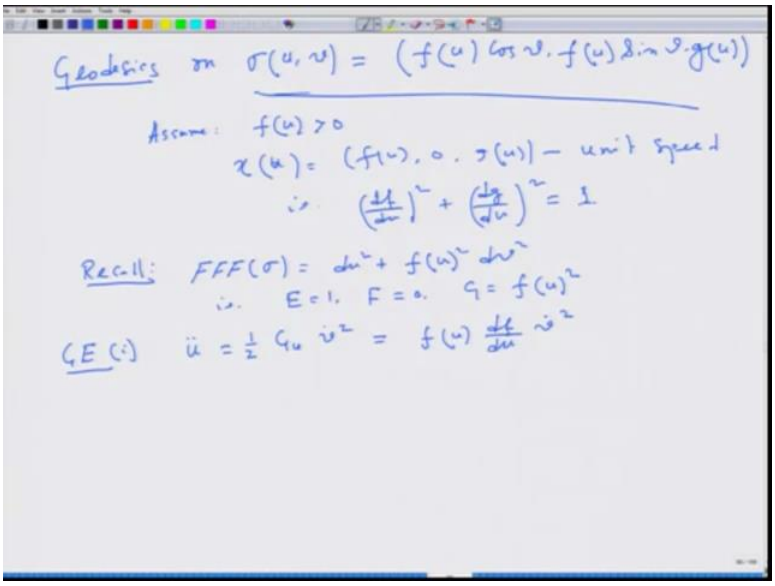

Geodesic Equations

$\gamma(t) = \sigma(u(t), v(t))$ is a geodesic on $\sigma(u, v)$ iff it satisfies

$$\begin{cases} \text{(i)} \ \frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2) \\ \text{(ii)} \ \frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2) \end{cases}$$

— GE

Proof: $P \in \sigma(u_0, v_0)$ $\quad \gamma(t_0) = P$

$T_P = \text{span}\{\sigma_u, \sigma_v\}$. γ is a geodesic iff

$\ddot{\gamma} \perp T_P \iff \ddot{\gamma} \perp \sigma_u$ and $\ddot{\gamma} \perp \sigma_v$

note $\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$

i.e. γ is a geodesic iff

$\frac{d}{dt}(\sigma_u \dot{u} + \sigma_v \dot{v}) \cdot \sigma_u = 0$ — (i)

$\frac{d}{dt}(\sigma_u \dot{u} + \sigma_v \dot{v}) \cdot \sigma_v = 0$ — (ii)

Geodesics on $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$

Assume: $f(u) > 0$

$\gamma(u) = (f(u), 0, g(u))$ — unit speed

i.e. $\left(\frac{df}{du}\right)^2 + \left(\frac{dg}{du}\right)^2 = 1$

Recall: $FFF(\sigma) = du^2 + f(u)^2 dv^2$

i.e. $E = 1, \ F = 0, \ G = f(u)^2$

GE (i) $\ddot{u} = \frac{1}{2} G_u \dot{v}^2 = f(u)\frac{df}{du}\dot{v}^2$

GE (ii) $\frac{d}{dt}(f(u)^2 \dot{v}) = 0$

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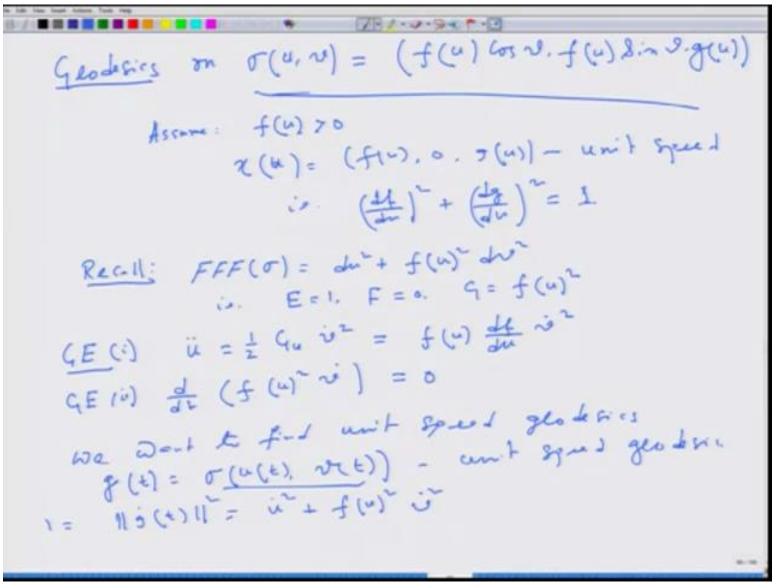

So, that is U dot dot left-hand side equal to so GU V dot square but G is FU square, so this is simply FU half is there so, FU df du, applying chain rule, V dot square. What happens to the geodesics equations 2? F is 0 G is FU square, this is 0, this is 0, GV is there, it will be same as, so it is saying d dt of FU square V dot equal to 0. The right-hand side and 0, right.

GV is 0 because G is a function of U only, so this right-hand side becomes 0. G is FU square V dot ddt of this thing is 0. Okay. So, you want to find unit speed geodesics, so let G be such unit speed geodesic. What it means? 1 equal to G dot T square, now calculate what is it, Sigma UT VT is given to us, so this will be simply U dot square + FU square V dot square.

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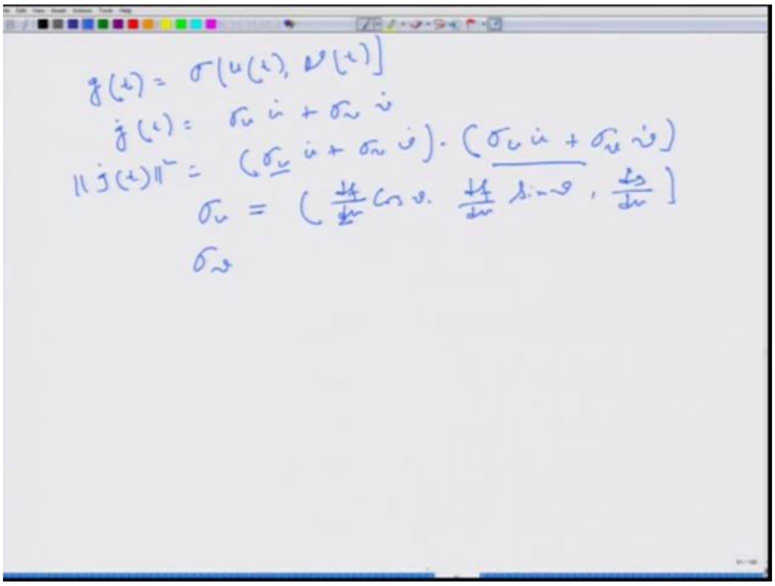

Why is so, I mean, you can check. GT is Sigma UT VT, so G dot T is equal to Sigma UU dot + Sigma VV dot, so G dot T norm square is Sigma UU dot + Sigma VV dot inner product will same, Sigma UU dot, Sigma VV dot, but now see, you calculate Sigma U, Sigma U is what? Sigma U is FU, Sigma U is prime, so df du cos V df du sin V and then dGU du. Similarly calculate Sigma V, calculate the cross product terms of you will get this expression.

I mean, you have to use that gamma is unit speed, so you may wonder what happens here is that df term comes up but when it is multiplied, there will be df df du square and because of sin and cos, there will be dg du square but we have already assumed that df du square + dg du square is equal to 1, so use that, you will get

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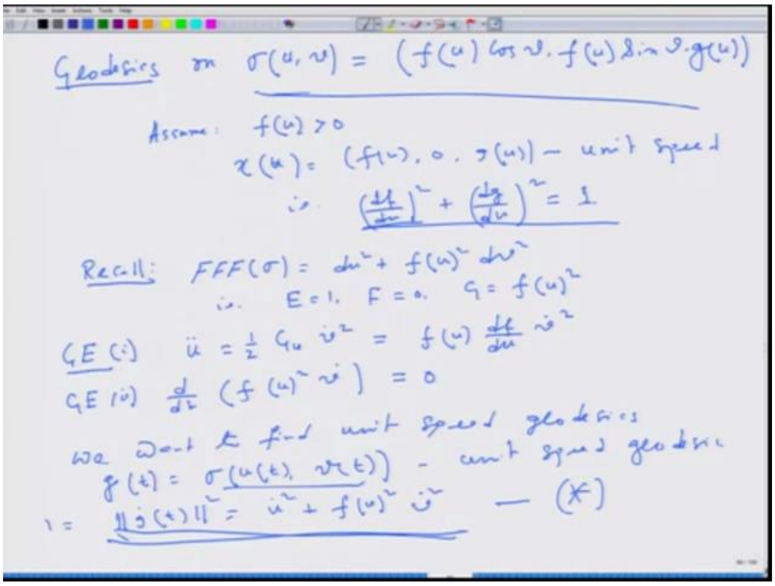

this condition is there. Let us put this condition as star. Okay. So, you do the calculation yourself, no problem. No, let us continue, have we find the geodesics.

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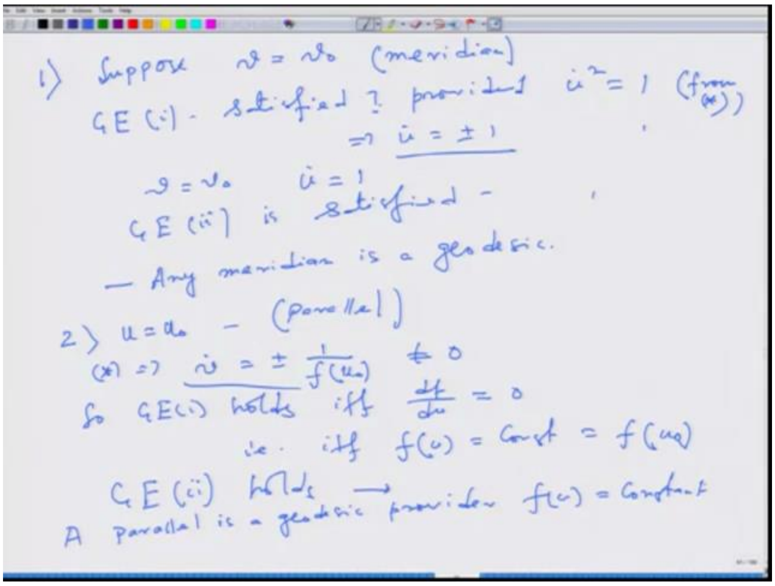

1st, suppose, I will go very humbly, I will look at V equal to V0, that is a meridian, here in the picture, V equal to V0 is a meridian. So, V0 some constant, okay. Is GE1 satisfied? Satisfied by this fellow? So, if I have V equal to meridian, so this right-hand side is 0 because V dot is 0 because V is constant. So, I have U dot dot equal to 0 and U dot square equal to 1, Satisfied provided U dot square equal to1 from star, that is U dot equal to + - 1, so suppose U dot is + - 1, so I have V equal to V0 and U dot equal to1 or -1.

Is GE 2 satisfied? Well, sure, because V dot is 0, so this side is 0, this is 0, so GE1 and GE2, both are satisfied, you understand what is happening. If I have V dot equal to 0 and U dot equal to + -, U dot equal to + -1, then GE 1 satisfied. And if V dot equal to 0 GE 2 is automatically satisfied because V dot is 0, right-hand side is 0. So, this says GE 2 is also satisfied. Says what? This is if and only if condition, any curve satisfying GE 1 and GE 2 is geodesics. So it says any meridian is a geodesic, provided have taken gamma to the unit speed, this is very important.

Okay, let us look at the parallels U equal to U0, some constant, so this gives a parallel. What will happen if U equal to U0 in star? This is 0, so I will have from star will give us F dot FU square V dot square equal to1 which gives V dot equal to + -1 by FU0. This should not be equal to 0 but that is not equal to 0, we have assumed F0 equal to 0 always. So, where is GE 1 holds? Look at GE 1, V dot equal to + - V dot square equal to 1 - FU square, U dot equal to 0, so this holds, G1 holds if and only if df du, this has to be 0. That is if and only if FU is constant, but this constant has to be FU0 because I fix U at U0.

So, then GE 2 also holds, I write in front of this, hence a parallel is a geodesic provided FU is constant. So, my curve gamma here, this is my curve gamma, this FU should be a some constant C,

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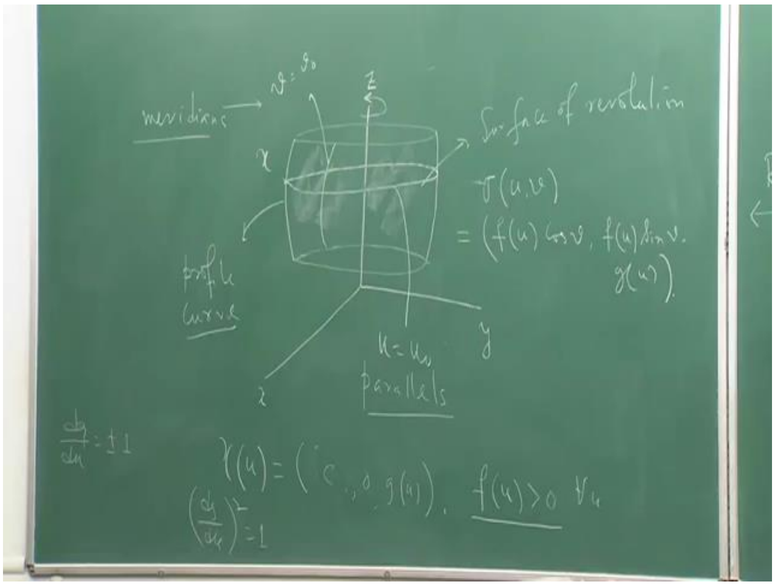

so that, but that will force GU square, right. So, GU is also, and GU I want a smooth function, so GU will also be forced to be constant. No no no no, sorry. That will force dg du square equal to 1, so dg du equal to + -1, so U to be a straight line, G to be a straight line. So, parallels in general are not constant, are not geodesics unless this gamma U, the 1st part FU part is constant and GU is just a straight line.

So, when it is, so what will be the situation?

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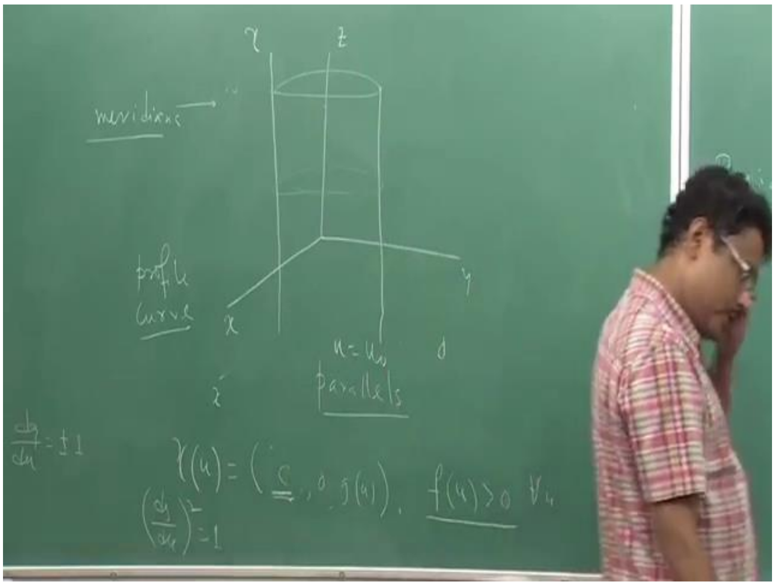

The situation you see as, okay, FU has to be constant and GU has to be a straight line, so this gamma curve has to be a line and in that case what I will get, I will get a perfect cylinder. So, only in the case of cylinder, parallels are geodesics and we have proved it last to last, last to last lecture, so here in cylinder, the parallels are actually geodesics, it has to be because now we have also proved through geodesics equations.

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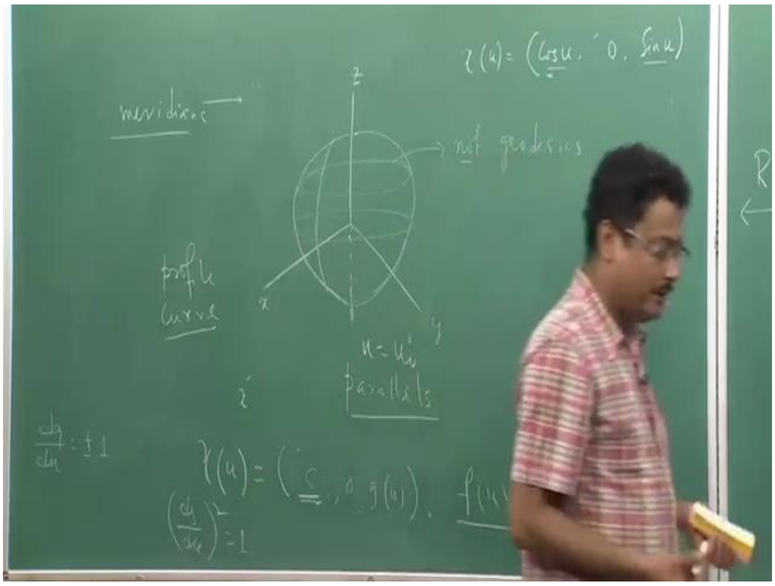

Okay, this is also a reason why something happens in the sphere. Well, for a sphere, I should take this to be the curve, so gamma U I should take cos U sin U, sorry 0 sin U. Here the FU part is not constant.

Well, I have some conditions, right. I mean I should not touch the Z axis, okay does not matter, because after all, if I remove little bit, I do not complete the curve, the parallels are not going to be changed. I will still get the sphere, maybe with a hole through the North pole to South pole. So, I will get the globe, our globe in general. But here the parallels are like this, but they are not geodesics but meridians are geodesics. So, these are not geodesics because FU is not constant and G is not a straight line.

This has a far-reaching consequence for finding the line of flight. Suppose I have a long distance flight, it will try to find, it will try to follow the geodesics because as you will see in one of the next, next lecture maybe that geodesics are actually the shortest path between any 2 points. Now, for long distance flights, so we try to fly along the meridian, to join along the meridian so that they have to travel shorter path, because if they travel around the parallels, they will have to cover much more because the shortest path is always the meridian, along the meridians. Okay.

So, this shows on a surface of regulation, I have meridians parallel, meridians are geodesics and parallel are geodesics only for cylinders.

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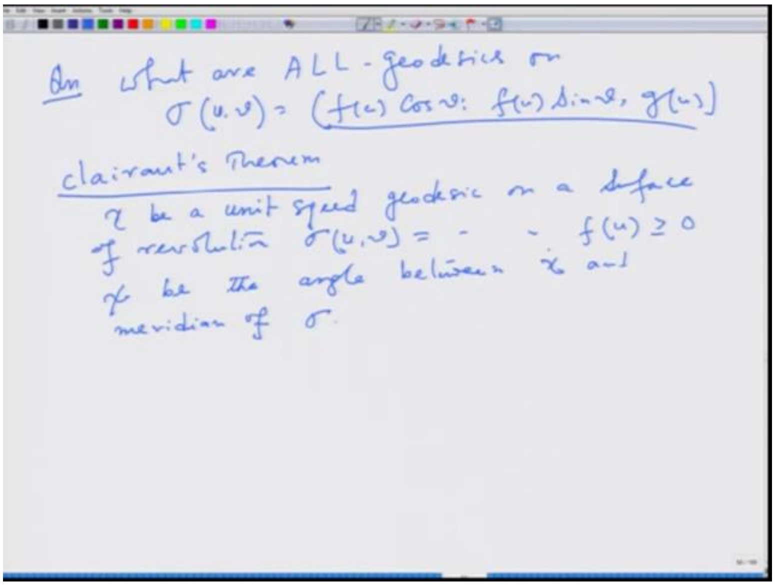

So, in general, the question remains, what are all geodesics? Assumptions are same, gamma is unit speed, okay, I cannot answer it, we cannot answer it from this geodesic equation alone now. So, special case we can, looking at meridians and parallels. For that we need a deeper theorem, it is called the Clairaut's theorem, to prove the geodesic equations again. But let us see the statement.

So, gamma be a unit speed geodesic on a surface of revolution, this is for surface of revolution only, okay, equal to this fellow with assumption, FU is I can take greater than equal to 0, does not matter for Clairaut's theorem. Psi be the angle between gamma dot and meridian of Sigma. So, I have taken a point, at that point I am looking at, I am trying to find out geodesic passing through that point.

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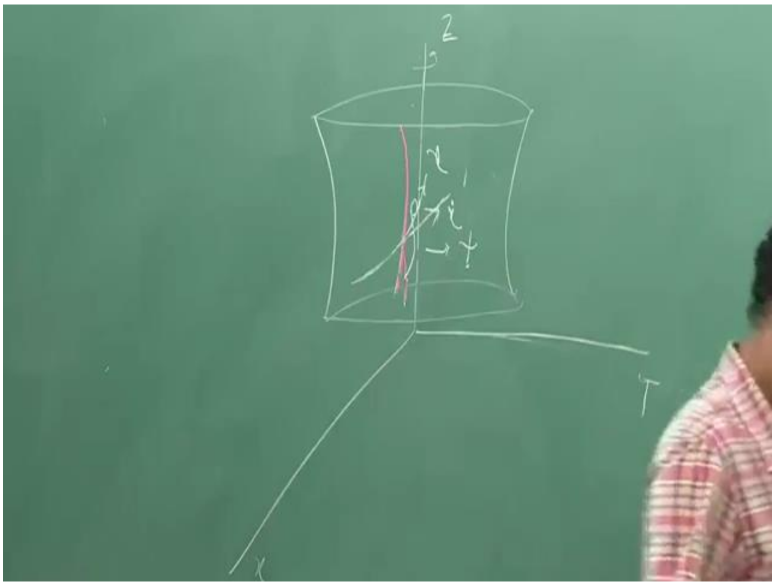

Let us take a very simple curve, I am rotating it along Z axis, okay. Here is a meridian passing at, passing through some point P, okay. R gamma be another curve which is also passing through P and maybe this relation and Psi is this angle.

So, this is the tangent on the meridian, maybe I will use a different coloured chalk, this is a tangent on meridian, this meridian and at P and this is gamma, this is tangent to gamma, this is gamma, tangent to gamma, gamma dot at point P, so I am looking at this angle Psi.

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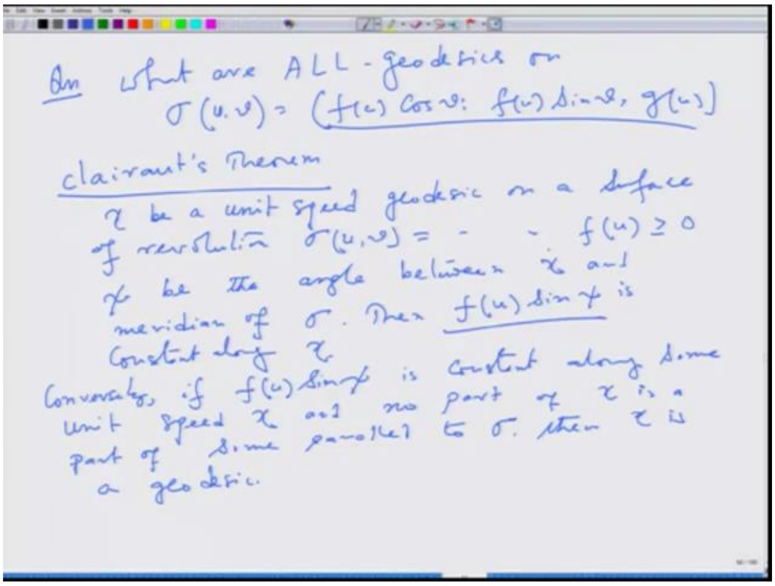

1st conclusion is then, FU sin Psi is constant along gamma. Conversely, if FU sin Psi is constant along some unit speed curve, unit speed gamma and no part of gamma is a part of some parallel to Sigma, then gamma is a geodesic.

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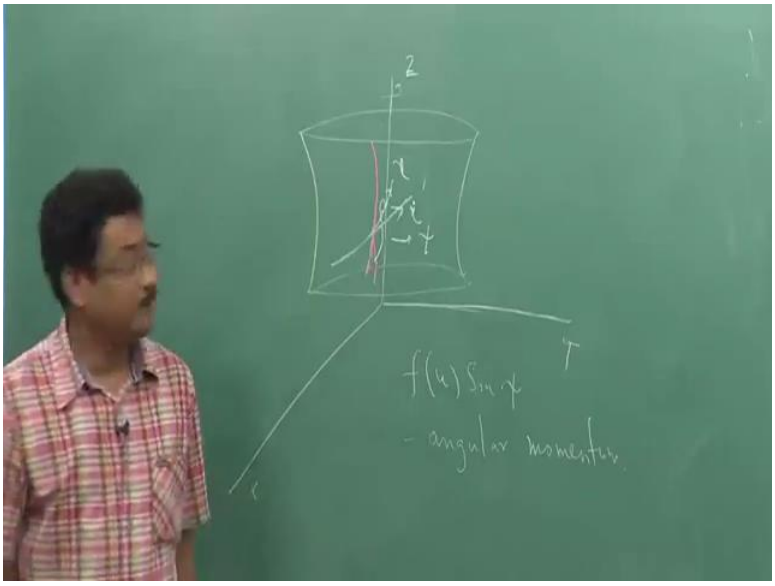

Okay, what is this quantity? Suppose you are travelling along a geodesic, what is FU sin U? If you are travelling along a curve gamma, then FU sin U, sin Psi, Psi is this angle, this is just the angular momentum in kinematics and what it says, to be a geodesics, angular momentum must be preserved. Otherwise, and if angular momentum is preserved, if you are travelling along a curve, preserving the angular momentum and you are not travelling parallel because parallels are ruled out from the previous observation. Then, you are actually travelling through at geodesic.

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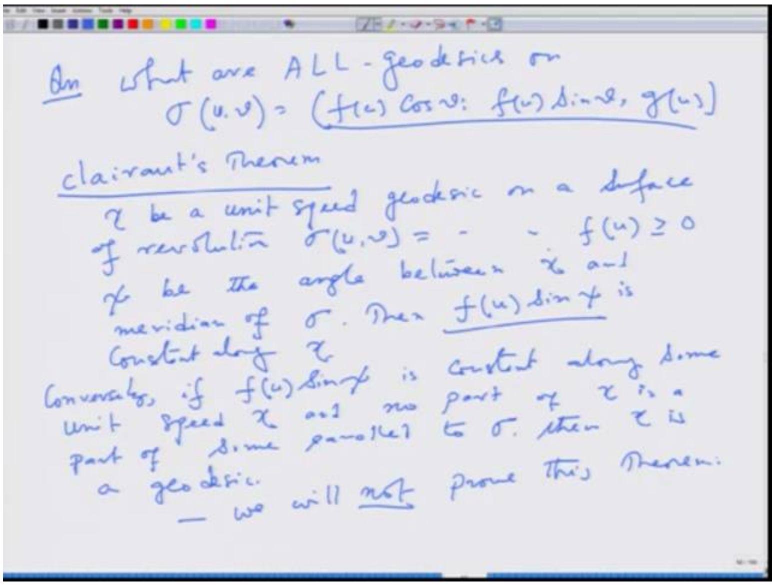

We will not prove this theorem, but use it find geodesics on surface of regulation like this. For today’s class lecture, I will end with 2 exercises and the next lecture we will consider a particular type of surface which is very important and which staffs one area of geometry geometry, so called hyperbolic geometry, but for today.

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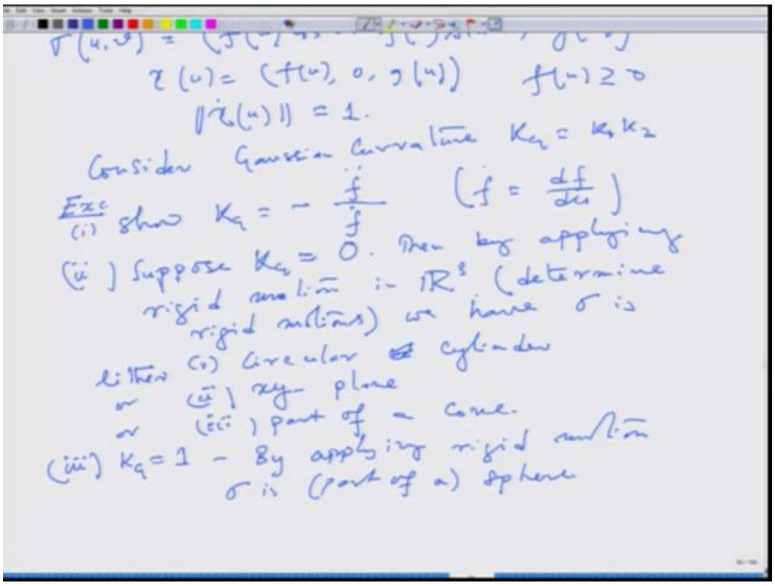

So, again consider FU Cos VFU sin V GU with all the conditions, so gamma U FU0 GU FU greater than equal to 0 gamma dot U is 1, unit speed, everything, FG smooth.

Consider the Gaussian curvature, what we defined this week, KG is product of 2 principal curvatures. Exercise 1st, show KG is - F double dot dot F dot, so F dot is df du, with respect to U. This will be easy. 2nd is also easy but be careful while doing this part. If KG is equal to 0, that is Gaussian curvature is 0, then by applying rigid motion in R3, so determine the rigid motion, determine the rigid motion. Not the… Because there are cases, we have Sigma is either a circular cylinder or XY plane or part of a cone.

So, Gaussian curvature is 0, then I can only have of 2 rigid motions, 3 surfaces, circular cylinder, XY plane and part of cone by surface of revolution of course, I mean surface of revolution can be only 3 these 3 cases. 3rd part of exercise, suppose the Gaussian curvature is constant 1, +1, then by applying rigid motion, so, of 2 rigid motion, Sigma is part of a sphere. So, sale is the only surface of revolution which has constant Gaussian curvature 1. Remember for fear we saw K1 and K2, both are 1, for cylinder, K1 is 1 and K2 is 0. Okay.

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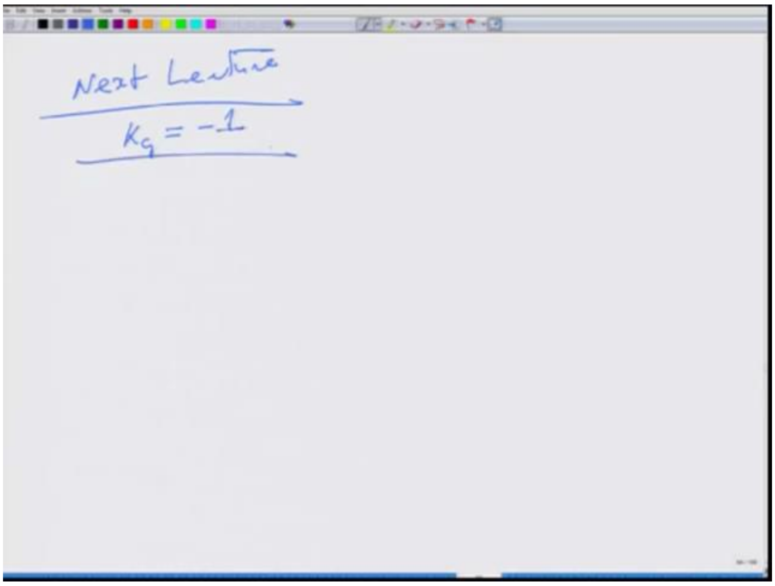

And, next class, next lecture, so this is for next lecture, this is what I said, that the beginning of the new geometry, we will consider the surface with constant Gaussian curvature negative.

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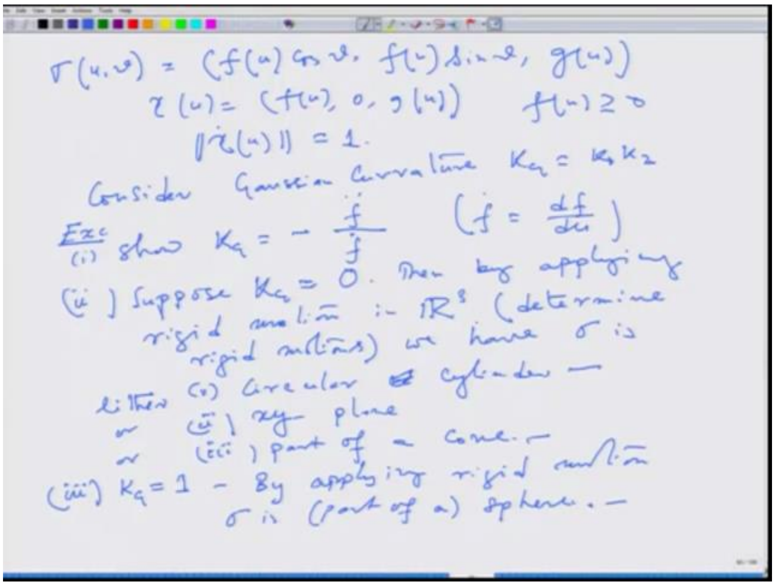

And we will see one such in fact prototype of such a surface and try to calculate geodesic on that surface. For instance, you know, for circle cylinder, XY plane, part of a cone and sphere, we know all the geodesics, for circular cylinder we know the straight-line, circle or circular helix, XY plane, straight lines, part of a cone, these are meridians. In fact these are meridians and for sphere, of course we know the great circles, but here things will be quite different. We will see it in the next lecture, thank you.