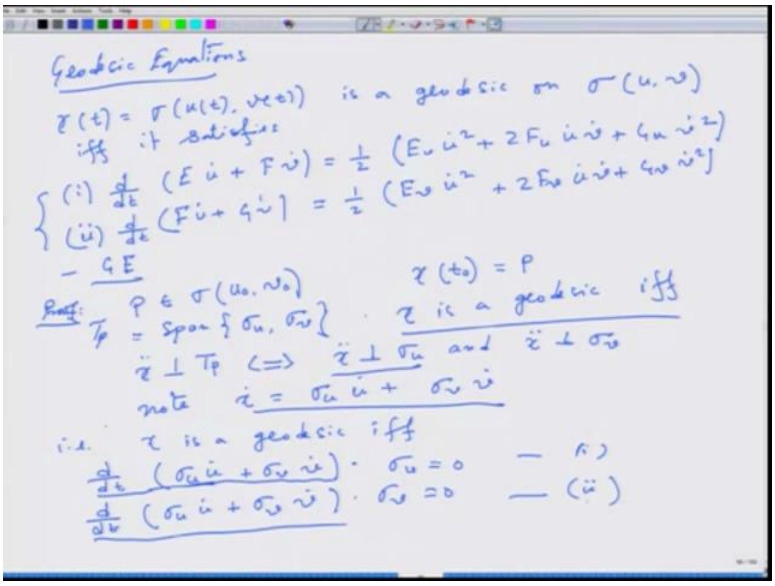
Curves And Surfaces.
Proffessor Sudipta Dutta.
Department Of Mathematics And Statistics, Indian Institute Of Technology Kanpur.
Module-Iv.
Surfaces-3: Curvature And Geodesics.
Lecture-18.
Existence Of Geodesics, Geodesics On Service Of Revolution.

(Refer Slide Time: 0:17)



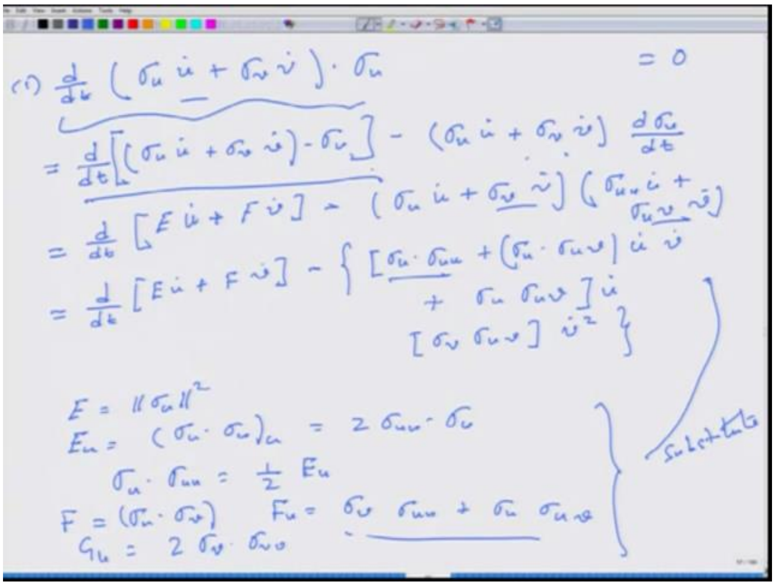
So, today we will start by recalling geodesic equations which we derived, which we stated yesterday. Recall that suppose gamma T is a curve, is a geodesic on Sigma U V if and only if it satisfies the following 2 equations which we call geodesic equations. Right. What are they? 1st one was d dt of E U dot + F V dot equal to half of E U U dot square + 2F U U dot V dot + GV V dot square. This is equation 1. And equation 2 was d dt F U dot + G V dot equal to half of E V U dot square 2F V U dot V dot + G V V dot square. Just look at equations once again. These 2 are the geodesic equations. Okay.

We will refer to them as G1 and G2. Okay. How do you prove it? So, let us today start with a quick proof of this fellow. Proof is very easy, although it looks difficult, this equation, after all let us take a point P on the surface, Sigma U0 V0, I want to and gamma is gamma T0 is P. Now, I know the tangent plane at P, this is spanned by Sigma U Sigma V, correct. And by definition, gamma is a geodesic if and only if, what happens… Okay, let us continue the proof. So, gamma is a geodesic if and only if by definition, what we have… Gamma double dot is perpendicular to the tangent plane, right.

That is if and only if, since it is spanned by gamma U, sorry Sigma U and Sigma V, gamma dot dot should be perpendicular to Sigma U and gamma dot dot should be perpendicular to Sigma V. If one vector is perpendicular to both the basis vector, it is perpendicular to the plane. So, that shows what, that I must have, now note, recall, I mean we have, this is equal to Sigma U U dot + Sigma V V dot. So, this concludes that is gamma is a geodesic if and only if, what happens, I must have d dt gamma U U dot + gamma V V dot dot with Sigma U equal to 0. This is equation 1, this will lead to geodesic equation 1. And d dt gamma U U dot, sorry Sigma U U dot Sigma V V dot Sigma V equal to 0.

How do I get it, this gamma dot dot should be perpendicular to Sigma U, so, but you see, this is gamma dot, this is gamma dot, so gamma double dot is this fellow, right, this fellow is gamma double dot, so gamma double inner product with Sigma U must be 0 because they are perpendicular. Similarly gamma double dot dot Sigma V equal to 0 because they are perpendicular, okay.

(Refer Slide Time: 5:40)



Now look at equation 1. 1st part equation d dt Sigma U U dot + Sigma V V dot dot gamma U equal to 0. This is my if and only if condition for geodesics. Now, I write this fellow as little bit different from, I write it as, okay, dot Sigma U but if I do this, what will happen?

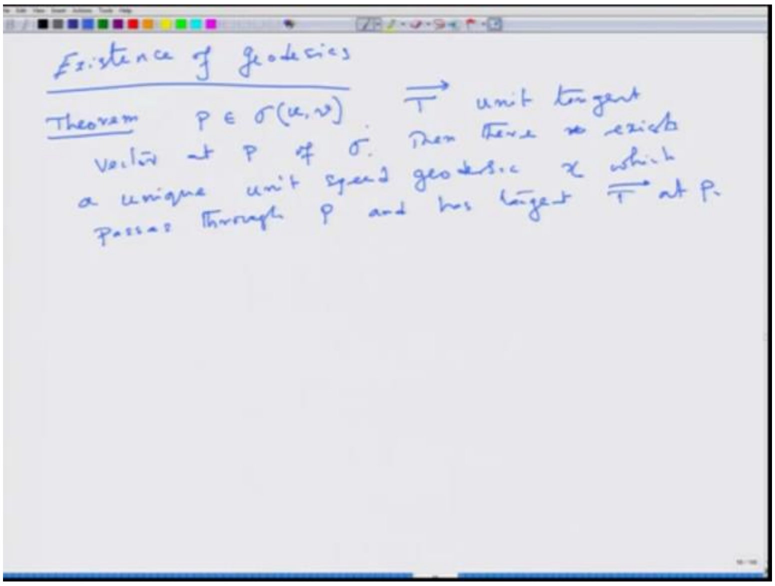
If I apply chain rule to this, 1st will be this one and there will be a 2nd term, but 2nd term is not there here, so I have to, you see if I apply Leibnitz rule to this fellow, I will get this + this. This term + this term, so I write this term as this - this. Is it all right? Correct. Now, look at what, what happens here… Sigma U dot Sigma dot, so d dt remains here, if I multiply, if I take the inner product Sigma U is Sigma U, that is my definition Sigma U norm square which is U. U dot + Sigma U Sigma V which is F V dot -, now this term, I write it as Sigma U U dot + Sigma V V dot, Sigma U d dt is again Sigma U U U dot + Sigma U V V dot, chain rule again. So, this is equal to ddt of d dt of E U dot + F V dot - okay, let us put the terms together with U dot and V dot, what I will have. So, I collect the terms with U dot and V dot, so with U dot I will have 1st terms Sigma U dot Sigma U U, then + Sigma U dot Sigma UV, okay.

U dot V dot + Sigma U Sigma UV with U dot okay. And the next will be, V dot term will be Sigma V Sigma UV V dot square, this and this. All right. Just check it is all right. 1st term, this term comes from Sigma U Sigma V U dot U dot, so U dot square, but I take one U dot out, this U dot, so I keep this here. Now what is E? E is Sigma U square, norm of Sigma U square, so what is EU, EU is Sigma U Sigma U U which is 2 Sigma UU dot Sigma U, this term. So, this term is Sigma U dot Sigma UU is half of EU. Correct.

Similarly what is FU? F is Sigma U dot Sigma V, so what is FU? That will be Sigma V Sigma UU + Sigma U Sigma UV okay. And similarly GU is, so GU is 2 Sigma V Sigma V dot dot. So, put everything together, so substitute this here, you will get the geodesic equations. This is the proof for GE1, for GE 2 you follow the same argument. Using, so this is for GE 1, for GE 2, you use this relation, gamma dot dot, so this equation and do the same thing what we did here. Okay. But as I said last time, in general solving these geodesics equations 1 and 2, this may be quite hard.

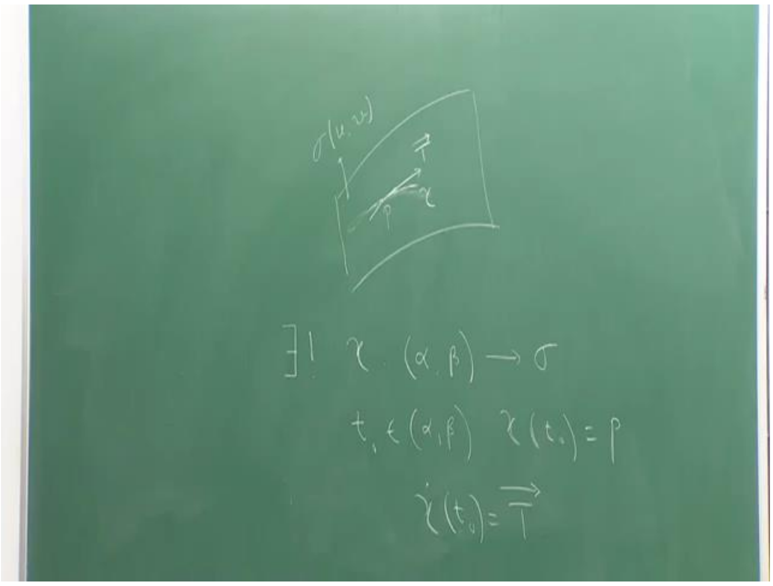
We did one example for cylinder where E and G was 1 and S was 0, so in that case the equation was very easy, it becomes linear equation, but in general is my not be. Right, because square terms are involved. So, solving the geodesic equations may not be easy, it may not be very, yah, may not be easy but it gives something.

(Refer Slide Time: 12:02)



It immediately gives existence of geodesics. So, this I state as a theorem. So, I take a smooth surface and a point on that, let T be unit tangent vector at P of Sigma, okay. Conclusion is, there exists a unique unit speed geodesic gamma which passes through P and has tangent T. So what is happening here?

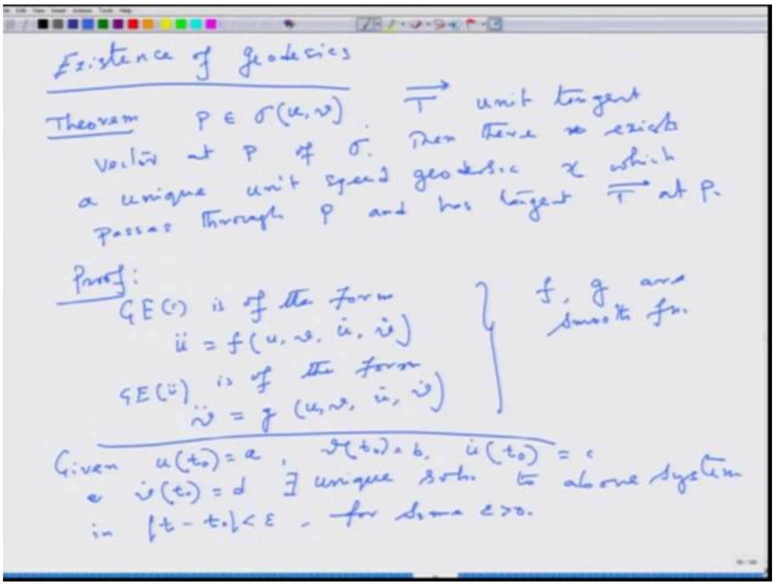
(Refer Slide Time: 13:52)



You take any smooth surface patch, okay, take any point P, what the theorem says, that fix a direction, fix a direction means unit tangent vector for the surface Sigma U, this is Sigma UV. Then there exists unique, so what this is saying, that exists a unique gamma, let us call some Alpha, beta Sigma, for some T0 in Alpha, beta gamma T0 is P, so it passes through P.

And gamma dot at T0, so given any point on the surface and given a direction, there exists unit speed geodesics with in that direction. Proof of this is very easy using geodesics equations.

(Refer Slide Time: 15:21)

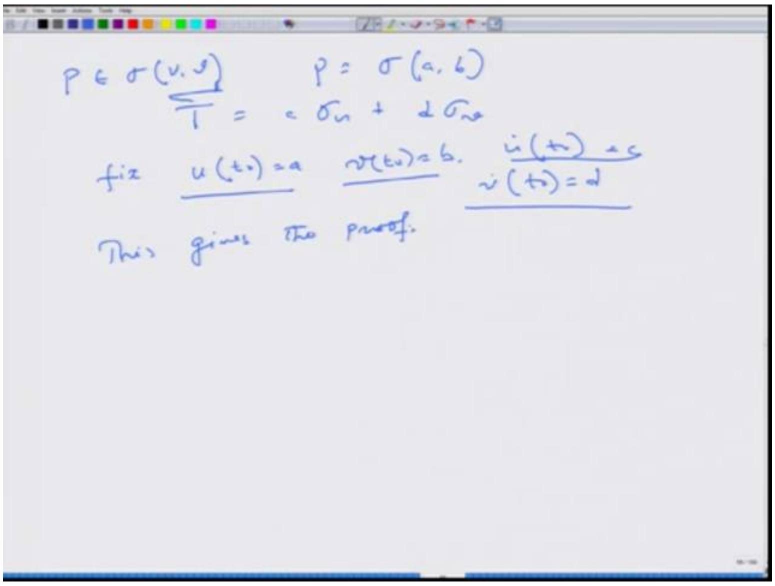


Recall the geodesics equations again, look at it, 1 and 2, okay. I say, so the proof is simple use of what is called existency or existence of uniqueness theorem of 2nd equation, not linear differential equation, differential equation. So, I say geodesics equations, or maybe I will abbreviate, G1 is of the form, which form? I said second-order differential equation of this form. Correct. And G2 similarly is of the form, so second-order differential equation involving in V dot dot.

Where, here, F and G are smooth functions. Now, if you look at any differential equation, good differential equation book for instance called Coddington and Livingston, they will give you existence of, unique solution of this provided I give, so this will have, so existence and uniqueness will tell you given U T0 equal to let some number, a fixed number A, BT0 equal to B, U dot T0 equal to C, V dot T0 equal to some D, something is number A, B, C, D, whatever it be.

There exists unique solution to above system in some neighbourhood of T0, some neighbourhood of T0, you will get unique solution to this above system. This you can, this theorem you can find in Coddington Livingston introduction to differential equation. So, what we will do now? I have given a point P, right.

(Refer Slide Time: 18:12)

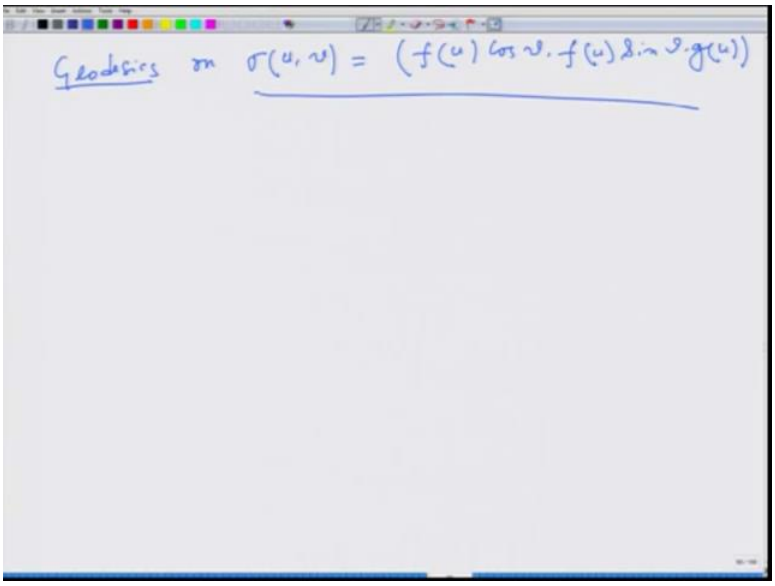


So, what I will do, I will just take P Sigma UV P to be Sigma AB, P I will take C Sigma U + D Sigma V because any T will be written by some C this way, P will be given by some A, B and then I will fix UT0 equal to A, VT0 equal to B, U dot T0 equal to C and V dot T0 equal to D.

So, what we will do, we know this kind of, this solution exists provided this happens. Now, for a surface, okay, P will be given by (())(18:55), T will be given by some C Sigma U + D Sigma V because T is a tangent vector. So, it belongs to the span of Sigma U and Sigma V, so I will fix UT0 equal to A, VT0 equal to B, U dot T0 equal to C, V dot T0 equal to D and that will guarantee my existence of solution, unit speed solution.

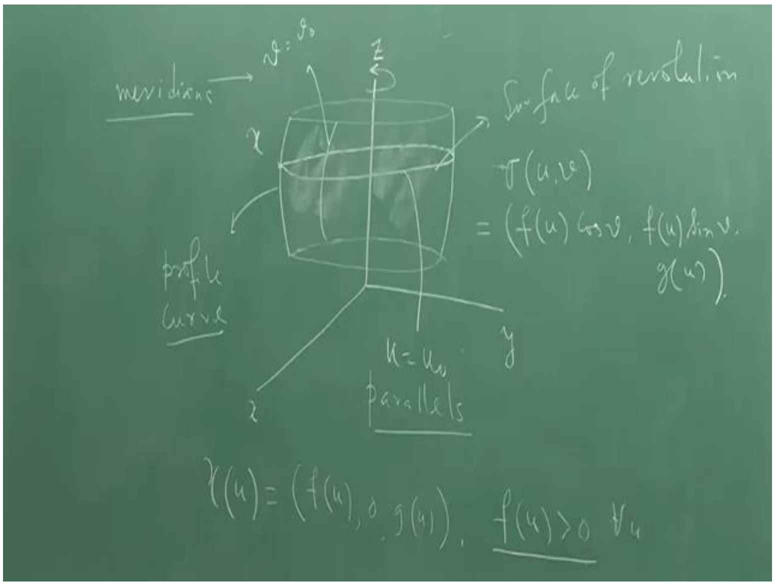
This, this gives the proof. So, proof is basically, you observe that geodesics equations of second-order gives the second-order system of equation and for such system, given initial condition, there is nice theorem available for us which gives you existence and uniqueness of solution. So, at least this geodesic equation is gives us the existence of geodesics.

(Refer Slide Time: 20:09)



In the next part, I will try to find out geodesics using everything whatever we have learnt on Sigma UV equal to FU cos V FU sin V G, what is this, you have to recall, this is what, this example we have done.

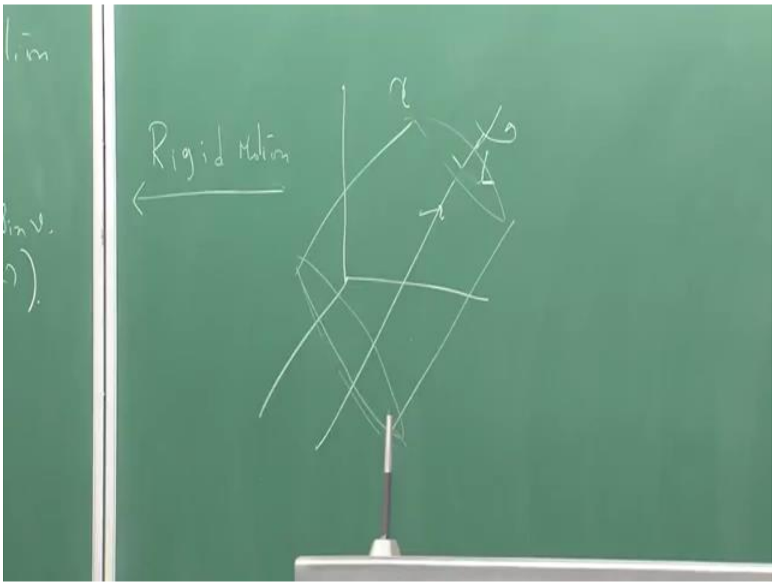
(Refer Slide Time: 20:45)

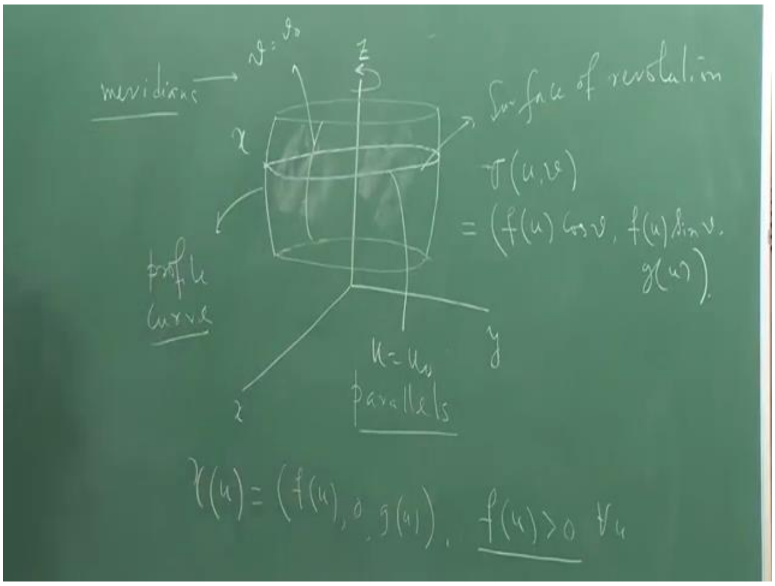


So, I take a curve gamma on the XZ plane, so gamma is a curve on the XZ plane. I take FU greater than 0 for all U, that is curve does not intersect Z axis, this condition means gamma does not intersect Z axis. Then what I do? I take, I just revolve the curve around Z axis, that will generate a surface, that will generate a surface and this is the surface of revolution, right. Parameterised by… Recall this curve gamma, this is called profile curve, we have given a terminology, curves which are parallel to the profile curve, that is V equals to constant, these are called meridians.

And if I take a point on gamma and look at the curve which is generated by rotation of gamma, that is U equal to constant, these are called parallels, just like on the globe you have meridians and parallels. So, here we want to find out all the geodesics, 1st of all using geodesic equations and secondly using some deeper theorem because geodesic equation alone will not give us the all the geodesics. And that is what we are going to do in the next lecture, so next lecture we start with this surface and try to find all the geodesics on the surface of revolution.

(Refer Slide Time: 23:50)





Recall if I do it for this surface, I have taken a particular curve gamma on XZ plane but after all, any surface of regulation about an axis, you take any curve wherever you want and you take a axis wherever you want and you want to rotate it with respect to this axis, you will generate a surface like this. But after a rigid motion of R3, rigid motion of R3 will turn this picture into this picture, or such a picture. That is I can take the curve on XZ plane and axis to be Z axis. Just nothing, right, just rigid motion on the plane. So, if I do it on the surface, I can do it for every surface of revolution. We will continue on the next lecture.