**Curves And Surfaces.**
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**Module-IV.**
**Surfaces-3: Curvature and Geodesics.**
**Lecture-17.**
**Geodesics.**

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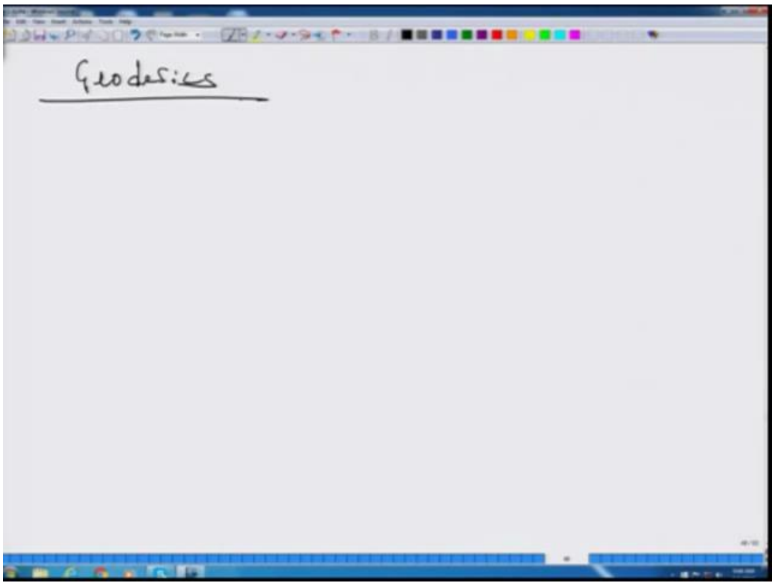


Okay, so today we will be talking about geodesics. Geodesics are some special class of curves on a surface S. So, suppose I am moving on a surface,

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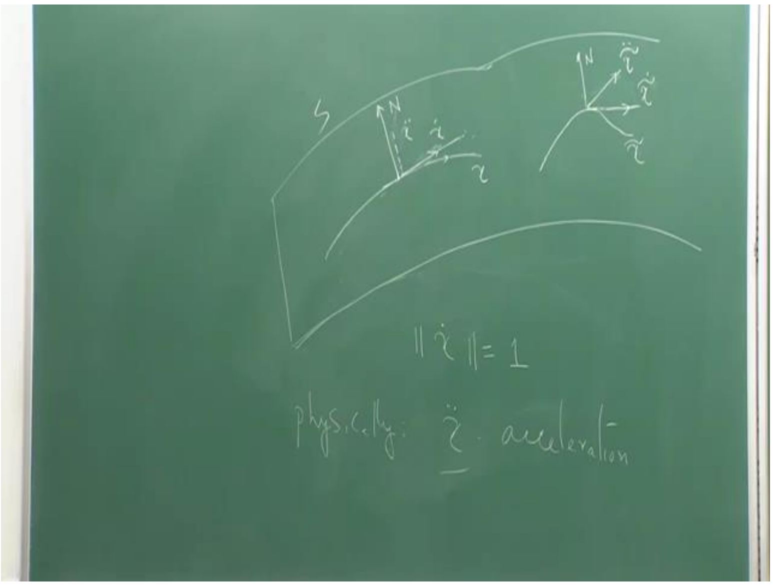

let us say I have drawn a surface patch on the board and I am moving on the plane gamma, sorry I am moving along a curve gamma. And suppose I move on unit speed. Okay, you physically all of us know that gamma dot dot, gamma dot is a speed and gamma dot dot, the double derivative, what is it, this is, this is a physical interpretation, it is called acceleration.

Now, if I am working on the surface, what portion, if I am working on the surface, then what component of the acceleration I can perceive? I can perceive the only component of acceleration which is parallel to the tangent at gamma, that is parallel to gamma dot. So, if I am moving along this, I can only perceive what is tangent to this, the way I am moving. Suppose we are travelling in a car, so how do you feel the acceleration, if you speed along the road, then along the road how much you are accelerating or your car is accelerating, that much you can feel. But it will have other components, right.

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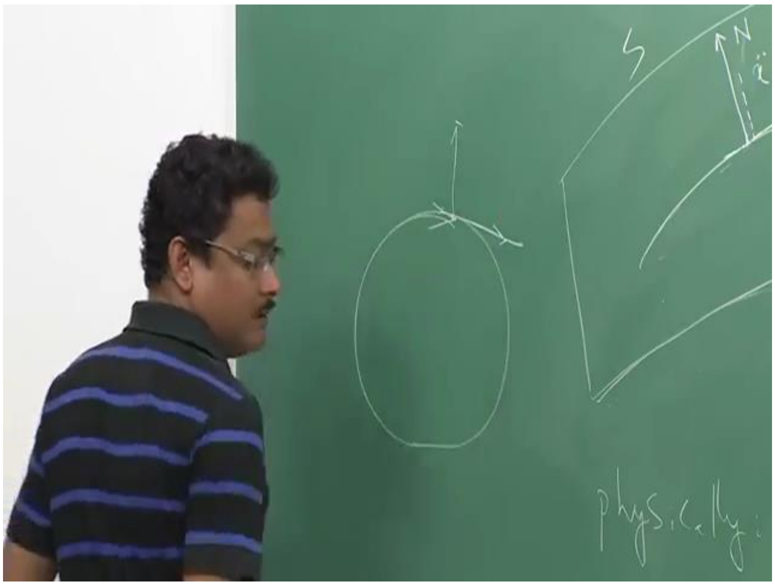


If I am moving along a circular path, I will always see, at a particular point, I will always see, if I am moving in a circular path on a surface, I can see only acceleration along this part. But this will have another component of course, that component I cannot feel.

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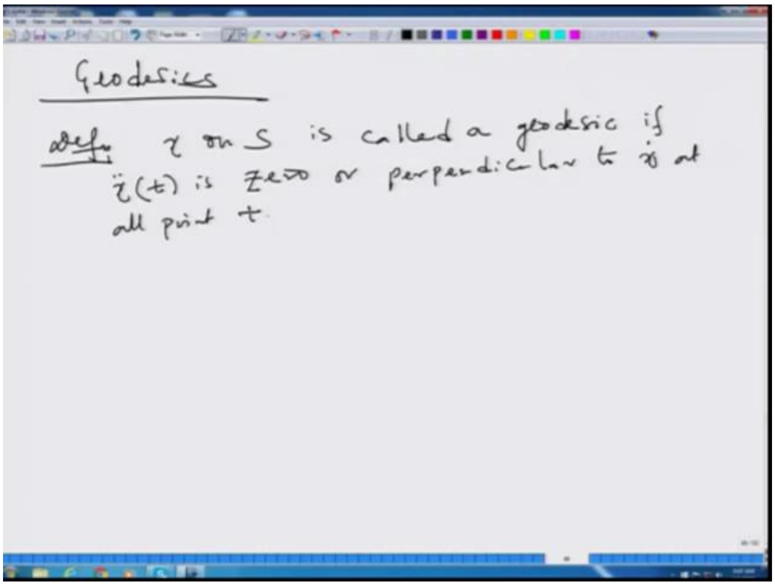


So, I will make a definition here and then discuss the consequence, gamma a curve on surface S, smooth curve is called a geodesic if gamma dot dot T is 0, this acceleration is 0, that is you are moving in constant speed or perpendicular to gamma dot at all points T. What does it mean?

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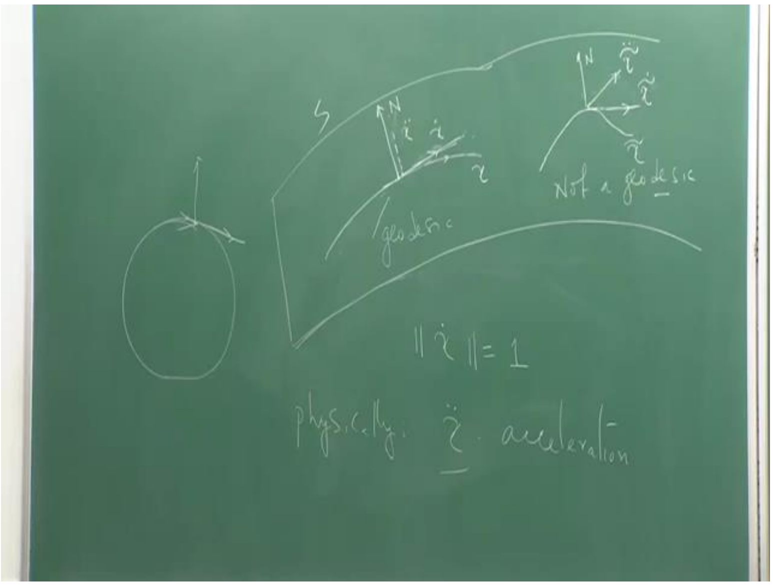


That if I move along a geodesic on a surface, I will not feel any acceleration because I can feel acceleration which is only on the tangential direction gamma dot, but if gamma dot dot is 0, there is no acceleration anyways. But in 2nd case, gamma dot dot is perpendicular to gamma dot, then I will not feel because I cannot see this component, I can see the component only this way.

So, in particular, this curve is geodesic, the way I have drawn it, whereas this curve gamma dot dot, gamma tilde gamma tilde dot dot is not perpendicular to gamma dot, neither 0, so this is not a geodesic. So, why we are interested in such a curve? Because this I am saying verbally, suppose a particle on a surface is moving along a curve and this particle is subjected to no force, except a force that is perpendicular to the surface, that is along the direction of N, shallow normal. Okay.

And the particle keeps moving, then it will only move along geodesics. Why is so? Because you recall the Newton's 2nd law, which says what, 2nd law of motion says force on the particle is parallel to the acceleration gamma dot dot, F equal to MG, that is the way we write, G is the acceleration in that case but I am already given this gamma dot dot, the acceleration is perpendicular to the surface or parallel to the perpendicular to the surface, parallel to N.

So, then it cannot change its path, unless there is an acceleration, the particle does not change its path, it is a 1st law of Newton. It says that okay, particle moving along a line keeps moving, unless there is some force is acting on it. So, this is one interpretation.

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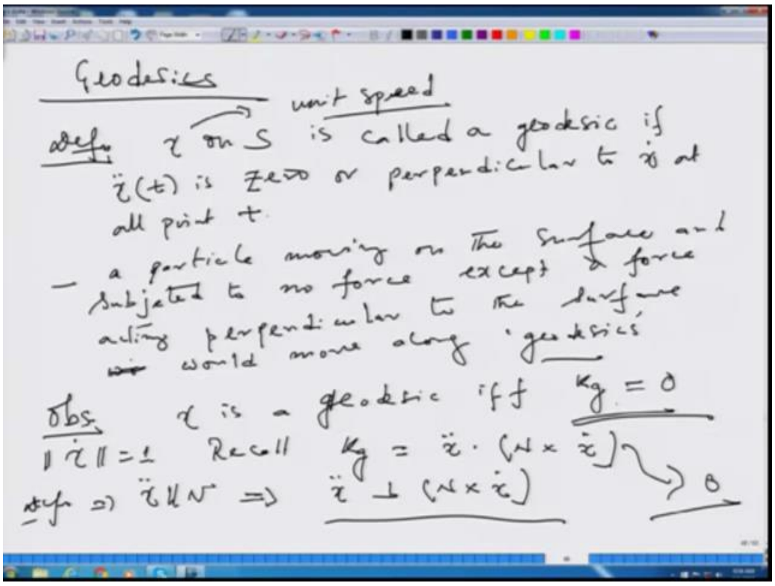


A particle moving on the surface and subjected to no force, except a force acting perpendicular to the surface will move along geodesics. Later on, maybe next lecture we will have another interpretation of geodesics, and that is very interesting one, which says that, if you look at a regular smooth surface, nice surface and if you take 2 points, so you can join the 2 points in various curves lying on the surface itself. The shortest one will be the geodesic, we will see why.

Now from the definition itself, the immediate observation, gamma is a geodesic if and only if the geodesic curvature is 0. So, that was interpretation, so, we talked about normal curvature, we did not talk about geodesic curvature. So, now we see what is the relation… From the definition it says it follows, I will show how, very easy. That gamma is geodesic if and only if the geodesic curvature is 0. So, if a curve has geodesic curvature, then it cannot have, it cannot be a geodesic. Why this is so? Well, we have gamma to be unit speed, so all curves are unit speed, again, I keep on forgetting writing that, this is unit speed curve.

But I have said it that this is blanket assumption that on any curve we discuss, in that we are talking about unit speed curve. So, gamma is unit speed, so gamma dot norm is one, that gives you, okay recall. What was KG? KG was gamma dot dot dot product which N Cross gamma dot. But by definition, implies gamma dot dot is parallel to the normal, that will imply gamma dot dot is perpendicular to this vector. Now, if the 2 vectors are perpendicular, then this has to be, this fellow has to be 0, dot product has to be 0.

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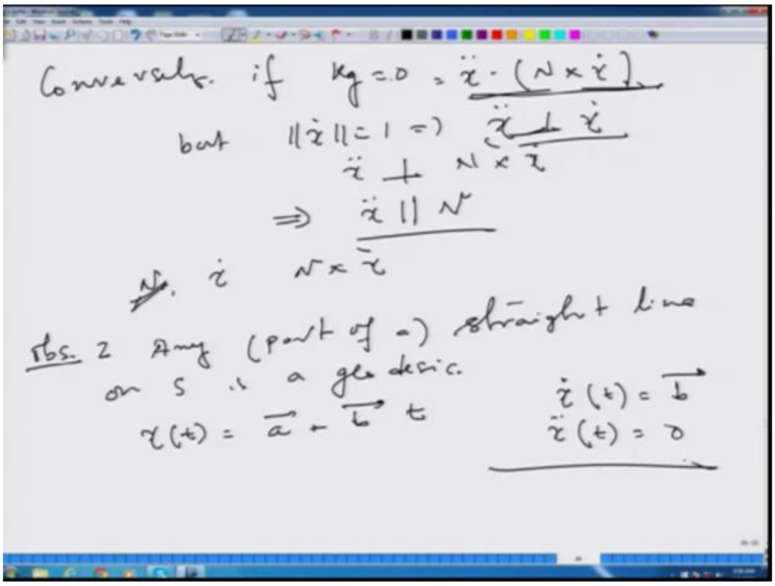


Conversely, if KG is 0, then this is equal to gamma dot dot N Cross gamma dot but gamma is unit speed, this implies gamma dot dot is perpendicular to gamma dot again and gamma dot dot is per, this shows that gamma dot dot is perpendicular N Cross gamma dot also, so this will show that gamma dot should be parallel to N.

I have 3 vectors, right, N, gamma dot and N Cross gamma dot, perpendicular vectors. Gamma dot with this vector is perpendicular, gamma is perpendicular to this, gamma is unit speed, so gamma dot is perpendicular to this, so it has to be parallel to this vector. Next observation, so let us call this observation 1. Next observation 2 is any part of a straight line on any surface S, regular smooth surface is geodesic. Why? Straight line looks like gamma dot is, at all points T is just this vector P, so gamma dot dot T is 0 and that satisfies the 1st condition of geodesic. That either you have gamma dot dot T is 0 everywhere or perpendicular to gamma dot at every point.

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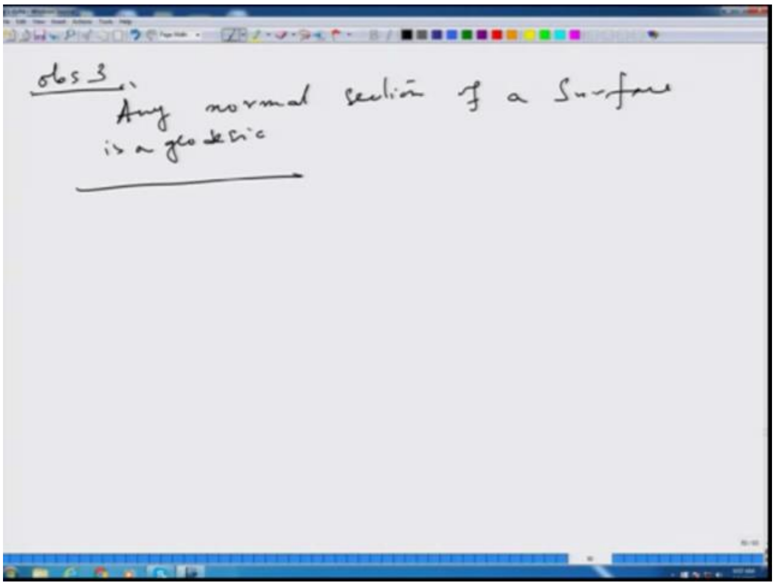


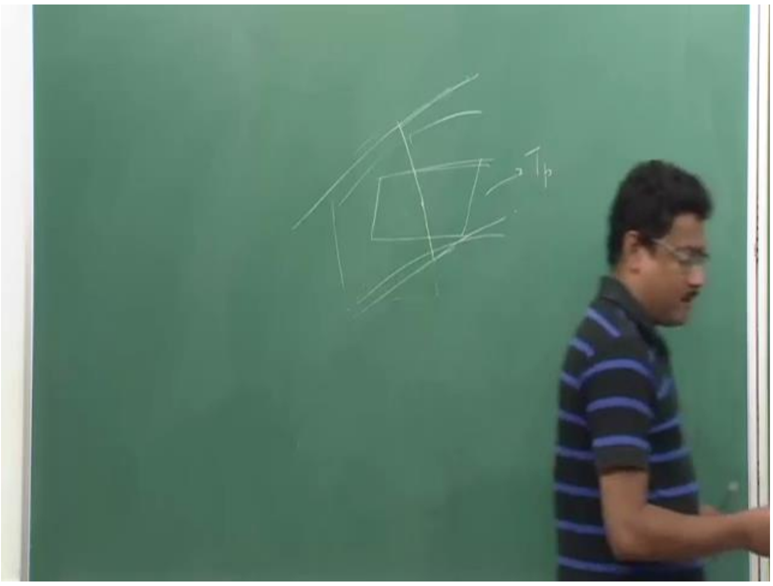

Observation 3, any normal section of a surface is geodesic, is a geodesic. What is a normal section? I think I have not defined it before. So, normal section is a curve like this. Suppose I have a surface and suppose this is a, at some point P, this is a tangent plane at P. Okay. Normal section is the plane which cuts the surface perpendicular to it. Let me in draw it. So, okay… So, you cut this plane perpendicularly by another plane, and goes this way.

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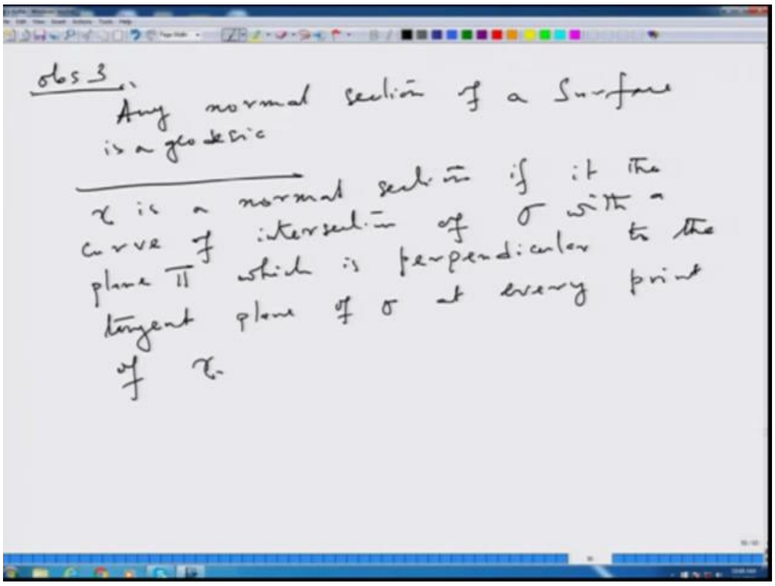


Now, the definition of normal section is, gamma is a normal section if it is a curve of intersection of the surface patch which I am looking at Sigma with a plane, let us say pie which is perpendicular to the tangent plane of Sigma at every point of gamma. Okay.

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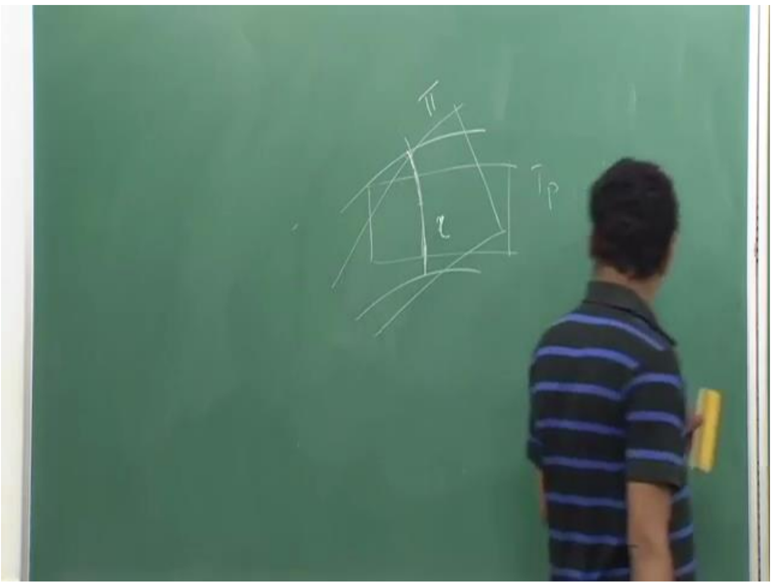

A better picture maybe, we can try to draw. So, you have the surface patch Sigma, you have a curve gamma, this is the curve gamma, so this is a tangent plane, so at a point P, at point P, suppose this is the tangent plane, the normal section will cut along this, cut the tangent plane perpendicularly at every point of gamma. So, 2nd thing will be called a normal section. Okay. Why it is so, why it is geodesic? This needs some work.

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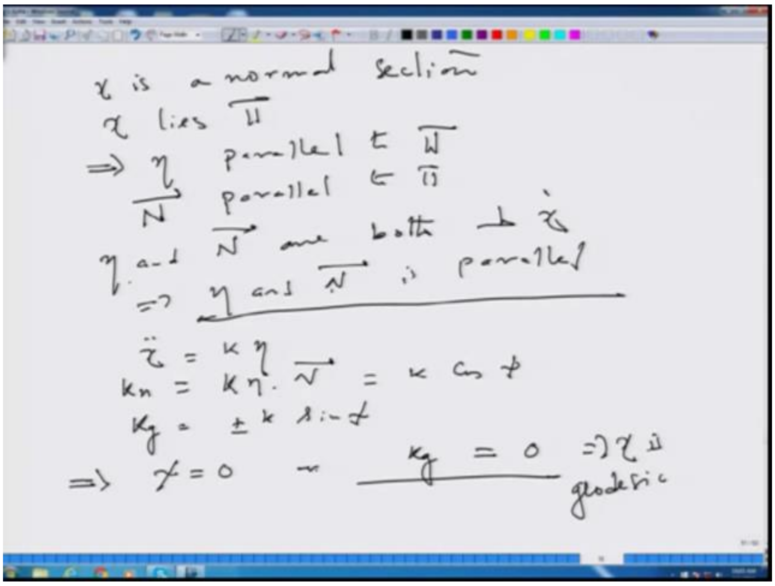

So, suppose gamma is a normal section, so gamma lies in the plane pie, right. This will imply the normal of, the principal normal, that gamma, is parallel to pie, standard normal of the surface is also parallel to pie but eta and N, standard normal are both perpendicular to gamma dot. So eta and this are parallel, eta and perpendicular to both gamma dot and they are parallel to the plane, one particular plane, 2 vectors which is perpendicular to one vector, both of them perpendicular to one vector and they both are parallel to pie. So, that implies eta and the standard normal, standard normal and the principal normal of this curve, they are parallel at each point of gamma.

But gamma dot dot, I know, this is some K times eta, so the normal curvature, and the normal curvature was what, K times eta dot this which is K cos phi, Phi is the angle between N and gamma dot and KG is + - K sin Phi, right. But gamma is a normal section, so this psi equal to 0, yes, psi equal to 0, therefore KG equal to 0 and we have seen geodesic curvature 0 means gamma is geodesic. Okay.

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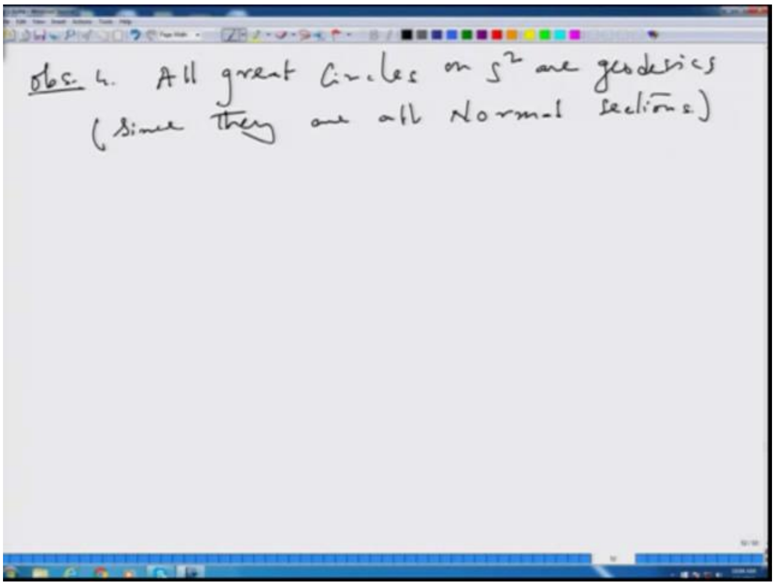


So, this observation will also give us, this next observation, all great circles of the sphere are geodesics, since they are all normal sections.

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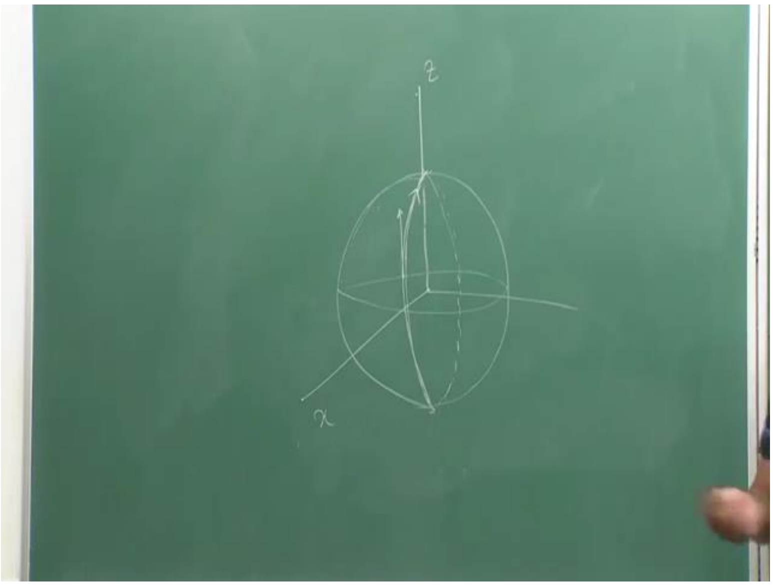


I should draw the sphere, you will see it immediately. Here is my sphere, I will take a great circle, right. If you cut, so suppose, take the standard sphere, so great circle looks like this, right, longitudes.

So, if you cut the sphere with a plane perpendicular to… So, if I take this curve, its tangent plane is this way, suppose I am moving this way, so tangent plane is the, the tangent plane is, tangent plane is XZ plane, so if I cut it through YZ plane, then I will get, which is perpendicular to XZ plane, so if I cut it through YZ plane, I will get a surface. So, you think of a globe, here you cut it normally by just a cardboard, then what you get is a great circle and that great circle has to be geodesic because my construction is on normal section.

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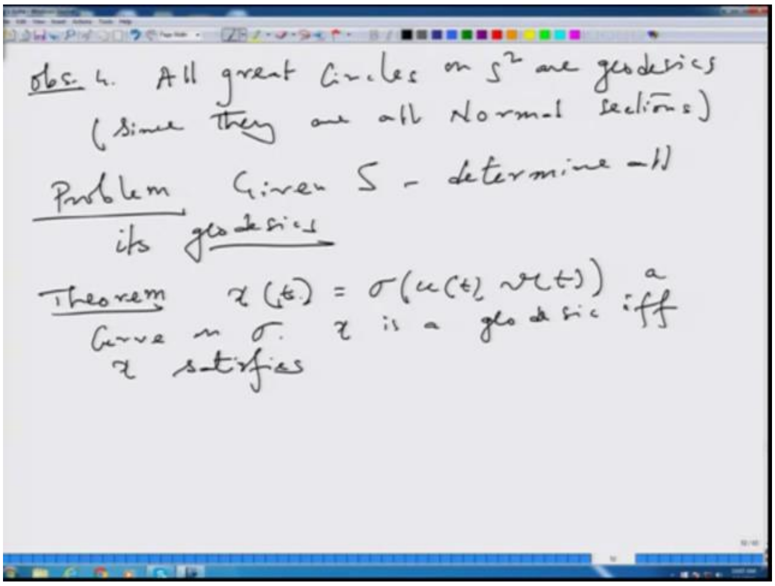


Now, given a surface, the problem is here… Given a nice smooth surface S, determine all its geodesics. Okay, theoretically you can but you will soon see it is very difficult job. There are good theorems, one of them is like this, which will tell you what are the geodesics. So, let us take gamma T a curve on a surface patch Sigma, curve on surface patch Sigma. Theorem says very clearly, gamma is a geodesic if and only if, so this distance, and since saw you… The 1st problem but you will soon see the difficulty… if and only if gamma satisfies, okay, I will write the equation, it will satisfy the question on the next page.

So, given any regular, smooth surface patch, I take a gamma gamma T,

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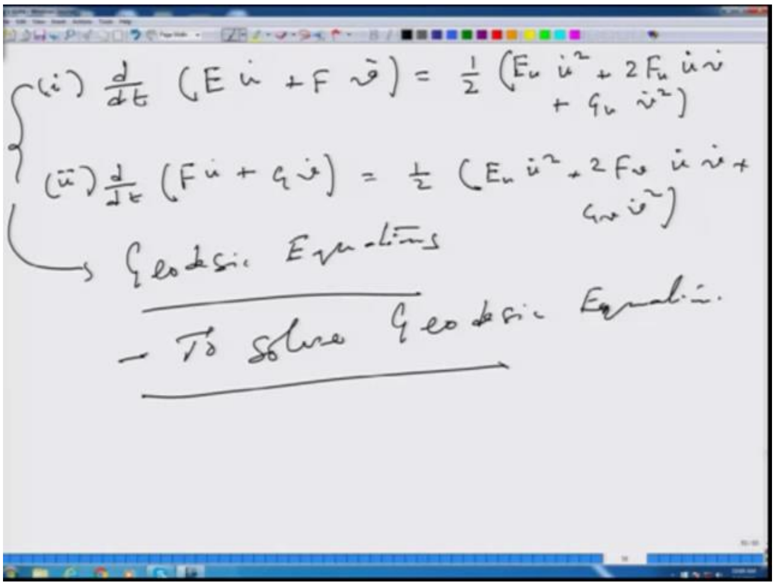


it will be a geodesic if and only if gamma satisfies equation 1 d dt E U dot + F V dot equal to half E U U dot square 2 F U U dot V dot + G U V dot square, 1st equation. 2nd equation is d dt F U dot symmetrical GV dot is equal to half E U, same, U dot square 2 FB U dot V dot + GV V dot square. These 2 equations are called geodesic equations, okay. Proof, I will write down next time, proof is very easy. But the problem is who will solve them? Okay. How we will solve for U and V from here?

So, the problem is to solve the equation. Well, if your surface patch is nice enough, then I can do it, perhaps. But not all the time E H U dot V dot will be very nice to, nicely behave, right.

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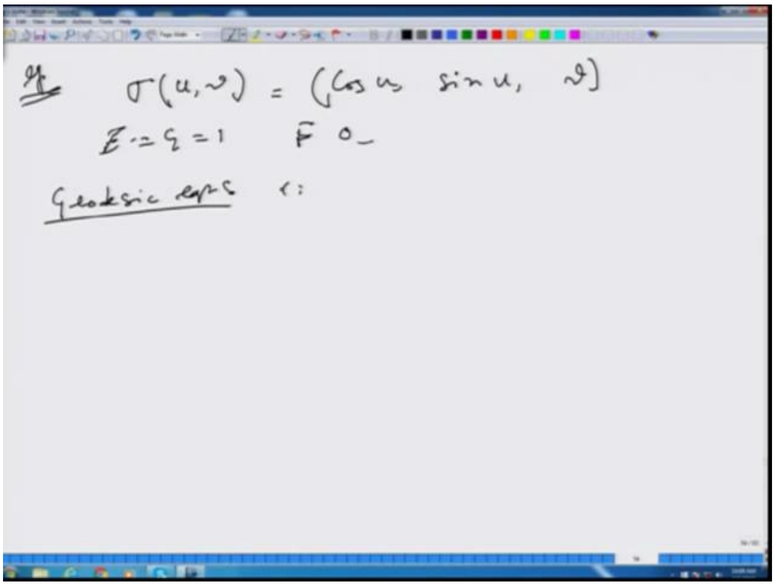

For instance, I will give an example. Very simple example, so that I can calculate, actually use geodesic equation. Okay, so, what is E, we have calculated before, E G, both are 1, F is 0, so geodesic equations are,

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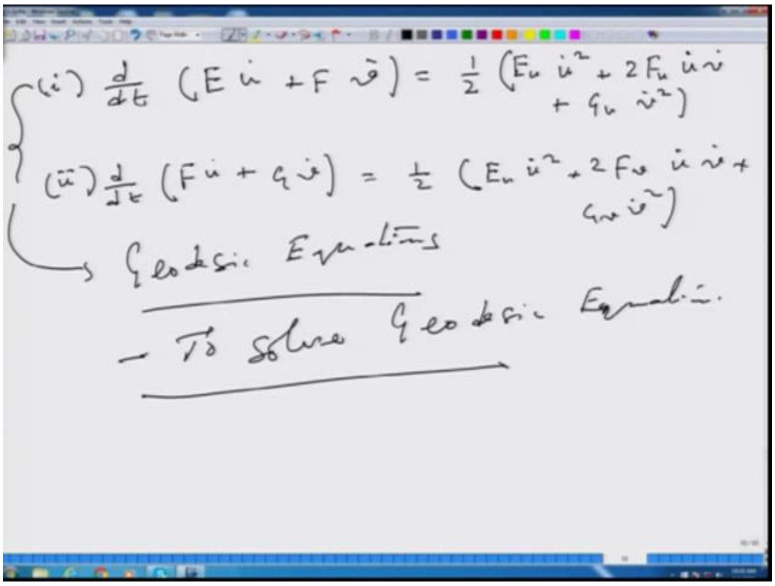


go back d dt E F is 0 E U 0 because this is identically one. So, right hand side is 0, both the cases and I get what, just get U dot dot + V dot dot equal to 0. And 2nd equation is E F, both are 1, so d dt of U dot + V dot, this is U dot dot + V dot dot, right hand side is 0.

And 2nd equation F is 0, sorry G is 0, F is 0 U dot, F is 0, so U dot does not occur, so just G is 1, so only V dot dot, right-hand side is again 0.

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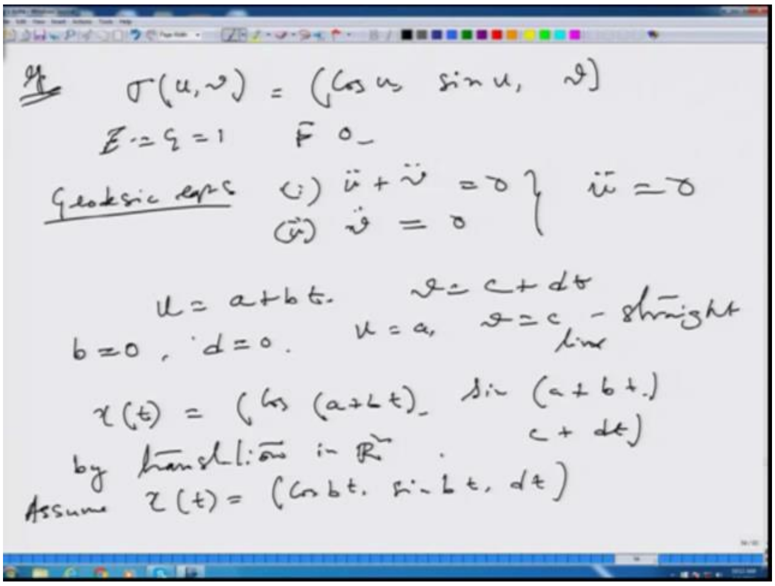

So, V dot dot is 0. But 1 and 2 together will also give me U dot dot equal to 0. So, what is there? U equal to A + BT, B equal to some C + DT, straight lines, UT and VT are like this, okay. I solve U and V in terms of T. Now, cases are, if B equal to 0 and D equal to 0, then I got U equal to A, B equal to C, so this is a straight line. Okay. Now, what is gamma T in general, gamma T if I put back, this is cos A + BT sin A + BT and V is C + DT. Okay.

So, by translation, so translation in R2, so rigid motion, nothing changes, we may assume gamma T, assume gamma T is just cos BT sin BT dt.

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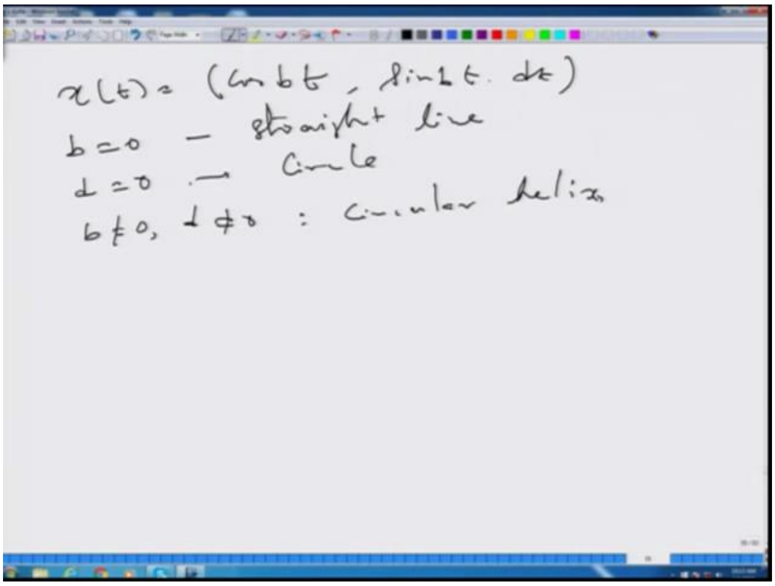


Gamma T is looks like cos BT sin BT dt. Okay, B equal to 0, this is straight line, this is 0, this is one, 0 dt. D equal to 0 cos BT sin BT, this is 0, so this is circle and B equal to, B not equal to 0, D notequal to 0, both are nonzero, then it is a circular helix.

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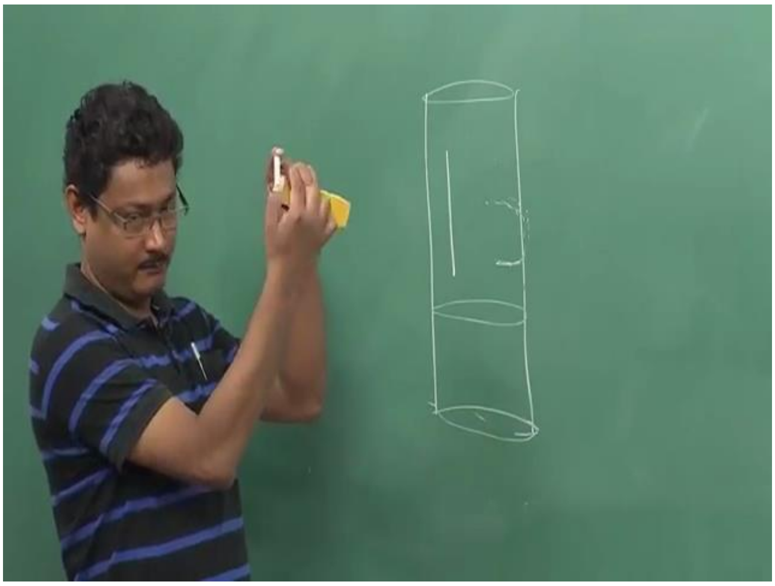

So, geodesic cylinder are straight lines, this circle or circular helix. What is happening here is actually, so this is a cylinder, if you want to have a geodesic between these 2 points, that will be straight line, I had geodesic between this point and this point living on the same plane, that will be circle. Geodesic between this point and this point behind, then it will go like circular helix. So, these are the…

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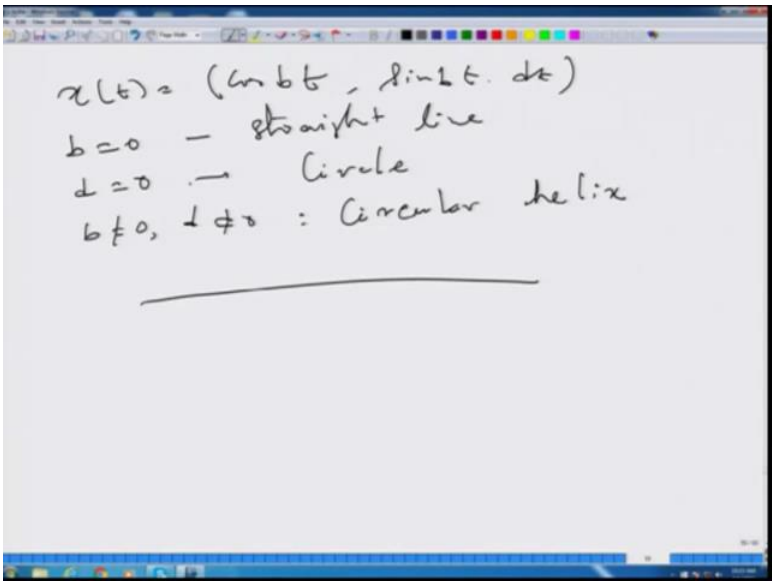


This is circle, circular helix. Okay. For this case it is very easy to solve but in general or may not be so lucky, we have to see some more techniques. So, that will be the end for today's lecture, tomorrow we will see, try to develop some more techniques to calculate geodesics on other surface. Thank you.