**Curves And Surfaces.**
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**Module-IV.**
**Surfaces-3: Curvature and Geodesics.**
**Lecture-16.**
**Regular Surfaces Locally As Quadratic Surfaces.**

Welcome to the last module of this course curves and surfaces. This week we will be mainly dealing with the geometry of the surface. So far we have developed certain quantities related to curvature of a surface.

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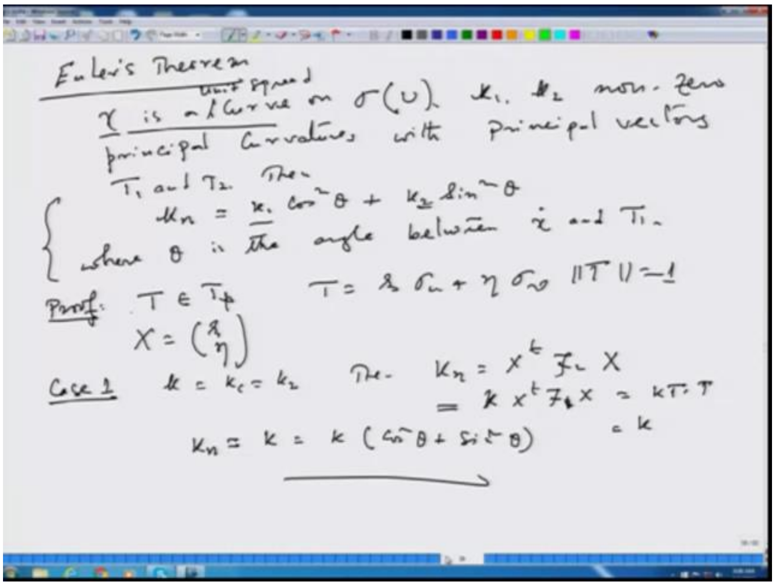


Last time I did this theorem, Euler’s theorem and we saw that normal curvature can be realised from, so normal curvature can be expressed in terms of principal curvature.

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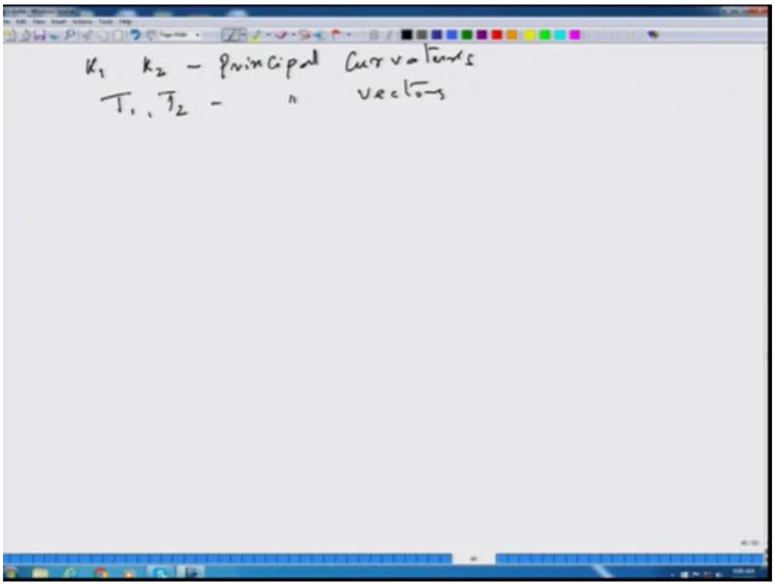


So, recall those quantities K1, K2, these are principal curvatures. And along with that I have 2 principal vectors T1 and T2.

Now to understand things in little bit easy way, what I can do, whatever my surface patch is on the three-dimensional space, maybe it is here,

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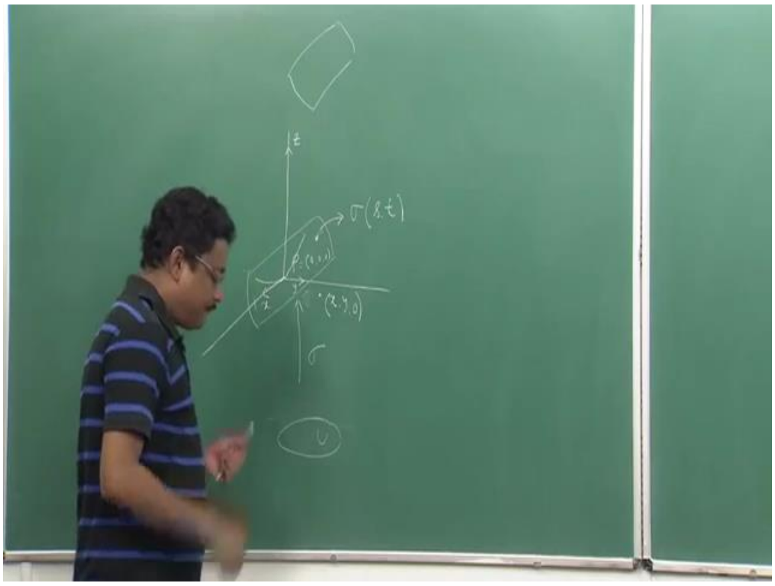


I can apply some rigid motion, so I will write it, applying rigid motion, maybe one more, and doing reparameterisation, we may and will actually, we will assume, assume what?

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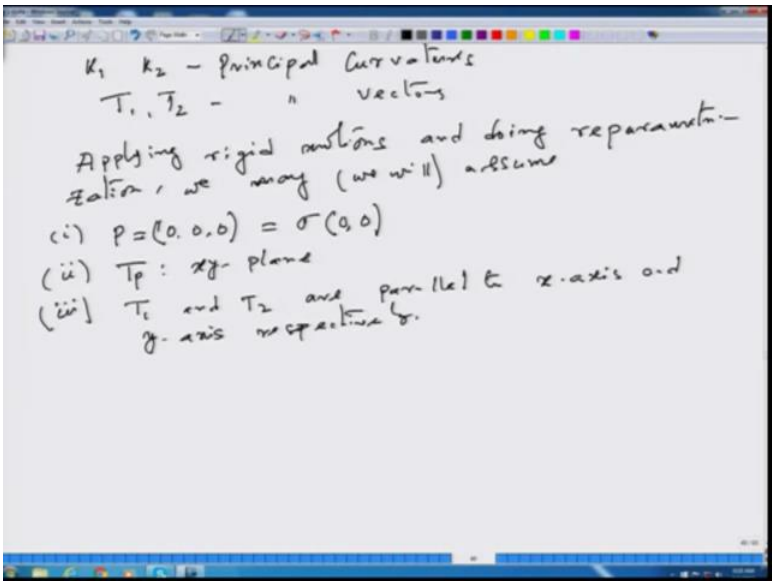


That I am looking at a point P on the surface which is 0, the origin and that point is actually comes from the patch, it comes from the patch, on the patch it is Sigma, 0, 0. We will assume the tangent plane at P is XY plane and one more, T1 and T2 are unit vectors parallel to x-axis and Y axis respectively. Okay.

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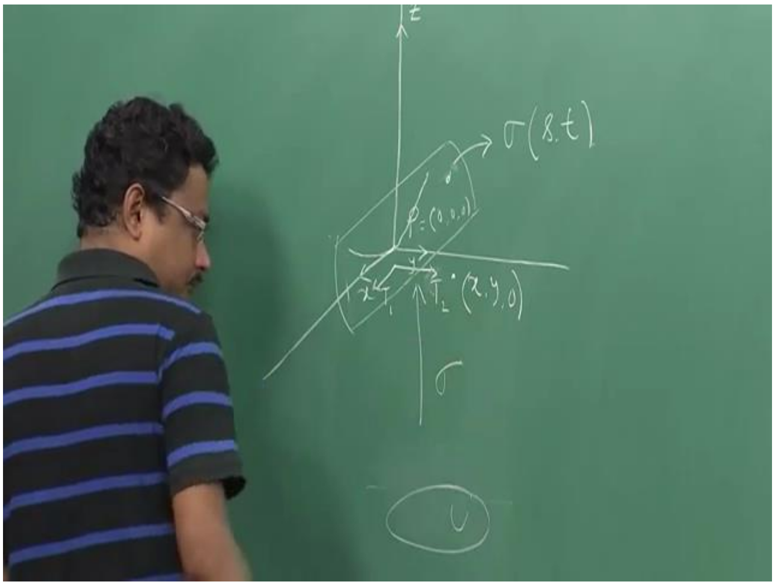


They were not be the vector 1, 0, 0 or 0, 1, 0 but it says at what point the principal vectors are parallel to X T1 is parallel to x-axis and T2 is parallel to the y-axis. So, T1 and T2 are perpendicular, that we know, if K if K1 not equal to K2. What we are going to show here with this assumption, okay, 1st let us convince that we can do it, that whatever the surface is, by applying some rigid motion, we can always bring it back to near origin. What is our idea here is to analyse a particular point, so I have taken a small surface patch, analyse a particular point Sigma ST around in terms of these vectors.

So, how the surface looks like around this point P and we will see that with this kind of assumption, any surface, any regular smooth surface you take around this P, particular point which I have, which can be fixed, we can fix any point, the surface is a quadratic surface. I have not talked about quadratic surface, I planned to do it in the last lecture possibly. But, anyways we will see it is going on.

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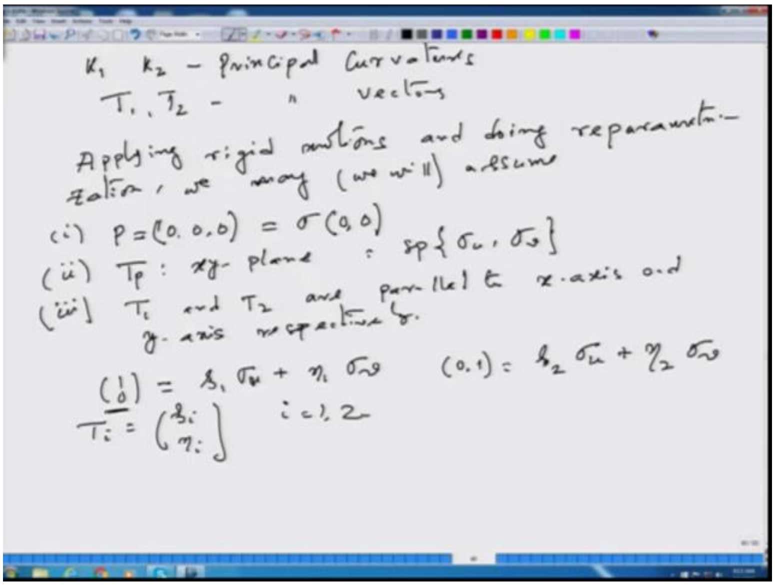


Okay, now, so let us take, start here. Since T1 and T2 are parallel to X axis and TP is XY plane which is of course pan of Sigma U at P and Sigma V. So, the vector 1 0, what will happen to that, this in the tangent plane TP, so it can be written as Zhi 1 Sigma U + eta 1 Sigma V because Sigma U and Sigma V span XY plane, right. So, 1, 0 is a vector in XY plane, so it can be written as a combination of this.

Similarly 0, 1, that vector also I can write as Zhi 2 Sigma U + some eta 2 Sigma V. And let us put T I equal to Zhi I eta I, as usual as before.

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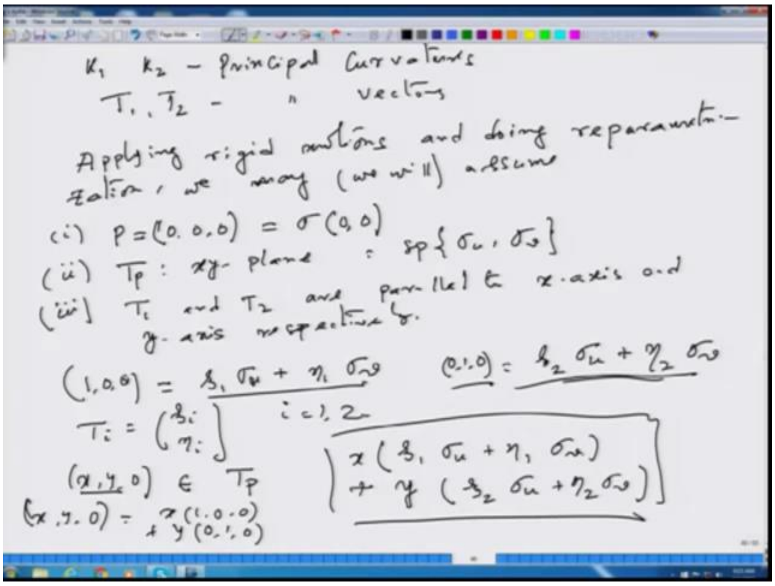


Okay. So, what about what happens to the particular point above the plain, XYZ… X , Y , 0. Now, X, Y, 0 I know, this is a point in the tangent plane, right. So, it can be written as X… why is it so? Because X, Y, 0 can be written as, look at where the point X, Y, this is 10 X into 10, Y into 01 but 10 and 01 I have written in this form.

So, this is in R3, so this is 1, 0, 0, this is 0, 1, 0 because X, Y 0 is X 1, 0, 0 + Y 0, 1, 0, and now 1, 0, 0 and 0, 1 I have substituted , so now it will look like this. Okay.

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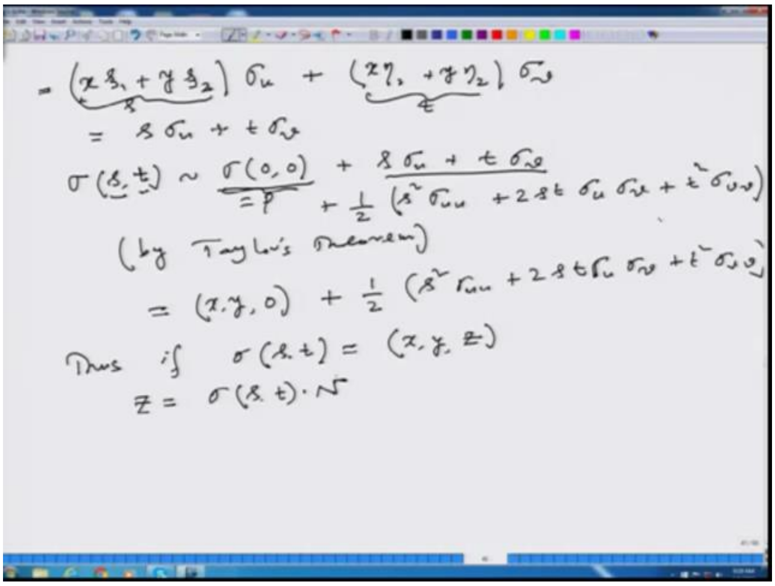


So, let us write it, is equal to, look at the expression, I collect the term with Zhi Zhi and eta terms differently, so I write it as X Zhi 1 + Y Zhi 2 Sigma U + X eta 1 + Y eta 2 Sigma V.

Alright. Okay. So, let it be. What… Some S Sigma U, I call this fellow S, and this fellow S and this fellow T, T Sigma V, okay. So, now this point, if I look at Sigma ST, then applying Taylor's theorem, I can approximate it by around Sigma, 0, 0, S and T are very small, so around P.

So, remember this is equal to P, right, the origin + S Sigma U T Sigma V + some second-order terms, S square Sigma U U + 2 ST Sigma U Sigma V + T square Sigma V V V. So, this is by Taylor's theorem $2^{nd}$ term, okay. This is by Taylor's theorem, this expression means, there is some error part is there, I am not writing equal to, I am writing approximately the same. If you want to write Taylor's theorem exactly, then what you should do, you should write equal to here and some + + some error term here. Okay.

This is, now I write XY 0+, this is 0, right this fellow is 0, origin, this is X, Y, 0 from the previous page, X, Y, 0 I have written in this form, Sigma U + Sigma T, so this fellow is X, Y, 0 + whatever remains. Thus, if Sigma ST in terms of the surface coordinates is X, Y, Z on the plane, then what is Z, this is Sigma ST dot N because XY I have taken the tangent plane, so the Z coordinate I should get, if I get Sigma ST dot product with the normal.

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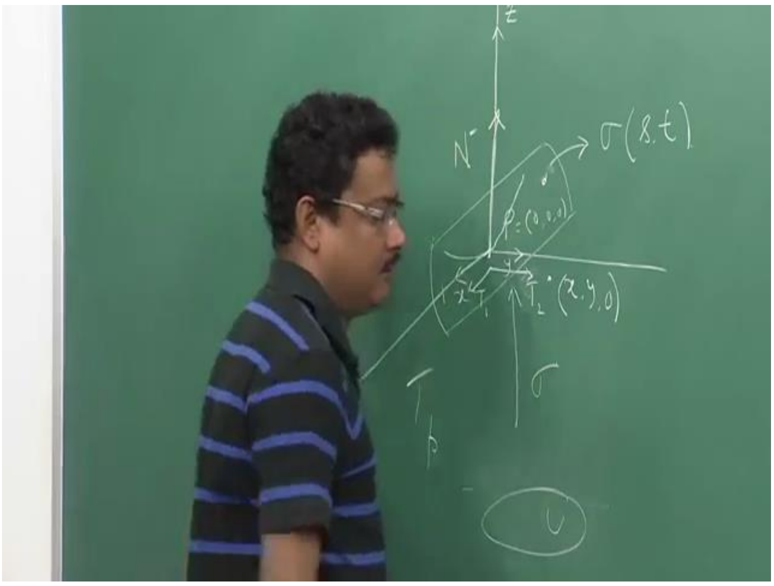

Because this is the tangent plane, this is tangent plane with XY plane, so the normal is along this coordinate. Correct.

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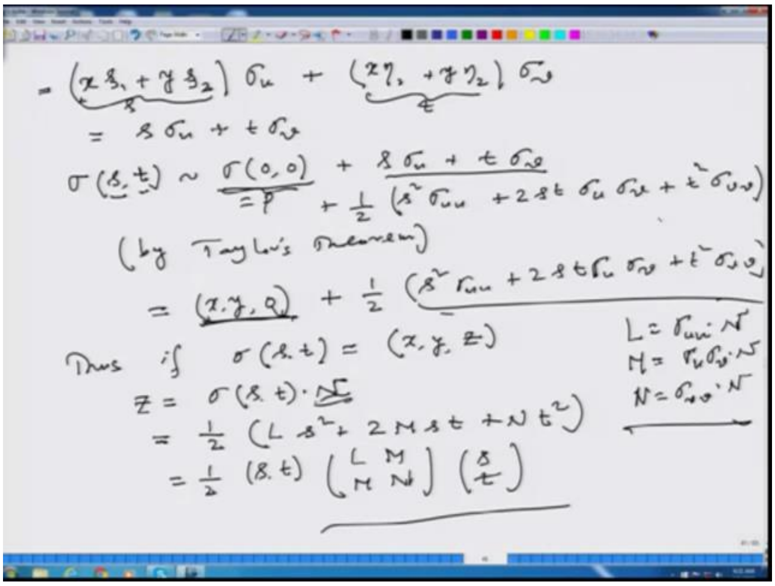


Which is we know half of LS square 2M ST is less M T square. Or half ST LM, the 2nd fundamental form comes in here, you understand how it follows, if I take the dot product of this fellow with N, this is in the XY plane, N is on the Z axis. So, dot product of N with this fellow is 0 and dot product of N with this fellow is S square, then N into Sigma U U, that is our L by definition, recall L is Sigma U U dot N. M is what? Sigma U Sigma V dot N and what remains N, N if Sigma V V dot N. So, 2nd fundamental form comes up here because only this term remains and I have just written in the matrix form. Alright.

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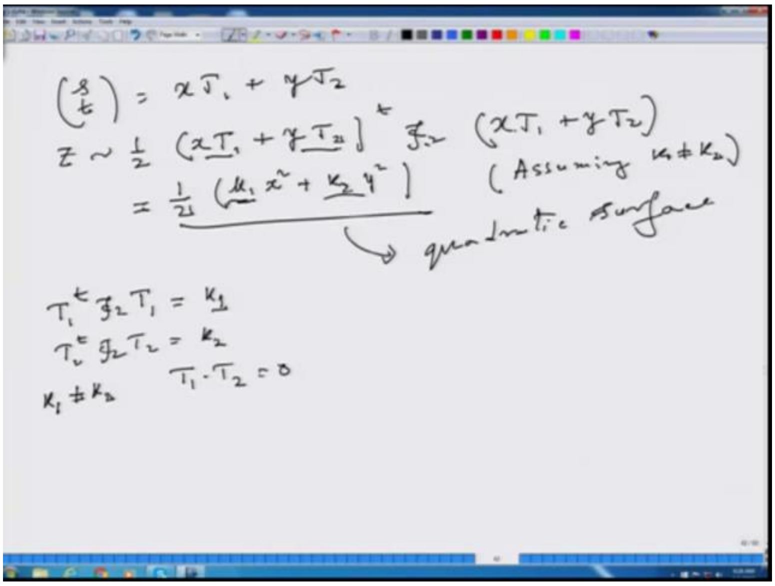

So, if I right now ST as X T1 + Y T2, then my Z becomes half of X T1 + Y T2 transpose F2, F2 is this matrix, this is F2, X T1 + Y T2. What is it by the way? Half of XT1 transpose F2 XT1, T1 transpose T1 gives you T1 F, T1 F 2 T, T1 transpose, T1 transpose F2 T1, how much is that? That is precisely K1. So, this is K1 X square YT2 and K2. Rest terms, cos products terms if I take K1 not equal to K2, then you know T1 and T2 is 0. Okay. So, if this is assuming K1 not equal to K2. I should not have written equal to because I have a tilde sign, approximately. So, JB is actually, this one is approximately a surface.

So, this is a quadratic surface, quadratic because only quadratic terms are involved. So, what it says that if I have a regular surface, then around a point it looks like as a quadratic surface and the quadratic equation of that is determined by the principal curvature K1 and K2. So, according to that because that, so behaviour of K1 and K2 determine what will this curve look like, what will this surface look like,

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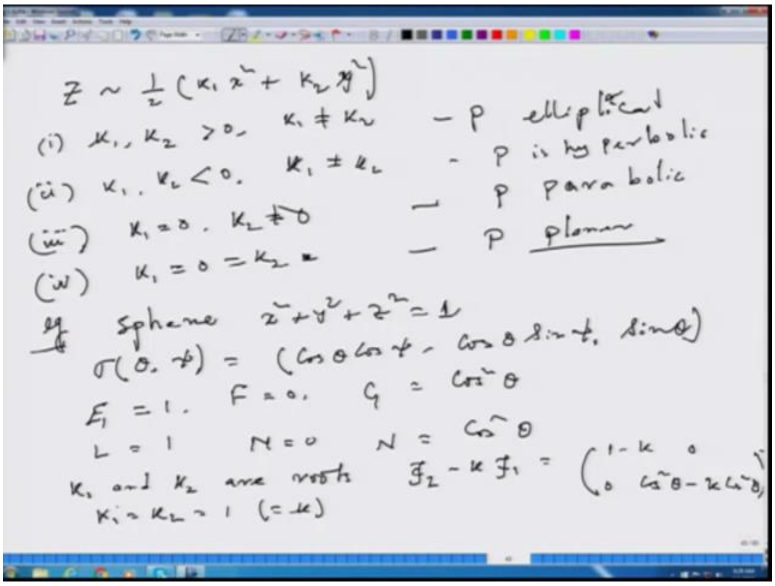


so in particular if I have Z is half of K1 X square + K2 Y square, so suppose I have K1 and K2, both greater than 0 and K1 not equal to K2, then this is a question of a Ellipsoid, then this point P is called elliptical. We will call this point elliptical.

If I have K1, K2 less than 0 and K1 not equal to K2, then this P is called hyperbolic point. So, around that point P, what does it mean, why elliptical in this case, $1^{st}$ case around P, the surface looks like a part of an ellipsoid. Here the surface looks like a part of a hyperboloid. $3^{rd}$ is, possibility is K1 equal to 0, that is possible, right. And K2 not equal to 0, then what happens, P is called a parabolic cylinder, P is called parabolic. It is just K1, where is that, yah, K1 0, so Z looks like that K2 Y square. So, it is a parabolic, X X moves free and Y and Z move like parabola, so it is called parabolic cylinder.

And of course there is another possibility, K1 equal to 0, K2 equal to 0, then you know what this is, P is a planar surface, Z is 0. Z moves around P, the surface is flat. Okay. Let us see some examples of these kinds of points. What happens to our most canonical example, sphere? You know in cylindrical coordinates, this is given by Cos Theta cos phi cos Theta sin Phi and sin Phi and sin Theta, right. You calculate E1, this will be in 1, this we have done before perhaps, and G is that cos square Theta, N is 1 common M is 0 and letter M is again cos square Theta.

K1 and K2 are roots of which matrix? We have done it right, F2 - K F1 which is 1 - K 0, 0, cos square Theta - K cos square Theta. What are the roots? So, K1 will be 1, that will be same as K2, which is we denote by K. So, just write this way. So, K1 equal to K2 equal to1 and say it is equal to K, so K is 1. In this case what will happen? Z looks like X square + Y square but other terms will come up, right.

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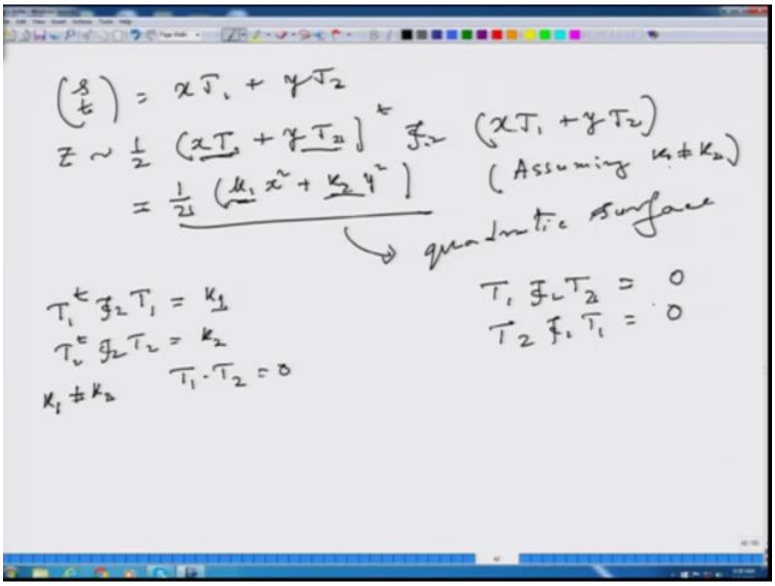


So, cos term will not go away here because K1 is not equal to K2. What will happen to cos term? If K equal to K equal to K1 equal to K2, what happens to the cos terms? T1 T1 F2 T2, we have done it before. Similarly T2 F2 T1, we have seen, look at the proof of the Euler's theorem proposition before that, this is always 0.

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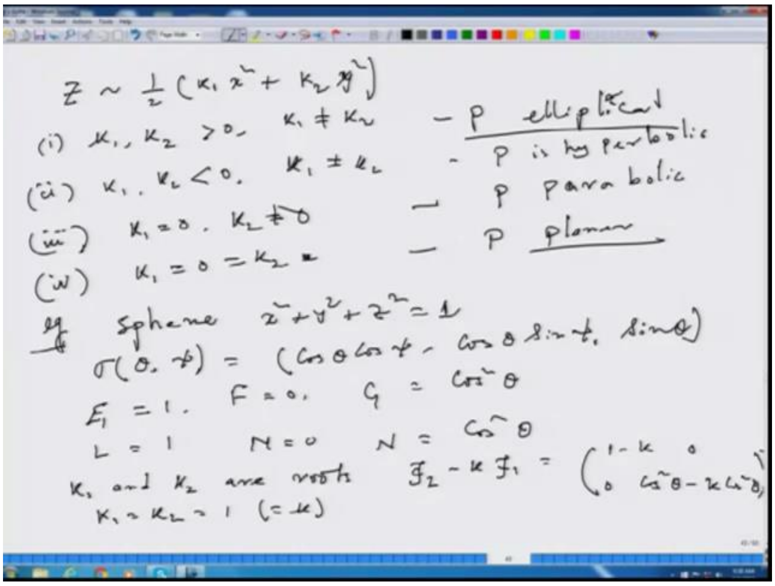


So, it will be X square + Y square, so P will be looks like it just a sphere, it is part of a sphere. And so everywhere around, every point of P on sphere looks like a sphere, that is what the example says. Better example could be…

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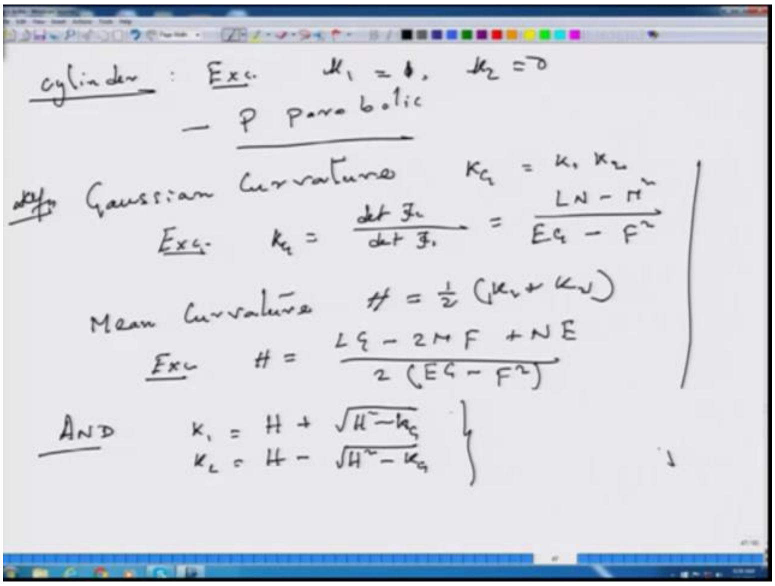


Let us take a cylinder, you can most… And for that I will give an exercise, you calculate K1, you will have that K0, sorry, K1 is 1, K2 will be 0, does not matter. In this case, in cylinder, P, all parabolic, that was the case, right. Cylinder around the point P, if you look at any point P, it looks like a quadratic surface paraboloid, so any point is parabolic. Okay.

To end the lecture today, I define 2 more quantities, so definition. Gaussian curvature, what is Gaussian curvature? It is just K G is just product of K1 K2, which will be, K1 K2 are the roots of F1, F2 - K F1.

So here comes an immediate exercise that KG is actually determinant of F2 divided by determinant of F1. In terms of fundamental form it is LM - M square, SG - S square. So, this fellow is called mean curvature, it is only denoted by H. This is the average of these 2. And the exercise is in terms of fundamental forms is LG - 2 MF + NP divided by the determinant of fundamental, $1^{st}$ fundamental, 2 into $1^{st}$ fundamental form. And K1 is H + A square - KG and K2 is H - A square - KG. Solve this exercise and remember this formula, these formulae, that is it for today. Tomorrow onwards we will talk about geodesics.