Curves And Surfaces.

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Module-III.

Surfaces-2: First Fundamental Form.

Lecture-14.

Curvature Of Surfaces.

Okay, today we will start about analysing the geometry of surface. We have seen the definition of 1st fundamental form and we have seen how it indicates with isometry and conformal maps.

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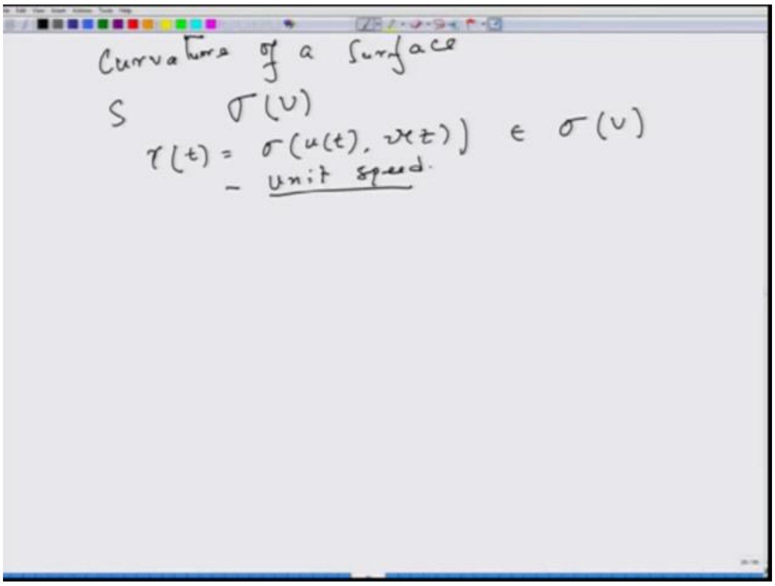

So, today I will define curvature of surface. So, surface for us is always regular, smooth surfaces. Now, as we have seen in last 2 lectures that to analyse the geometry of surfaces, we actually need to work on the surface, that is we have to analyse the curves on the surface. So, let us take a surface S, let us take a surface patch on that Sigma U and let us take a curve, Sigma T, Sigma UT VT on this surface patch, okay.

That I will we will always take unit speed for, so for today what I will do, I will define certain quantities which are related to the curvature of the surface and we will today 1st lectures and tomorrow's lecture will be mostly on calculation. Once we have some calculations, some theorem in hand, then we can actually go for the geometry part. Now, all this calculation, I will always assume, we are talking of unit speed curve, I emphasise on that. So, while on assignment, solving assignment or doing, writing the exam, if you see curve not unit speed, you make it unit speed and then do the calculation, otherwise quantities will not match. Okay.

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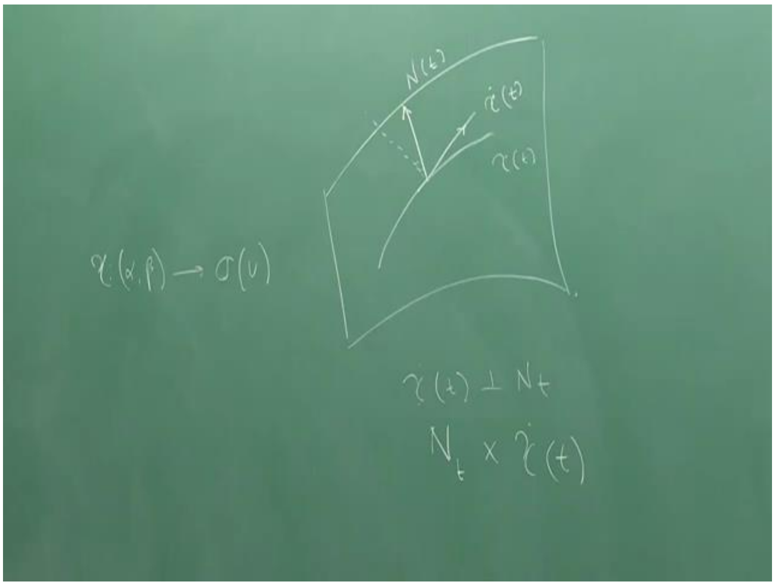

So, what I know? Okay, let us start with the picture, so I have a surface patch out there and I take a curve gamma T here, right. So, at a particular point T0, gamma is defined from, okay (())(3:00) T0, I have a tangent vector, unit, this is unit vector because I have taken unit speed. I have a normal N, it will also depend on T of course. And which is perpendicular to gamma T, gamma dot T. And there is another vector which is drawn on the other plane, this is cross product of these 2 vectors which is also perpendicular to both of them. Right.

So, I have 3 vectors and okay, let me, just to avoid confusion actually one text usually this way, otherwise you will have forgotten the sign. What I wrote earlier is a negative of this thing. So, what is happening now?

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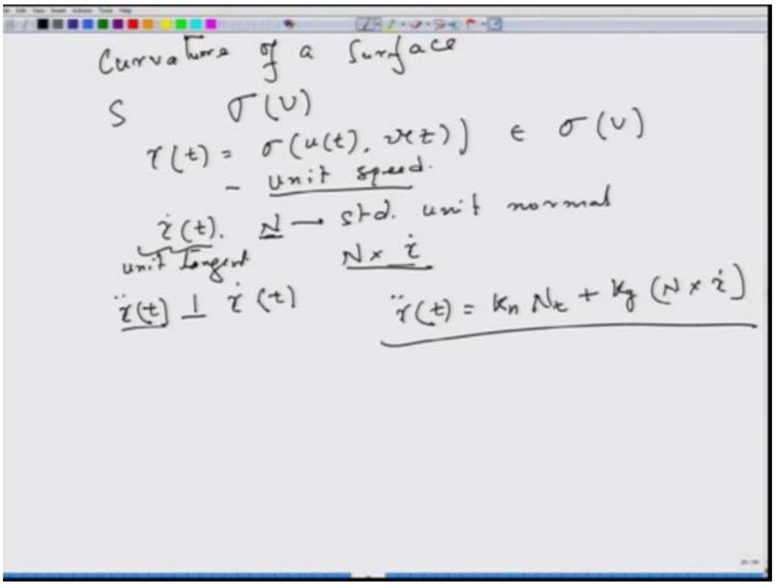

So, I have 3 vectors, gamma dot T, standard unit normal, so this is the unit tangent, this is standard unit normal, okay. And I have this vector N cross gamma dot and these 3 vectors are perpendicular to each other, so these 3 unit vectors form a basis of R3 where there is our surface S.

Now we look at this vector gamma dot double T, this was the quantity we remember, norm of this we call curvature in we talked about curvature of curves. Norm of this gamma double T is actually the curvature of gamma. I want to associate this to the curvature of surfaces. Okay, gamma double T, gamma with unit speed is perpendicular to gamma dot T, okay. These 3 vectors forms the basis of R3 and this fellow is perpendicular to this, so therefore gamma doubled T will be some, let us write KN NT + KG, why I am writing N, I will just explain. Is that okay, this will be a linear combination of these 2 vectors, what is that in the picture?

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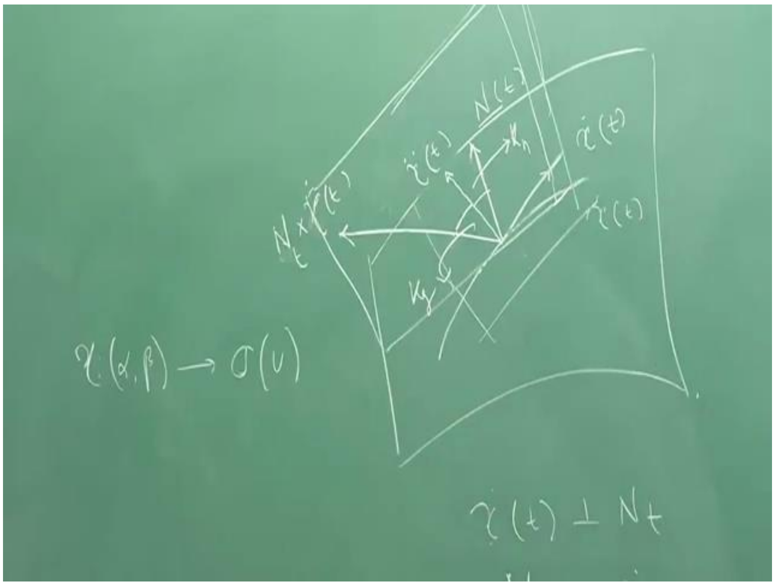

So, one is when gamma double T is there, it is perpendicular to, so this is along this line, let us say again.

So, it is on the plane, so look at the plane generated by Eta T cross gamma dot T and Eta T, the plane containing this one, these 2 vectors. So, here is my gamma double dot T, so should I, should draw little bit bigger picture… Normal, this is tangent and maybe this is a vector which is coming outside of the board, Eta T + gamma dot Eta T cross gamma dot T. This vector is coming outside of the board. So, gamma double T, double dot T is a vector which is lying in a plane generated by this one and this one. Right. So, it will be combination of this and that, this vector and that vector, so here is my cos Theta of this angle which I call K Eta and this angle I call KG. Okay.

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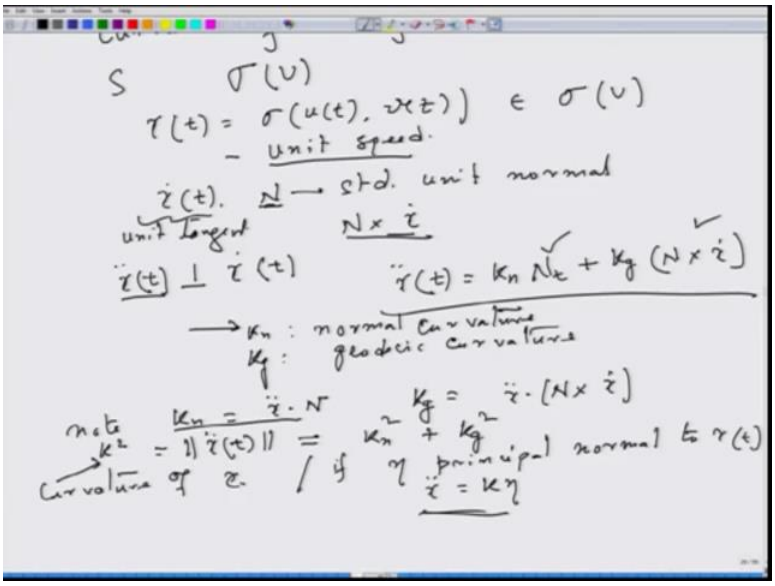

This KN, this is called the normal curvature. Again, this is called the normal curvature and KG is called the geodesic curvature. Okay. We will see how what is the geometric interpretation of these 2. 1st I will, we will, next few lectures will be concerning about normal curvature and then will be, when you introduce geodesic, you will see what is the geometric meaning of geodesic curvature. Just note normal curvature from this relation can be obtained by gamma double dot T dot product N, that is angle, right. These 2 all are unit vectors, so this is the angle.

And geodesic curvature is equal to gamma double dot dot product with another vector. And curvature of the curve, the curve itself is now… Correct, because these vectors are unit vectors, they are perpendicular to each other. So, this is usually, we call this, we call it curvature of, absolute curvature of the curve when we discuss about the space curve. So, this fellow is the curvature of gamma, okay. And we also know that if eta is a principal normal on the curve at the point P. Then we have the relation, you will call this gamma double dot equal to K eta. This was a part of Frenet Serret equation, what we did use in the 1st week.

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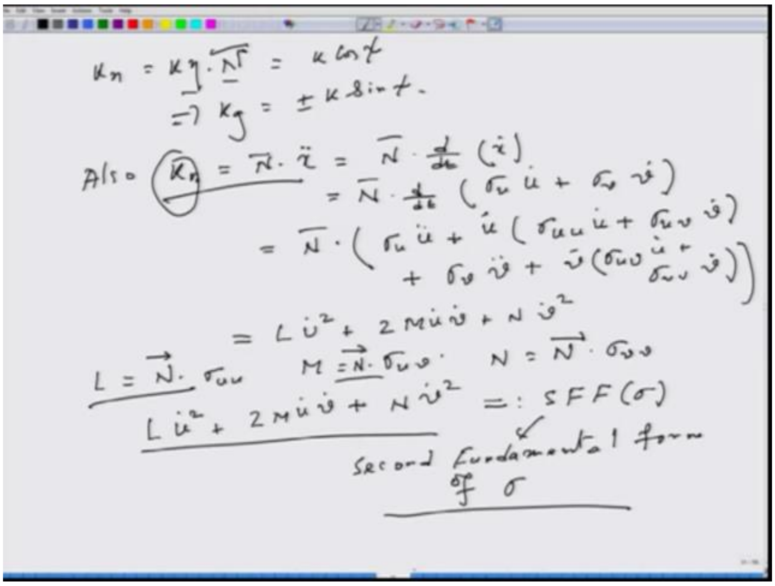

So, this will give us, now what is my K Eta, KN, KN is here, gamma double dot is here, so this will give me K normal curvature equal to K Eta dot N which is, this is a principal normal, this is standard unit normal, this is K cos Phi, so this is the angle between principal normal to the curve gamma and the standard normal at the surface NT and therefore, you are this curvature will be some + - K sin Phi. Also, note from this relation here. Also K normal curvature is N dot gamma double dot, does not matter K, so this is N dot d dt of gamma dot, I am writing it for some other purpose. N d dt, now I know what is gamma dot, we have done it various times, U dot + Sigma V V dot, right.

So, this is, I write it as now N dot Sigma U U double dot but then I have to have the U dot and with U dot I should have again Sigma U U U dot + Sigma U V V dot, then I will have it with V, so Sigma V V double dot + V dot Sigma U V U dot + Sigma V V V dot. Okay. So, this is actually now we will write as LU dot square + 2 M U V dot + N V dot square. So, what will be L them? You collect the term, this will be N dot Sigma U U, M will be Sigma U V dot, okay N dot and N will be, all right, there are 2 Ns, okay, this is the principal normal.

N is, just check the calculation is all right or not. Okay. So, this fellow L U dot square + 2 M U U dot V dot + N V dot square. This is known as SFF of Sigma, what will be FFF stands for? If you recall from another lecture, this is fundamental form and S for, we have already seen 1st fundamental form, so this one is called the 2nd fundamental form of the surface patch Sigma. See, in this expression, there is no gamma involved, so this is independent of the curve we have taken as in the 1st fundamental form, this depends on the surface only.

Parameterisation of the surface U, in terms of U and V, because this quantity L does not involve anything about Gamma because N is the standard unit normal to the surface.

So, what it does, this 2nd fundamental form? 2nd fundamental form, from this relation, what it does is, takes the curve, as in the 1st fundamental form what it does, it takes a curve, integrate you get the length. 2nd fundamental form does what? It takes the curve and gives this normal curvature. Okay. So, if you have a curve gamma which is UT VT at, so when UT and UT and VT are lying on the curve gamma, that will give us the normal curvature of the curve. So, that is, that is what it does. Any form as I said, the form I will not define, usually takes a curve on some special class of curve, allowed curves for us, it is always smooth curves and gives a real number. Okay.

Maybe little bit of more geometry I can explain but, okay let us see. We will come back to more geometric picture of 2nd fundamental form that it actually the, normal curve, analog of the normal curve, analog of curvature for the curves.

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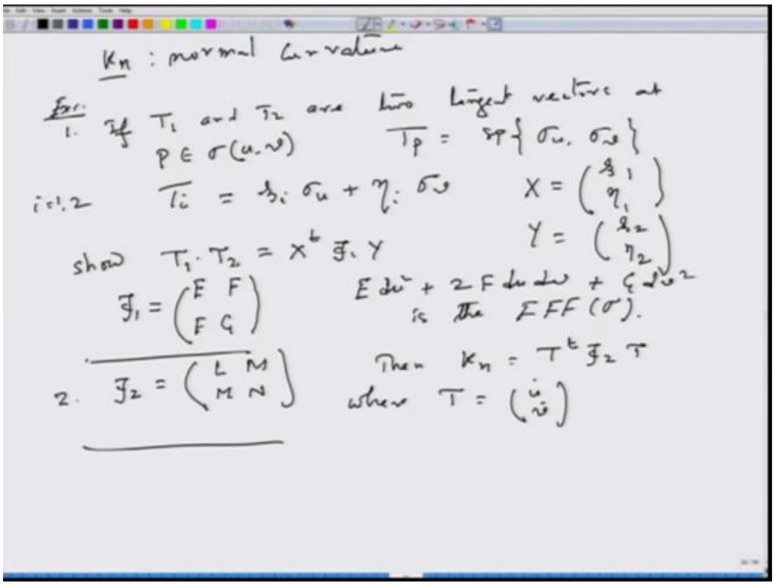

But for the time being, let us continue our discussion on this normal curvature little bit more. And I will start with these 2 exercises. I will put it in assignment also. But to proceed further on this course, you must solve these exercises, there were easy. If T1 and T2 are 2 tangent vectors at some point P on the surface, this means what? On the surface I have a curve passing through that and T1 and T2 are 2 tangent vectors, so you know on the, we have defined what is surface, what is called tangent plane at a point, right, for a surface.

Then of course my TI, I equal to 1 to 2 because you know the tangent plane that is Sigma U V or TP, this is spanned Sigma U and Sigma V, so T1 and T2, both can be written as some combination Zhi I Sigma U + Eta I Sigma V. Okay. So, let us take this vector X Zhi I Eta I. Oh sorry, Zhi 1 Eta 1 and Y to be the vector Zhi 2 Eta 2, then show T1 dot T2 is actually X transpose F1 Y. What was F1, F1 remember, we associate this 2 cross 2 symmetric matrix with respect to the 1st fundamental form. Where E V square is the 1st fundamental form of Sigma.

This is very easy actually. And next exercise is, if I write this, associate this matrix which is likewise for the 2nd fundamental form, then normal curvatures can be calculated this way, very easy, the verification, where T is a vector U dot V dot, it is there from the definition. Okay. Very good. So, this exercises are just verifications.

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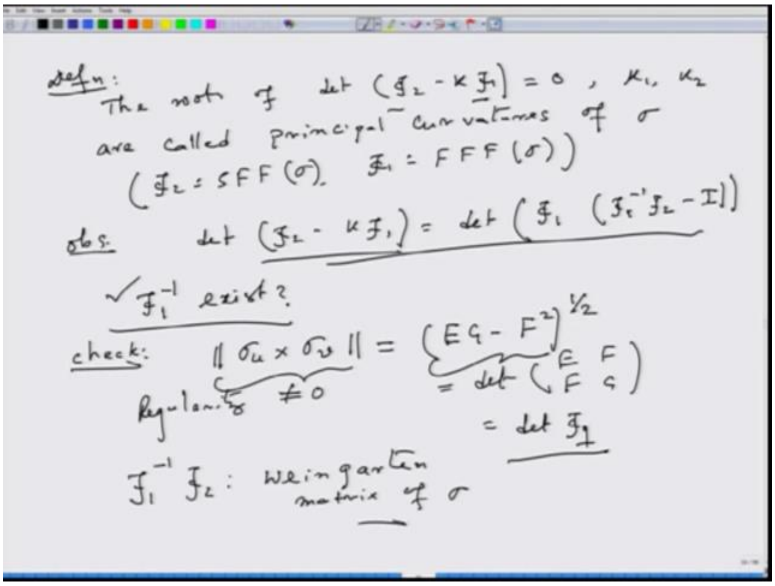

So, now I make, associate 2 more quantities, I said that we have to 1st this lecture and next lecture, we have to associate certain quantities and then we will analyse those quantities and see what is what is their integration on geometry.

The roots of F2 - K F1 equal to 0, so there is a 1st fundamental 2nd fundamental form, than the 1st fundamental form, so there will be 2 roots K1 and K2 are called the principal curvature curvatures of the surface patch Sigma. Recall F2 is 2nd fundamental form of this, this one is the 1st fundamental form of surface patch. Then you been see immediately, observation, that determinant of F2 - K F1 equal to determinant of, okay, F1 F1 inverse F2 - identity. What I have done? I have just multiplied with F1 and F1 inverse, I have just multiplied this on the left with F1 and F1 inverse. But who told me F1 inverse exists, why is it so? We have not checked it, right…

Sorry, F1, yes. Who told me this exists? Well, this is simple to check, okay. That is if you calculate Sigma U Sigma V, norm of this fellow, this is EG - F square square root 2. What is this? This is determinant of EF FG, which is determinant of F that is F1. But we state that we will always deal with regular surfaces. Regulatory condition of the surface gives us, this is always nonzero. So, determinant of F1 is always nonzero, therefore this F1 inverse exists, Therefore I can write this way, correct. This is an observation which we are going to use and I give a special name to this matrix F1 inverse F2, this is called Wein Garten matrix of Sigma. Okay.

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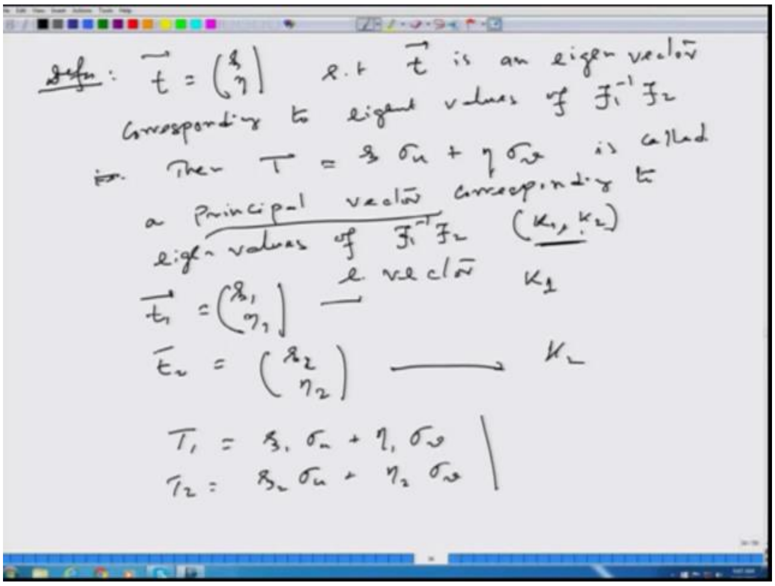

So, for today's lecture I will end with another definition. So, let us take T vector Zhi Eta such that it is, such that T is an Eigen vector corresponding to Eigen values of Wein Garten my Wein Garten matrix that is F1 inverse F2, that is when this vector T Zhi U + Eta V, this is called, just a definition we will see, okay just put called a principle vector corresponding to the **core** corresponding to that Eigen value… To Eigen values of, one of the Eigen values of this matrix, for Eigen values are K1 or K2, right. So, the principal, so what I mean here that I will have 2 2 vectors T 1 which is Zhi 1 Eta 1 which is an Eigen vector Eigen vector corresponding to be an Eigen value K1 and similarly T2 Zhi 1 Zhi Zhi 2 Eta 2 Eigen vector corresponding to K2. So, I will 2 principle vectors T1 which is Zhi 1 Eta 1 and T2 which is

Zhi 2 Eta 2. These 2 will be called the principle vectors. Okay. We will continue in the next lecture.