Curves And Surfaces.
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Module-III.
Surfaces-2: First Fundamental Form.
Lecture-12.
First Fundamental Form.

Okay, welcome to this lecture where we will actually begin our study on geometry of surfaces, as we did for geometry of curves. Now how do you get the feeling of the geometry for a surface S?

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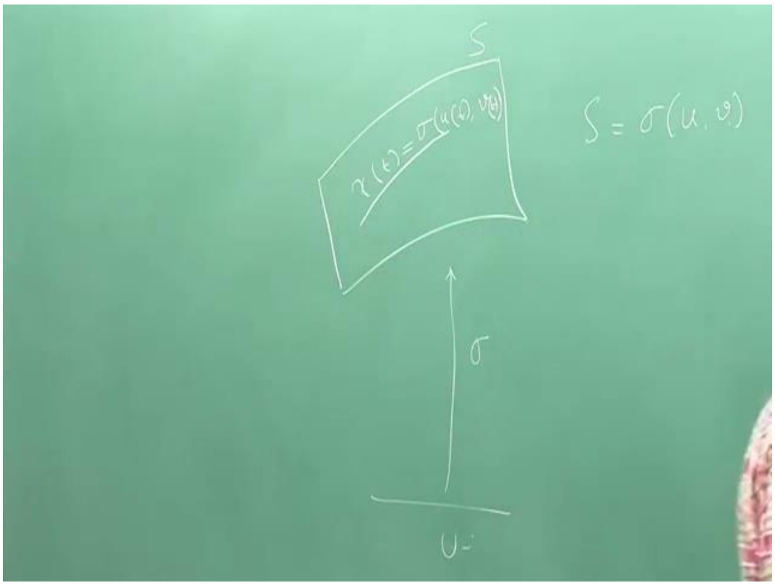

So, as we know it consist of, the surface will be consisting of different surface patches, so let us this be one such surface patch, here is my U. This is my surface S. Now to get the feeling of a geometry, intuitively what do you understand? If you want to have the feeling of that terrain, what you do? You walk on that terrain.

So, similarly for surfaces what you do, you try to take a curve gamma T which is on the surface, so it will be of the form Sigma UT VT, Supposing the surface patch are looking at Sigma U V. And we try to measure different quantities, relate quantities to surface along the curve.

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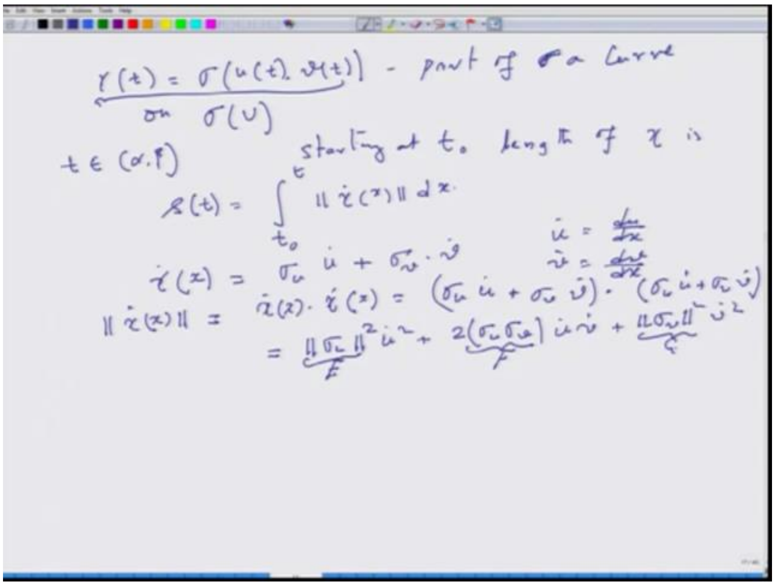

So, 1^{st} of all, 1^{st} of all okay, 1^{st} thing we note about a curve that this comes here, so I have gamma T is equal to Sigma UT VT, it is part of a curve, I do not need, part of a curve or full curve, part of a curve on Sigma U, right this is one surface patch you are looking at.

Now T was belong to some Alpha, beta and starting at T0, length of gamma is up to a point T is given by, we know from our discussion on curves gamma dot T dt. I should not be using T in this variable, maybe I put X, okay. Well, now what is this quantity, just have a look at gamma dot of X. This is, gamma T is this fellow, so this will be Sigma U U dot + Sigma V V dot were U dot denotes du dx, right. And V dot is, so I am applying chain rule here. So, what is this quantity, norm of gamma dot? Norm of a vector I can calculate if I take inner product with itself, so that is Sigma U U dot Sigma V V dot inner product with itself. Correct.

Let us explain this. What will happen before the multiplication? That will be Sigma U dot Sigma U, that will give you norm of Sigma U square and U dot square, cos term will come twice Sigma V U dot V dot + Sigma V square V dot square. Okay. Let us call this quantity V, this is dependent on E U, this quantity F and this as G.

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So, this is going to be really important, so I write it again for the surface patch, what is our denotion, this is a standard notation. V is Sigma U norm square F is Sigma U dot Sigma V and G is Sigma V square, remember this. So, corresponding to the surface patch Sigma, I define this quantity E, F and G, this defines on the same surface patch. On the same surface if you have 2 surfaces patches, this E, F and G will change.

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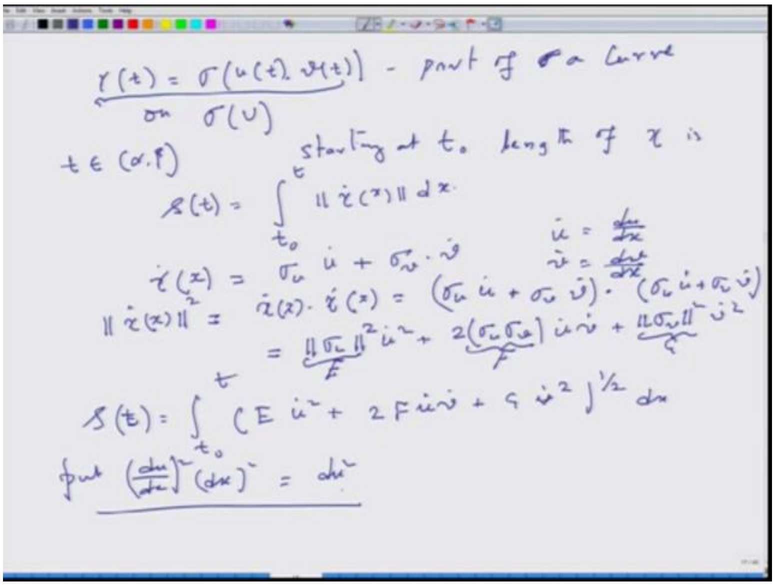

So, this fellow I can write it in E U dot square, 2F U dot V dot + G V dot square. Now, so S becomes, so ST becomes T0 toT, E U dot square 2F U dot V dot + G, so I am writing the same thing, G V dot square power half. Oh sorry, this was norm square, right, okay. So, I have to put a square root here.

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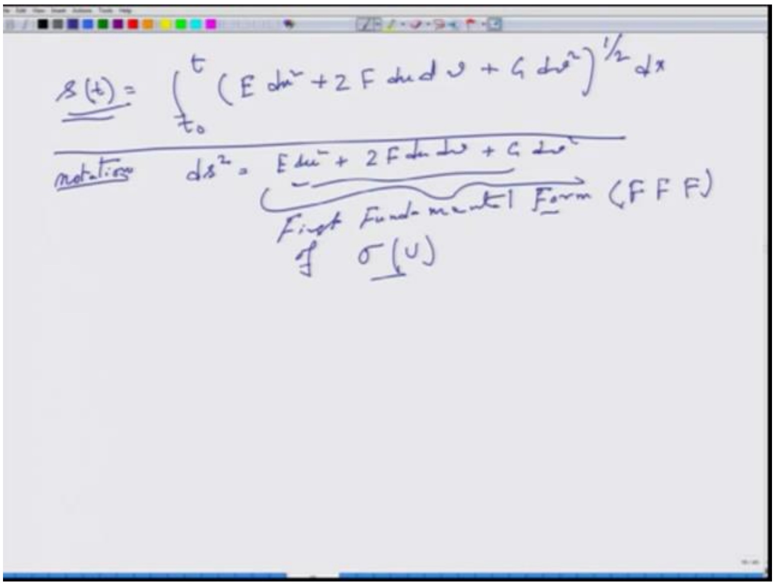

Now, put, if I put du dx, this was U dot square dx square, this is du square only, so this will match notation, what I get is, next page. XT equal to T0 to T E du square 2F du dv + G dv square power half d, correct. The usual form, we write this, so this is just a notation. d square is equal to E du square 2F du dv + G dv square. This fellow is known as 1st fundamental form FFF, this is the abbreviation we follow throughout of the surface patch Sigma. 1st fundamental form, okay, 1st time we are writing, assuming that 1st time we are seeing such an expression, so therefore 1st. Fundamental, it is really fundamental since that if I integrate this square root of the 1st fundamental form, then what I get, I get the length of the curve.

And why form? Form, well, form, I am not defining what is a form explicitly but any form what it does is that it acts on the curve, for instance here is R3 acts on the curve gamma to give a real number, in this case 1st fundamental form gives length of the curve. So, clear. However remember this expression, the 1st fundamental form, so we will see a 2nd fundamental forms, these 2 are very important as we see today itself. Now you will see immediately that what we have decided that we will, whatever properties on the surface you want to define, it has to be up to reparameterisation.

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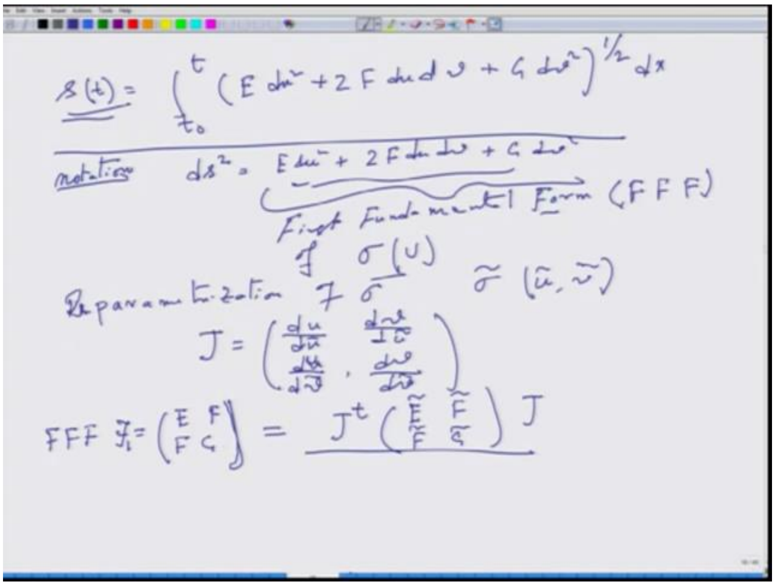

So, if I have reparameterisation of Sigma, let us say Sigma tilde, so that is Sigma tilde U tilde V tilde and then you look at the Jacobian matrix of Dell old by Dell new, right.

Dell U du du tilde dv du tilde du dv tilde dv dv tilde. You will see that 1st fundamental form if I write, if I write F F F in matrix form F F F, this is a very usual way of writing, just shortened way of writing the 1st fundamental form. Okay, this is usually denoted by F1, then this matrix and reparameterisation matrix will have this relation. So, 1st fundamental form is up to the Jacobian, changes of the Jacobian matrix, reparameterisation if I do reparameterisation. So, what are the things that keep remain same in the 1st fundamental form? Of course, length of the curve.

Why? Because in the length of a curve, that is from derivation, from derivation itself we have seen that actually in the 1st fundamental form when integrated gives you length of the curve and length of the curve does not depend on reparameterisation. So, that will remain unchanged. If a translate my surface, translation will give the Jacobian matrix JT. So, that will give, the 1st fundamental form is preserved by the translation.

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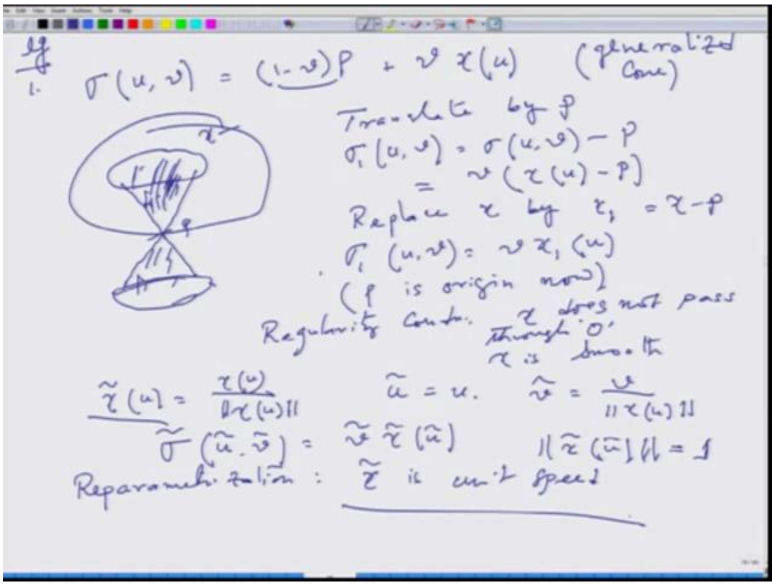

So, we are going to use that and see an example. So, let us example 1. Let us take the previous example. Sigma U V is 1 - VP + V gamma U, this was generalised cone, right. So, you have a curve gamma out here, you have a point, it is a ruled surface and, sorry, line passing through P and… Okay.

This is gamma, so that gives a surface, you have to remove the point P, so actually this is your surface and it has an opposite one if I take P, image of gamma with respect to P, I will have this one, so let us consider the upper part only without P. So, this is my surface. Okay, now as I said I can take P to be 0, why? Translate by P, so you get new parameterisation which will give to simplify V into gamma U - P. So, replace now, translate the curve also gamma by gamma 1 which is the translation of the curve gamma. So, I get, so it can have Sigma 1 U V is actually V into gamma 1 U.

So, now surface, this P is 0, P is origin now after translation, right. And I must know gamma must not pass through P, I know what, so we put the regularity condition, regularity condition last time we saw that gamma does not pass through the origin and gamma is smooth curve of course. I take this gamma to be smooth, to get the surface, regular smooth surfaces. Now let us make gamma tilde U, let us make one more change, so that instead of gamma, we consider gamma tilde, I make the changes U tilde equal to U, V tilde equal to V by gamma U. So, change of variable.

Then I will get my surface which is nice for U tilde V tilde equal to V tilde gamma tilde U tilde but in this case I have gamma tilde U tilde, this all is norm 1 because of this. Now I will do another reparameterisation. So, I am making things easier to calculate, reparameterisation so that gamma tilde is unit speed. That we know how to do it, we just have to change U tilde to something. So, we assume gamma tilde to be used its speed.

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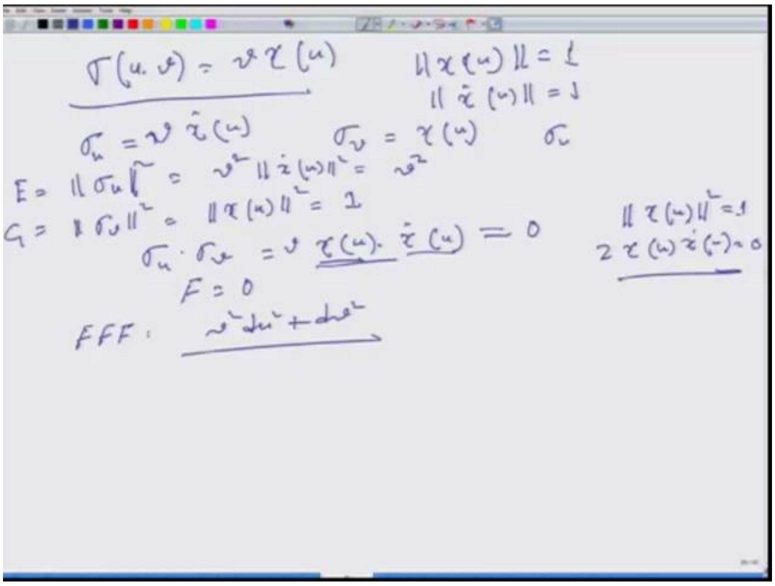

Now what happens to the 1st fundamental form? So, what is gamma U tilde? So, finally let us take the surface to be Sigma U V equal to V gamma U, where gamma U norm 1 gamma dot U norm 1 after reparameterisation I get this one. So, this page what I did, all this thing is reparameterised the cone in such a way that my surface looks like this.

So, now it is very easy, what is Sigma U? Sigma U is just your V remains same gamma dot U and what is Sigma V, Sigma V is gamma U. So, E equal to and okay, what is Sigma U V… Sigma U, okay… Let us calculate, Sigma U square, this is V square norm of gamma dot U square but we have assumed unit speed, so V square. Gamma V square, this is simply gamma U square, so this is again 1. Okay. And Sigma U dot Sigma V, this is V into gamma U dot gamma dot U, this is inner product is 0, why? Because gamma U square is 1, if you differentiate 2 gamma U gamma dot U is 0. So, that gives this, the cos term is 0. So, F is 0, this is G.

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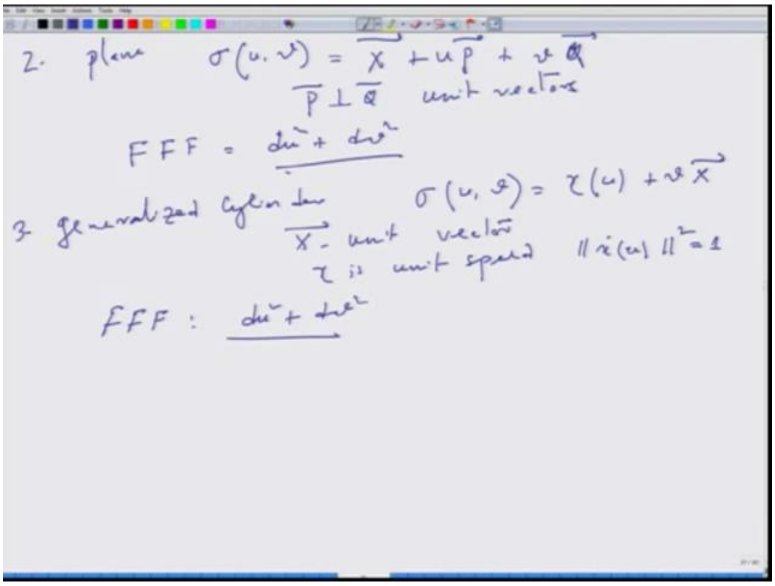

So, the 1st fundamental forms for the cone becomes V square du square + dv square. Let us go to another example, let us take the plane. Where P is perpendicular to Q unit vector we have done this example if you recall the last lectures. Here it is very easy to calculate, you can see FFF will be just, you calculate this Sigma U is just P and Sigma V is Q but both are unit vectors, so norm 1, so it will be du square dv square. Okay. Let us take the generalised cylinder, or the cylinder itself, this is gamma U + V X let us take X as unit vector. And gamma is unit speed, that is gamma dot U square equal to1.

If you calculate FFF here, you will get, this is again du square dv square. So, plane and cylinder has, they both have same fundamental form, it signifies something geometrically.

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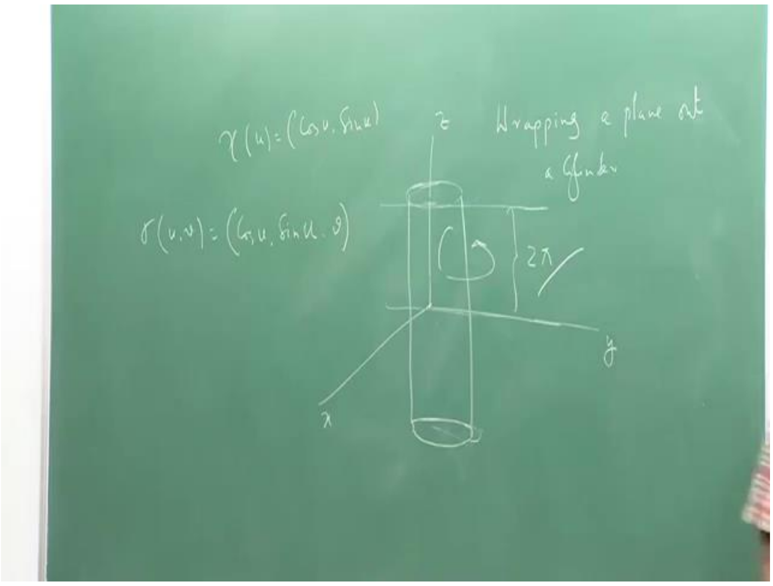

And I will try to recall you that what we did in one of the examples, suppose I take the standard cylinder here, cylinder of radius 1, correct. So, gamma is your curve cos U sin U and the cylinder is cos U sin U V. And you look at the YZ plane. This plane of length 2 pie, then we have done this thing before, that I can wrap the claim into the cylinder. So, recall this wrapping plane onto a cylinder. And this map, this wrapping map, this is very interesting map, this is what happens that if I have a curve was the plane, that will be wrapped and moved to some curve on the plane but their length will be preserved.

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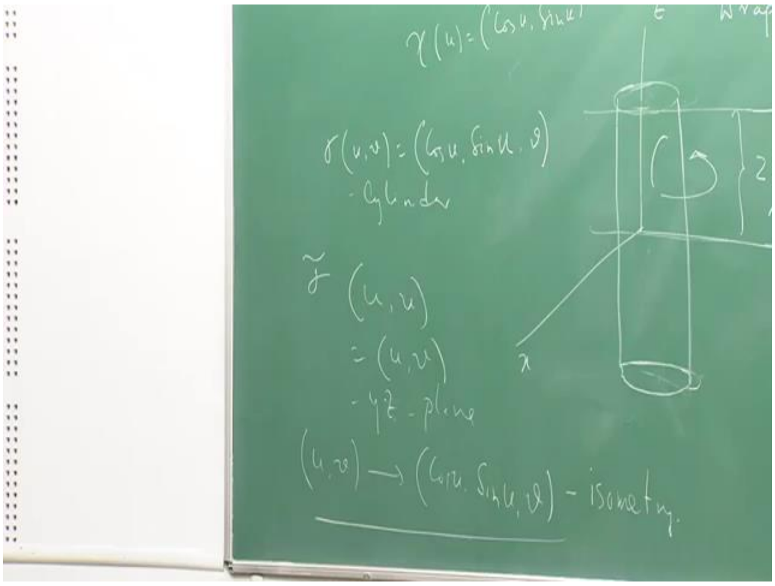

So, what it says that this wrapping map, whatever it is, we have done that before, that is actually a isometry. So, here in this particular case isometry is… this is the cylinder, this is YZ plane, so this is an isometry, what is isometry, that preserve the length of curves. Since this is isometric, we have 1st fundamental form, same, that this was the theorem I wanted to prove, then everything will become clear to you now. Okay.

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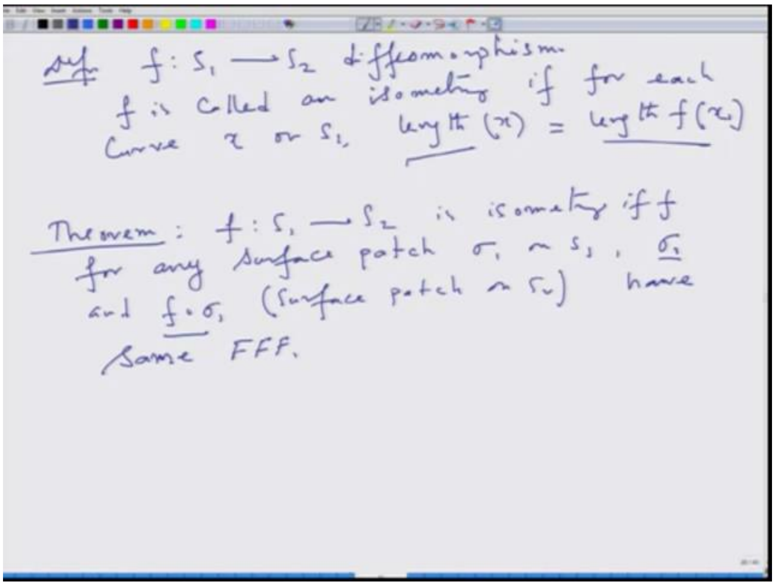

So, let us make a definition. I have 2 surfaces F from S1 to S2, diffeomorphism. In this I am talking about regular smooth surfaces, so I should take diffeomorphism. F is called an isometry if for each curve gamma on S1, length of gamma equal to length of F gamma.

So, F will take this curve to gamma, curve gamma, I have a curve F gamma, they must have, they have same length is called isometry and obviously the theorem we are looking for is, what I was trying to explain through this wrapping the plane, that these 2 in general, that is S1, the F from S1 to S2, this is isometric, if and only if for any surface patch Sigma 1 on S1, Sigma 1 and S compose Sigma 1, this is a surface patch I know. See this is a surface patch, this is the surface patch on S2, so, Sigma 1 and S Sigma 1 have same 1st fundamental form. And that is exactly what is happening in the case of plane in their life cylinder because this is in isometry and they have the same fundamental form. In this case both have single patch. Okay.

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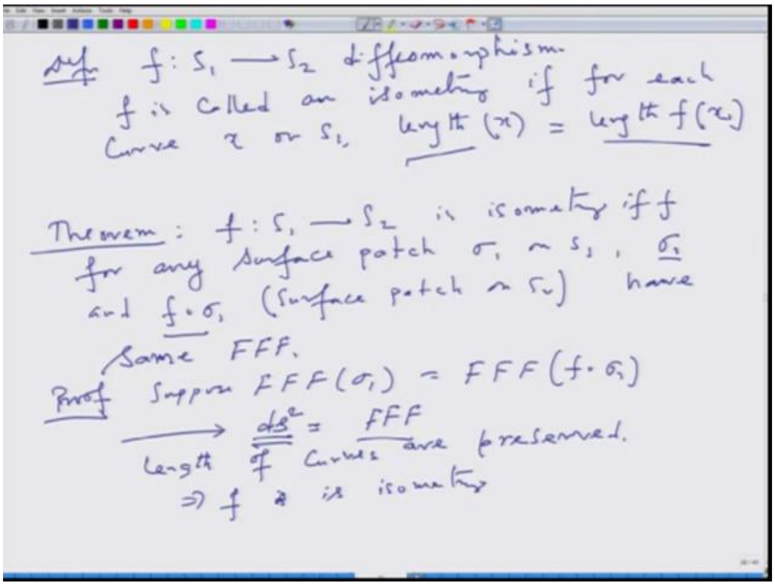

Okay. So, let us quickly try to see the proof. 1st part. Suppose FFF of Sigma 1 is same as FFF of F Sigma 1, then you see immediately that after all if I take a curve, this curve, then ds square is FFF, is given by 1st fundamental form and since fundamental forms are same, this ds, d square element is same, so length of the curve will be preserved. So, length of curves are preserved, hence F is isometric. So, this part is really straightforward. Okay. Now, the other way. The other way is little nontrivial, not really nontrivial, you have to do little bit work.

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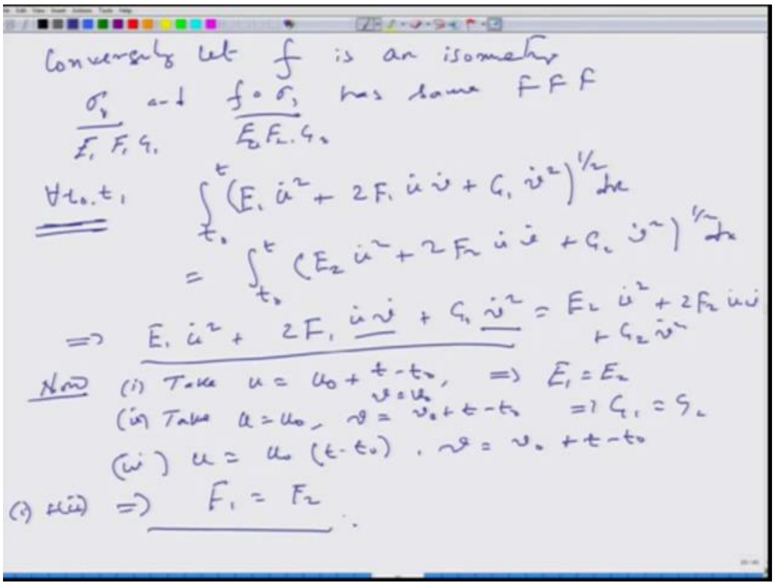

So, conversely let F is an isometric, what I have to show, the surface patches for Sigma 1 and Sigma 2 are same, sorry Sigma 1 and Sigma 2 to show Sigma 1 and F compose Sigma 1 has same FFF. Well, the FFF of Sigma 1, we will denote by U1 S1 G1 here with E2 F2 G2. Since isometry, then what I for all T0 T1 length is preserved… Okay. But this relation is true, this integrate equation is true for all T0 and T1, whatever the curve you take. That will imply these 2 are same. Here I should have treated U tilde, that is okay, U dot square + 2 F dot U dot V dot G dot V dot square.

But what I have to show, have to show U1 equal to U2, F1 equal to F2, G1 equal to G2, this, this does not give 1st fundamental forms are same. Since they are same, now I take special case. Take curve, consider this curve U equal to U0 + T - T0. If you do it, you will see this term will vanish and that will give you U1 is equal to U2. Similarly, oh, V is equal to V0, similarly take U equal to U0 and V equal to V0 starting at some, U0 and V0 are some fixed points, that will give you G1 equal to G2, now put them back or consider this curve, U0 T - T0 and V equal to V0 + T - T0, this curve, since this will this will imply 1 + 2 and along with 3 will give you F1 equal to F2. Okay. So, this is the more or less sketch of the proof, this proof is available in any book, in particular whatever the references I given you can check that. That is it for today, next time we will see another kind of map, conformal maps.