Curves And Surfaces.
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Module-II.
Surfaces-1: Smooth Surfaces.
Lecture-10.
Orientable Surfaces.

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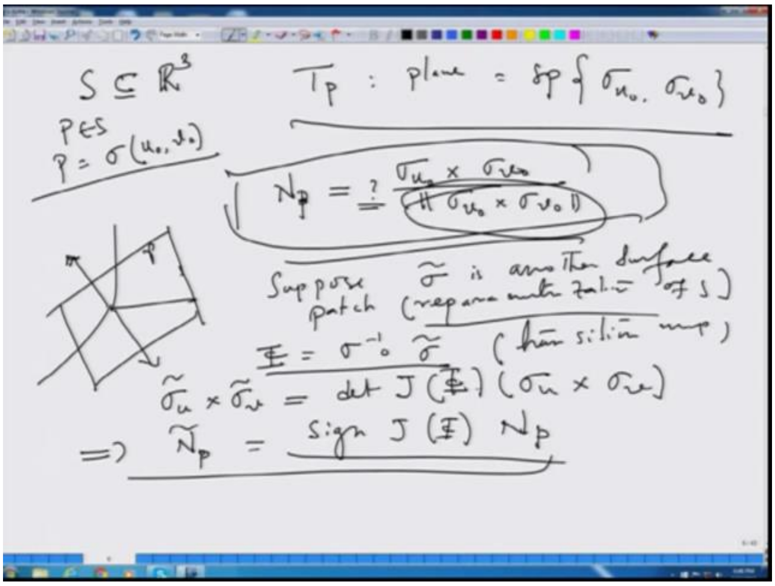

So, in the last lecture we were trying to define unit normal. And we come up with a kind of definition that okay I have the tangent plane TP which is a plane passing through plane which is span of Sigma U nought Sigma V nought P is… And then this fellow is uniquely determined by a unit vector which is perpendicular to it, unit vector perpendicular to this fellow is and which I have to define that to be the unit normal.

The problem is, as I said, given a plane this vector, I can call this one uniquely determines the plane P or this, the opposite one that also determines the plane P. So, here I need a choice of sign. What will happen here? Let us see, why over here I am making fuss about this sign. Suppose Sigma tilde is another surface patch or reparameterisation of S. Then the transition map Phi is Sigma inverse Sigma tilde, this is the transition map. Now, if I look at Sigma U tilde, Sigma V tilde, and we have decided any property of surface has to be invariant at the reparameterisation.

You can easily see that if I convert this thing in terms of Phi map, this will be determinant of the Jacobian of Phi, then this vector Sigma U Sigma V. So, calculate this, so in this case it is, NP tilde, this will be sign of the Jacobian NP because I have to divide by E, its norm. So determinant will get cancelled but sign will remain because sign in the denominator is always positive. Right. So, this NP is not really, this definition is not really invariant under reparameterisation. But I have to define it reparameterisation, yeah I have to define, if at all I have to define unit normal, then I have to define it restrict to parameterisation.

And, I will give an example today itself, that I may not, we may not able to do it, we may be able to it in many surfaces, many good surfaces we are able to do it but there are cases where we cannot do it.

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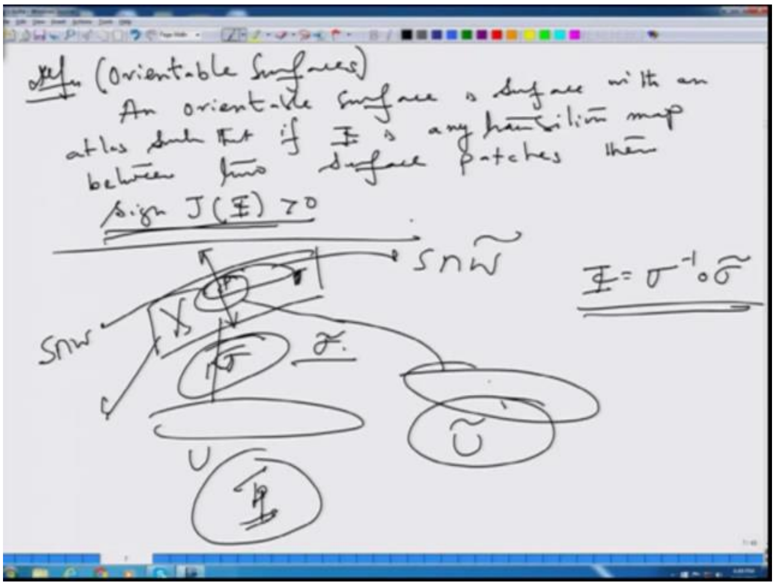

And thus comes the definition of Orientable Surfaces. So, let me write the definition. Surface with an Atlas, that means collection of surface patches, such that if Phi is a transition map, Phi is any transition map between 2 surface patches, then sign of the Jacobian of Phi is greater than 0. What it says, that I will have patches but whenever I do reparameterisation or there are surface, intersections of surface patches, the unit normal is, so uniquely determine, that sign of J Phi is uniquely determined.

So, consider this surface again. So, here is my U and suppose I have another U tilde, here is my point P. So, Sigma covers this part, Sigma tilde covers this part, okay. P is a point which is in between, so Phi is the transition map. Now I can define transition plane TP with respect to this Sigma as well as Sigma tilde. I know TP is invariant under reparameterisation. If I take Sigma or Sigma tilde, TP remains invariant. But should not there be case that when I take Sigma, then your choice of unit normal is this way and when I take Sigma tilde, your choice of unit normal is this way, this should not be the case, okay.

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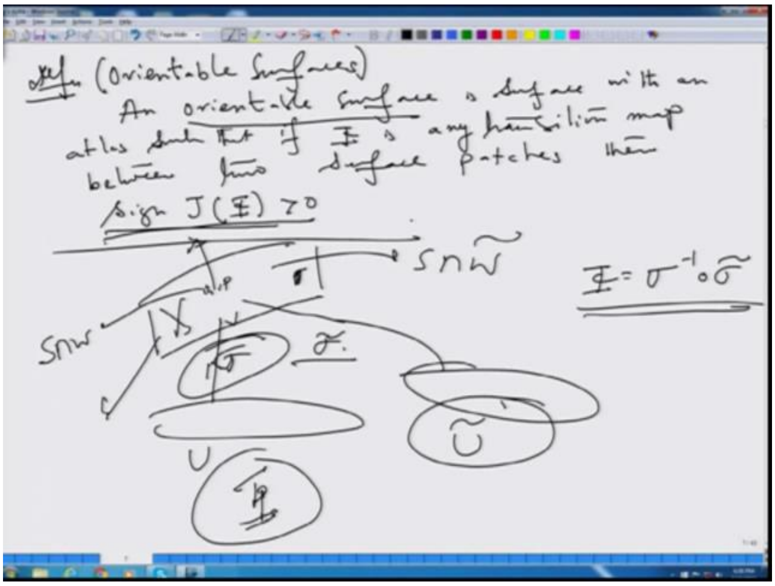

I must have the sign of the Jacobian or transition map Phi, so that if I take another surface patch and if I have intersection, then always the choice of unit normal is the same. So, that is called an Orientable Surface. I will give a very simple example and this example serves as a prototype of most of the surfaces because we know most of the surfaces can be realised as level surfaces.

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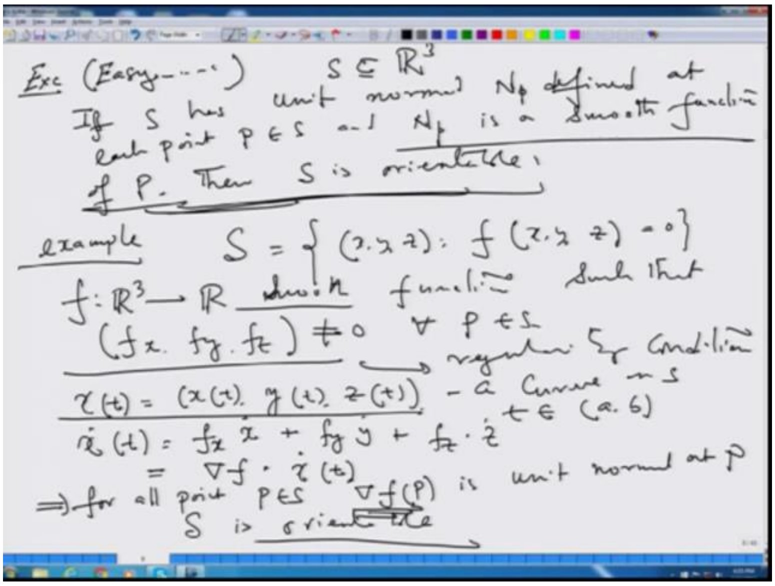

So, here is an example, before the example I put an exercise which is very easy, very very easy, but sounds difficult. Which says basically, if, so I have a surface S in R3, if S has unit normal NP defined at each point P in S and NP is a smooth function of P, then S is orientable.

Why so, because idea is that, if I have a smooth function, after all NP depends on sign, + or -. Right. But if it is a smooth function, it cannot just change the sign abruptly from one point to another. If it depends smoothly on P, the idea of the proof, idea of the exercise is that, it depends something is smooth, it cannot change the sign abruptly. From P to Q, if P and Q are close together, you cannot change sign. Then it will not be smooth in that case.

The smooth function is after all, you know, they are noise continuous functions. Okay, with that I can give the example which is, as I said a prototype. Consider a level surface, such that this condition I mean, this is for regularity, regularity condition. Okay. Let us take a curve gamma T on the surface. T belong to some interval A, B. What is gamma dot T? Very easy, FX X dot FY Y dot FZ Z dot, remember by the combination, dot always denotes the derivative with respect to T. So, this is Grad P Grad F dot gamma dot T. Okay. So, for all points, so for all points I consider such a curve passing through it.

This fellow is unit normal at P. And by the exercise, since F is a smooth function, I depends smoothly on P, so S is orientable. I think I wrote something wrong here.

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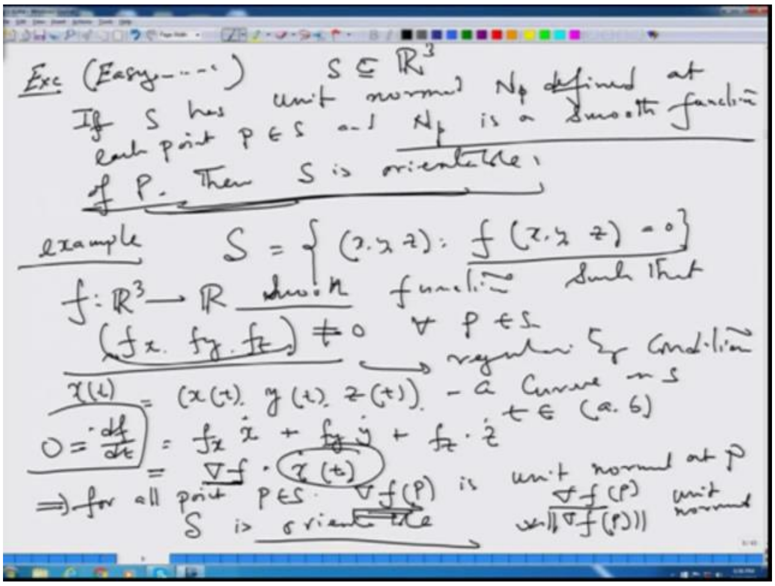

This is the curve gamma T and I take this is df dt which is 0 because it is a level surface. Right, this part is 0, df dt is 0 because it is a level surface but this is this thing, so this is Grad F gamma dot T, this will give the tangent plane. So, gamma F is actually Grad F divided by norm of grad F at P, this is unit normal. And being a smooth function, this is smooth function, I have A already assumed that this does not vanish, so this fellow is okay, denominator is never 0.

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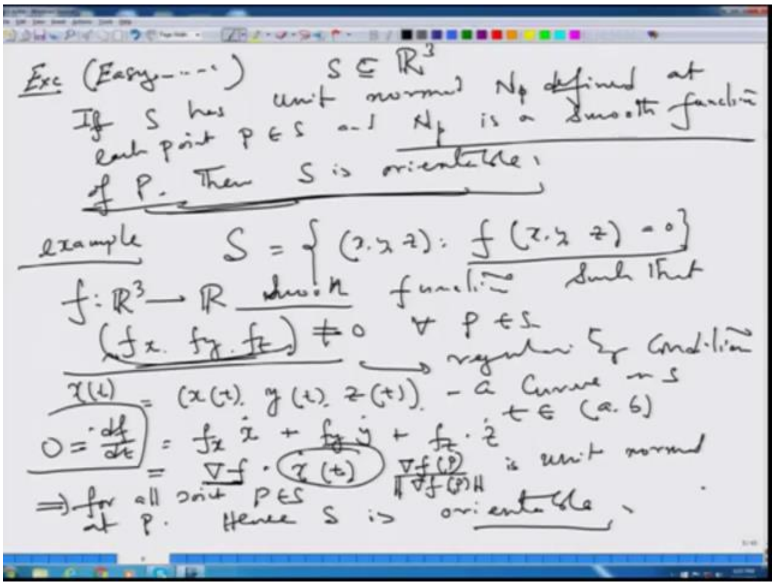

So, for all point P, not grad F at P, so let me write clearly here. Grad F at P by its norm, so this is a smooth function is unit normal at P, hence S is orientable. So, level surfaces for smooth functions, they are its orientable, provided it is regular, regularity condition is needed because I have to divide by the norm. And so I must have, this is non-0. And that was a condition, actually when I mentioned about this regularity condition that regulatory means, there exists a well-defined normal to each point. So, now you understand why this regularity condition is needed. Okay.

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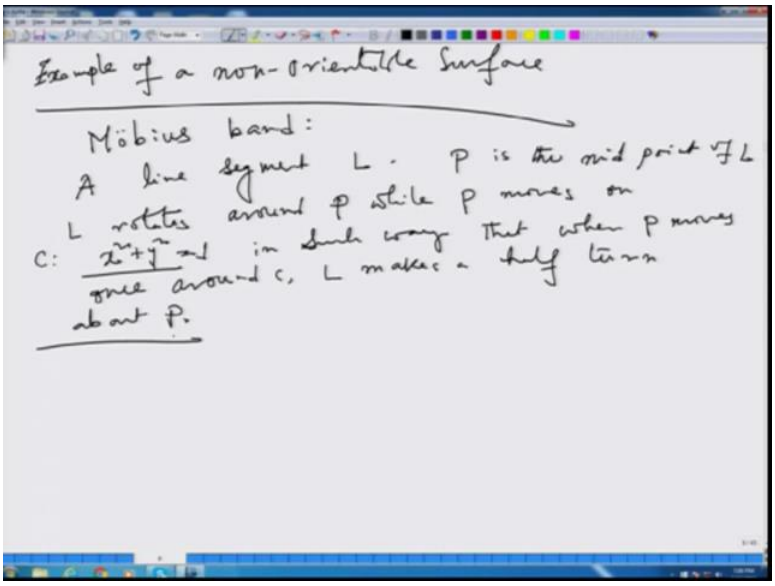

So, let us take an example, very famous example of a non-orientable surface. And perhaps you have done it, I mean you have seen this example in various form in your schooldays. It is the called Mobius band. I wanted to show you a picture of Mobius band because drawing Mobius band is difficult on board or this sliding pad but perhaps I will try to post it some link where you will see in the Internet how the Mobius band is generated. The idea of generating Mobius band is that, that suppose there is a line segment L and P is its midpoint. Now, L rotates, I take any circle, let us take the unit circle, L rotates around P, it rotates around P while P moves on the unit circle all unit circle for that matter.

In such a way, there is condition that when P moves once around C, L makes a half turn about P. So, suppose this is a line segment, just take unit length, it is on the x-axis, this is the middle point, so P is moving on the circle and at the same time this is moving that in such a way that when the P completes it, it makes the half turn around P. This is the full turn, right, I start like this, P moves around this, this is moving also and as P comes here at the initial point, this makes a half turn, it gets upside down. this is, I do not do how much is clear but you can search in the Internet, I will post a link as I said how the Mobius band is generated, there are pictures, I mean three-dimensional pictures are available in many video lectures were movie is bad is shown to how it is generated.

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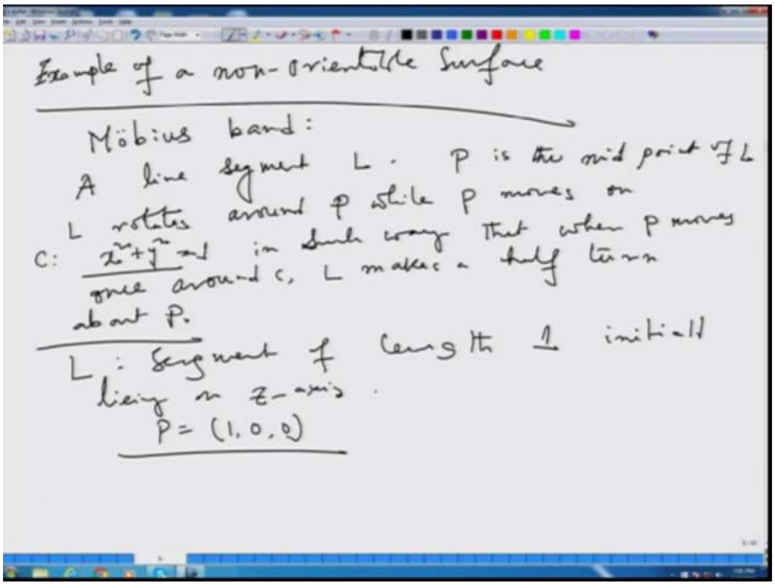

So, I take L to be line segment of length 1, does not matter, which is initially lying on Z axis. And the midpoint P is the point let us say 1, 0, 0, okay.

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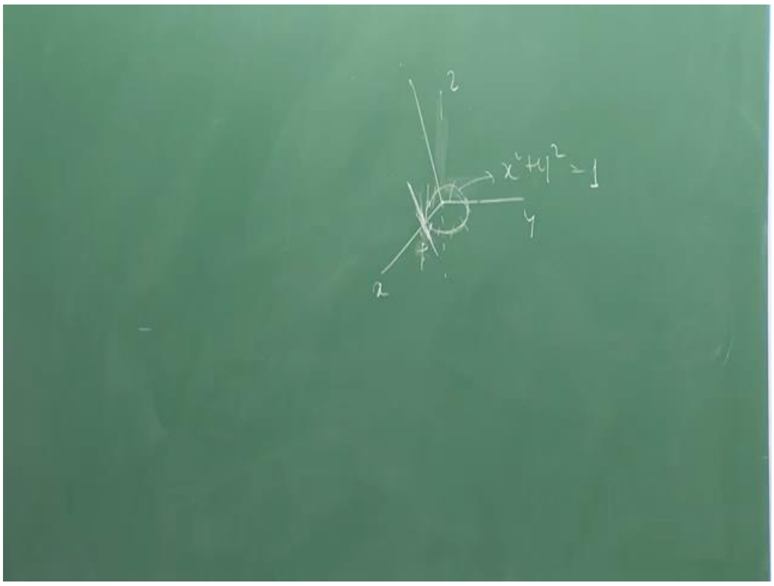

So, here is my P is 1, 0, 0, this is the point P, here is a line segment. And it is moving around this circle, the circle is a, the circle is in the XY plane, okay. Think it is like this, this is a circle on the XY plane, this is the Z axis, and P is lying alone parallel to Z axis, this is the point. Now as P moves, this fellow starts moving as well, so it moves like this and when P come back, it is again at the same position but upside down.

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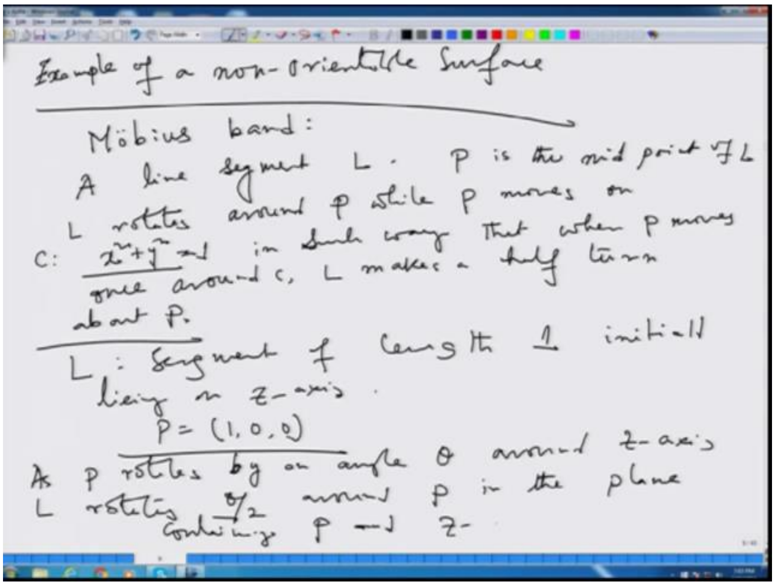

It takes little bit of time, so what happens, so as P rotates by an angle theta around Z axis, L rotates theta by 2 around P in the plane containing P and Z axis. P is moving, so the plane is also changing.

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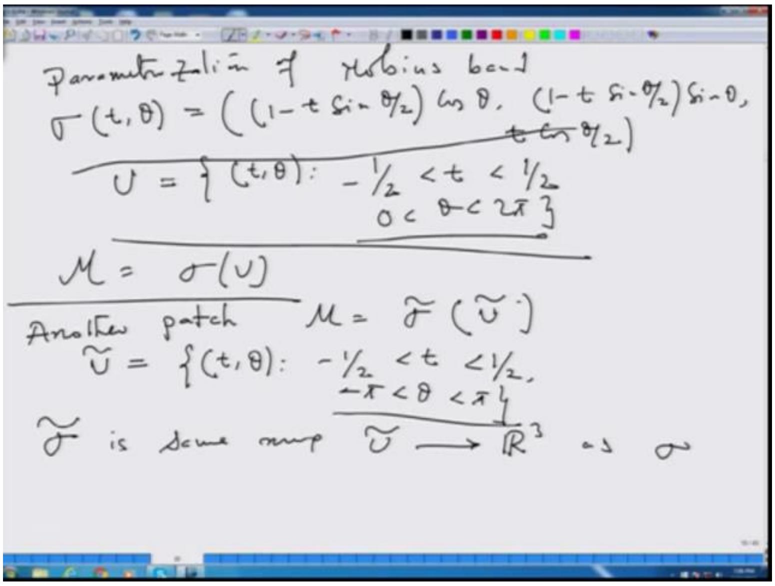

So, it takes little bit of thought that a parameterisation of Mobius band is where theta U belongs to… okay. So, Mobius band M is Sigma U. I can come up with another patch of Mobius band on the same U no, let us say U tilde, where U tilde is T theta, does not matter, right, if I cover it Mobius band by 0 to 2 pie, I also cover it for - pie to pie and Sigma tilde is the same map.

So, formula for Sigma tilde is same, Sigma tilde is also given by this map, so this is also covered. Okay.

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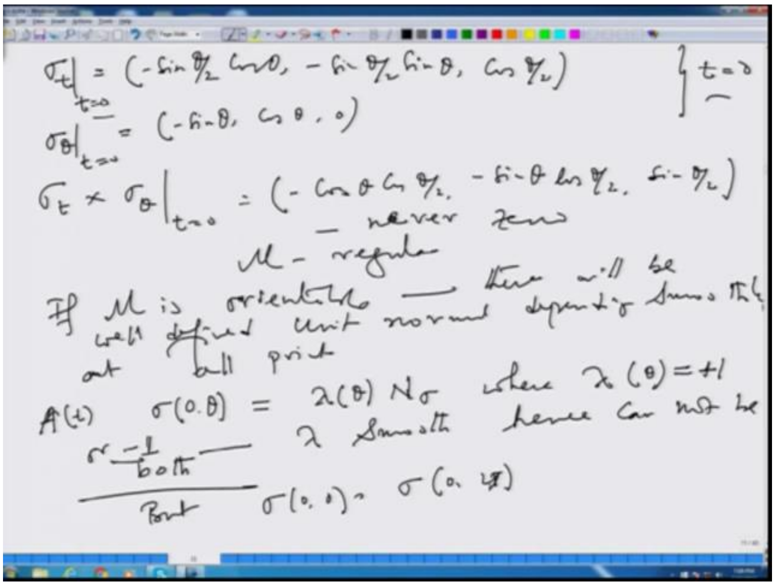

Why is this fellow not orientable? Okay, you just calculate, Sigma T, we will calculate at T equal to 0. So, let us see at T equal to 0 can I define unit normal? T equal to 0 Sigma T, just check P partly go away, there is a reason I am taking T equal to 2 and Sigma theta will be… So you calculate Sigma T Sigma theta, just do it. So, this is never 0 because cos and sin can never be 0 together, so M is regular. If M is orientable, then what will happen?

There will be unit normal, well-defined unit normal depending smoothly on the point on at all point. Okay. So, at in particular, if I look at F Sigma 0 theta because T equal to 0 I am taking here, Sigma 0 theta, this will be some lambda theta N Sigma where lambda theta equal to either +1 or -1 and lambda has to be smooth function, hence cannot be both. Right. Cannot be -1 sometimes -1 Sometimes +1, it is understood? But you see Sigma 0, 0 is Sigma 0, 2 pie as well.

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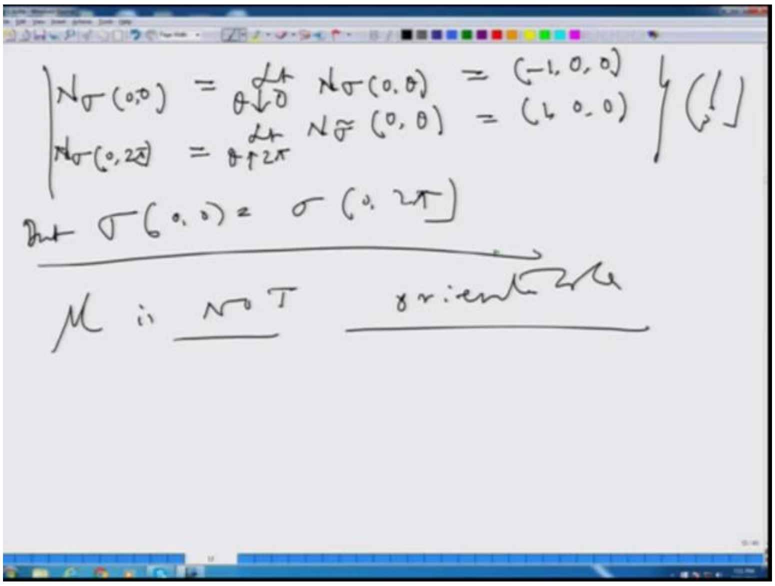

Now what you do, you look at normal F Sigma Sigma F 0, 0, if it depends smoothly, then if I come to theta, this should be, but that, if I calculate the 1st surface patch, this will be this, whereas it has to be also equal to 0 to 2 pie but now I calculate theta increasing to 2 pie along the other patch and you can see this will be this thing.

So, this contradicts, right, they do not match, so this, they do not match, so M is not orientable. So, this is an example of a surface which is not orientable, so next week we will be dealing with more geometric properties of surfaces, orientable surfaces but 1st I want to do some typical examples of surfaces how they are generated. One we have seen level surface, another we have seen sphere, next time I will do some more examples and then we will proceed to geometric. Thank you.