Curves And Surfaces.

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Module-I.

Curves In R^2 And R^3.

Lecture-01.

Level Curves And Locus, Definition Of Parametric Curves, Tangent, Arc Length, Arc Length Parameterization.

Welcome to the 1st lecture of this course which is curves and surfaces. So, this lecture as I said in my introduction, it is a kind of a follow-up of what I did in the 1st course on differential calculus of several variables. So, we will be using results from there and our viewpoint about here will be more geometric. I will not be working out everything very minutely and leave things to the assignments. Whomsoever attending the course, it is very important that you solve the assignment problems, you join the discussion forum, you have any problem, anything, you clear it out in the discussion forum and then go for the next module. Your each module depends on the previous one.

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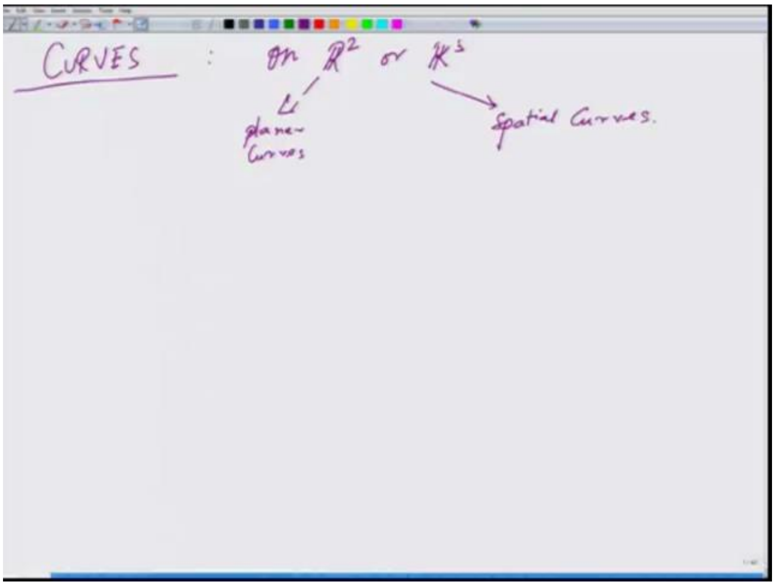

So, as I post it, the 1st module will be on curves. And we will be more or less mostly concerned on curves on R2 or R3. A curve on R2, this will be referred as planar curves while this R curve on space, for obvious reasons it is referred as spatial curve. Now all of us have some idea what curve means. Let me recall some example. While you are in school, then you are often, in coordinate geometry, you are often asked to find locus, locus of a point.

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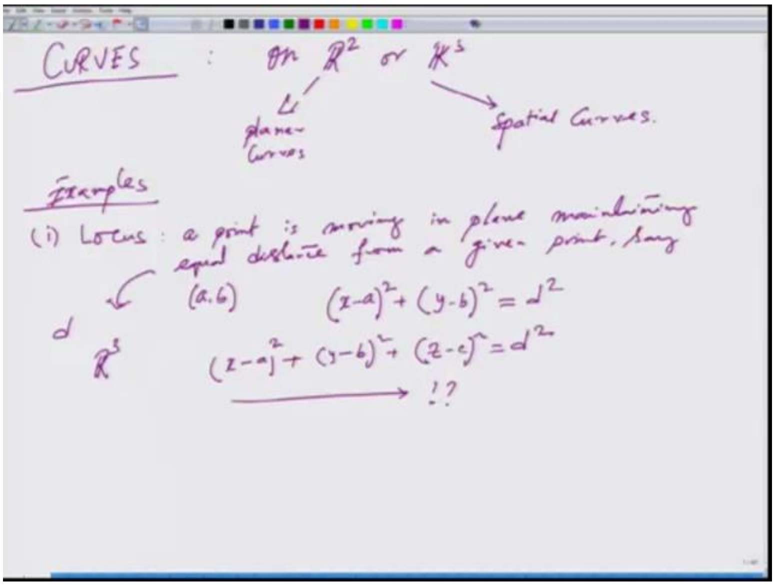

Let us see a very simple example, a point is moving in plane maintaining equal distance from a given point say a, b and the equal distance is, distance is d. And all of you know what is the locus of this point, locus of the point will be a circle with Centre a, b and radius d. Now, often you say that, okay, may not be the entire circle, the point may not move the entire circle, it is a part of the circle. So locus is a part of the circle. So, this is the first time you see in coordinate geometry examples of curves.

Similarly if you do it in R3 and let us say a point maintaining, moving and maintaining equal distance from a point a, b, c, what this will be?

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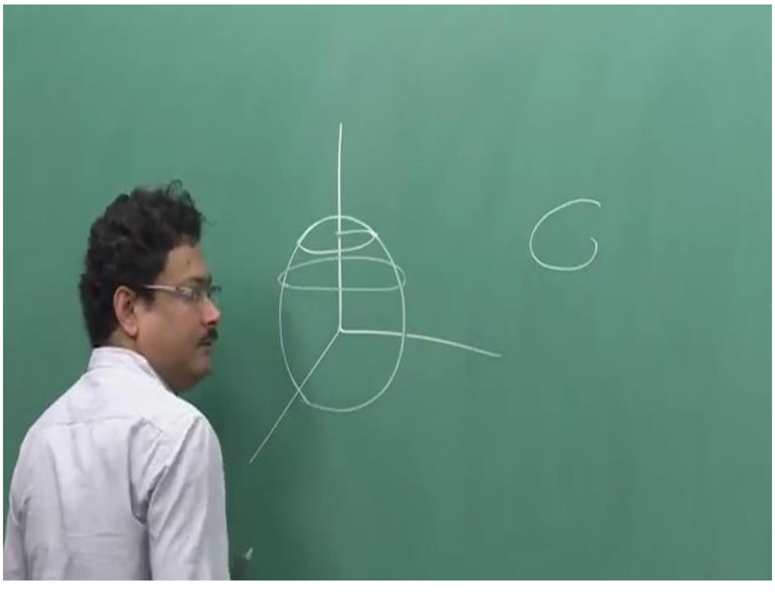

Well, this will be a part of a sphere which is obviously not curve. But suppose I am moving, so let us say here I have a sphere, so point is moving let us say keeping equal distance from the origin. And it can move on this circle and suddenly it will decide to change its course on the sphere and still move like this. So, finally if I look at in R3, I will get a curve like this. So there is a difference between geometry in R2 and R3, because R3 we have a extra dimension to move up in. So, this is the 1st example of locus.

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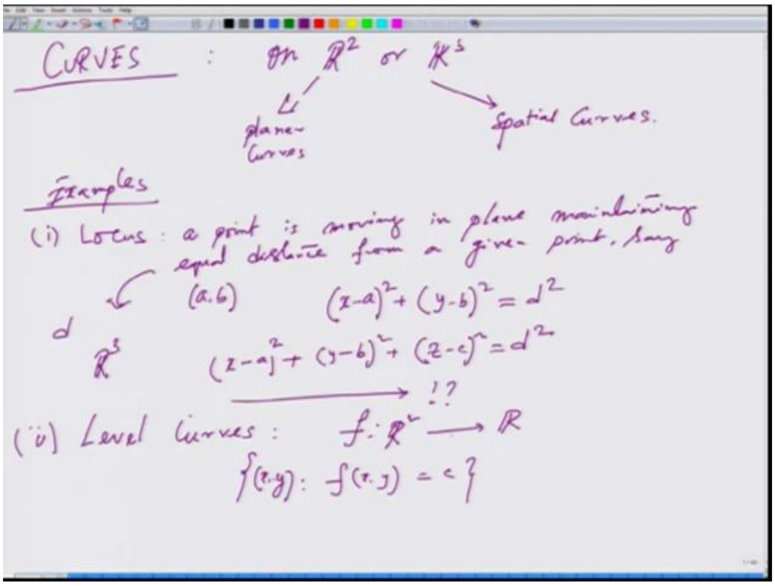

Next example you see in your calculus course, those are called level curves. That is I have a very nice function f from R2 to R and you look at all the spoils x, y, such that f of x, y is equal to some given c. So, these are called level curves.

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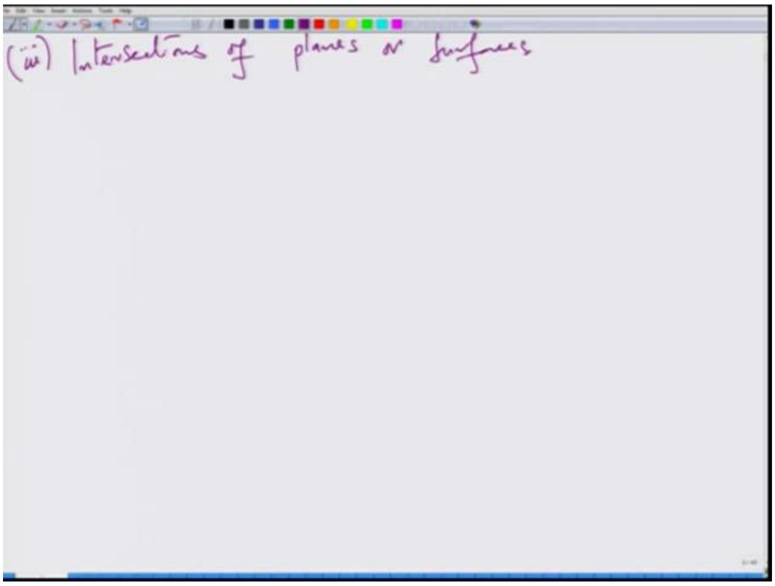

And more examples, you see intersections of planes or surfaces. That also gives rise to curves like this,

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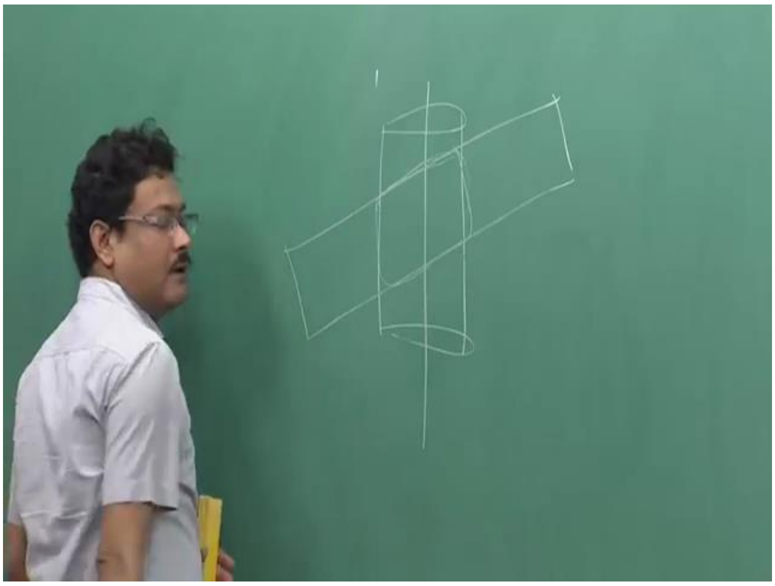

That suppose I have a cylinder here, out here and I have a plane cutting through it. The portion that belongs both to the plane and the cylinder, that gives rise to a curve in R3. So, you can imagine, a cylinder is there, a plane is cutting, the portion that belongs to the surface, outer surface of the cylinder as well as the also lying on the plane, that gives you a curve in R3.

Okay, now when we discuss curves, we have to give, we have to decide it mathematically, so I will consider curves in parametric form.

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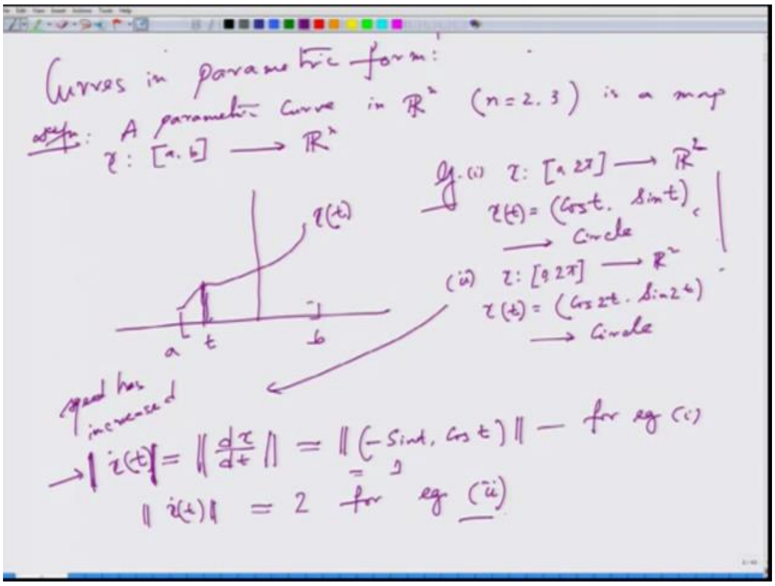

What does it mean? So, here is the definition. Parametric curve is simply, okay in Rn and for us n is either 2 or 3 is a map gamma from an interval a b to let Rn. So, this interval could be sometimes open and sometimes both closed. So, suppose let us take R2, here is my a b and I have a function which is defined from a to b, gamma T, so T is here, each point I point, I draw the graph and I get a curve gamma T.

Example, let us say gamma, interval is 0 to 2 pie to R2 gamma T equal to cosine of T and sine of T and that gives you a circle. Let us say example 1. Next I consider another function, on the same interval perhaps. Gamma T is, okay, this time I consider cos 2T, sin 2T. I will again get what? Think about it, you will again get a circle. But there is a difference between the 1st example and the 2nd example, what is that? Here, the speed of the curves has increased, what is speed? So, here we say that speed has increased.

In fact it is doubled. And what is speed? You know from your basic physics the speed is gamma dot T and its absolute value. So, this is absolute value of d gamma d T, which is in this case is absolute value, absolute value I mean in R2, absolute value is replaced by Norm. If I do it for this one, for example 1, this is - sign T, cos T for example 1, which is equal to 1. And gamma dot T for, is equal to you calculate, this will be 2 for example 2, because a 2 factor will come up. So, you see that this represents the same curve but the point, if you think like a point, if the point is moving on a plane, the point is moving at a speed trouble than the example 1.

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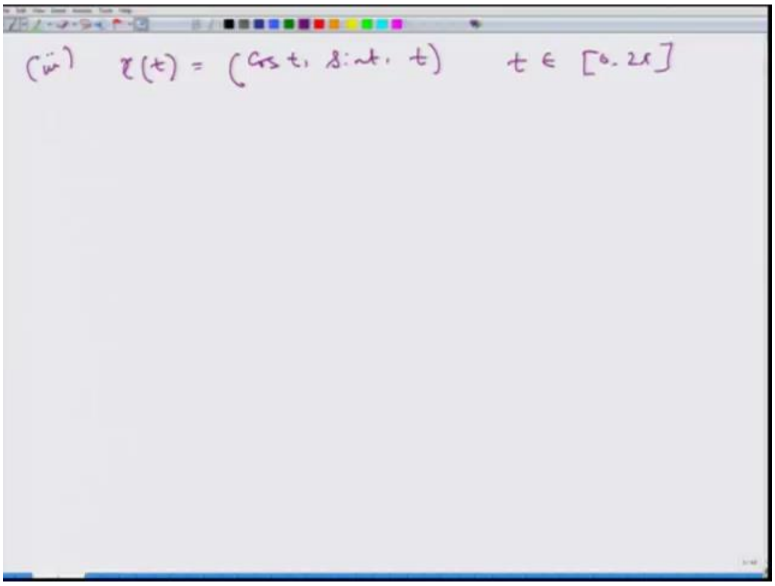

Another example, you can consider a spatial curve, gamma T may be on the same interval, let us say cos T, sin T T T in 0 to pie. How does it move? It will move so on the planar part it is a circle and it is increasing with T so that it will be a Helix. It will go like…

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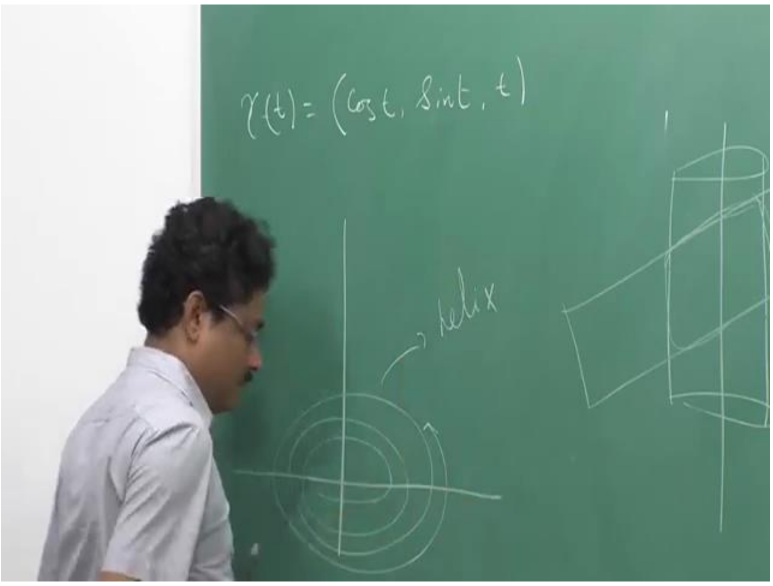

Very nice. Now, for some reason we cannot study all curves.

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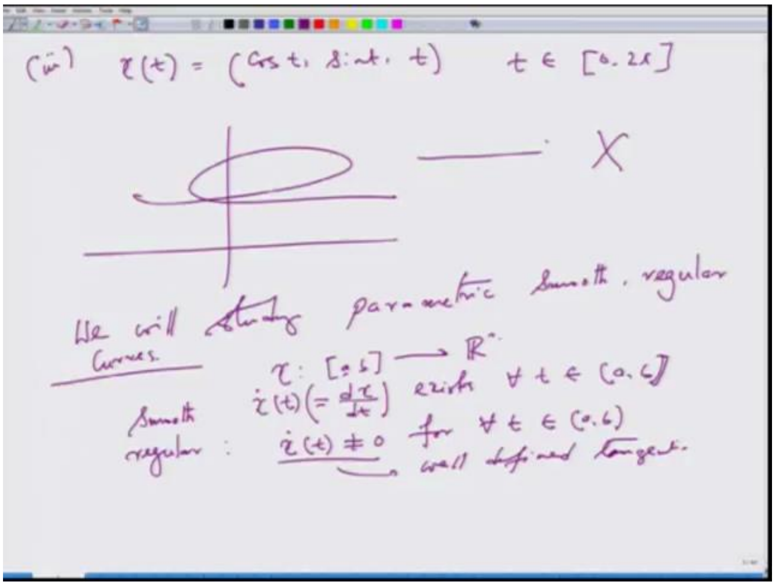

For instance, a curve in R2, suppose which is self intersecting, then studying this kind of curve needs special kind of technique or special kind of tools which are beyond the scope of this particular course. So, we will not study these kinds of curves, we will study parametric, smooth, regular curves. What does it mean? Okay, curve for us is a map from a b to Rn, this is a parametric curve, smooth means gamma dot T, gamma dot T throughout in this course denotes gamma d gamma d T exists for all T in a b. And regular means gamma dot T is not equal to 0 for all T in a b. That is derivative is non-vanishing, that is I have a well defined tangent. Okay.

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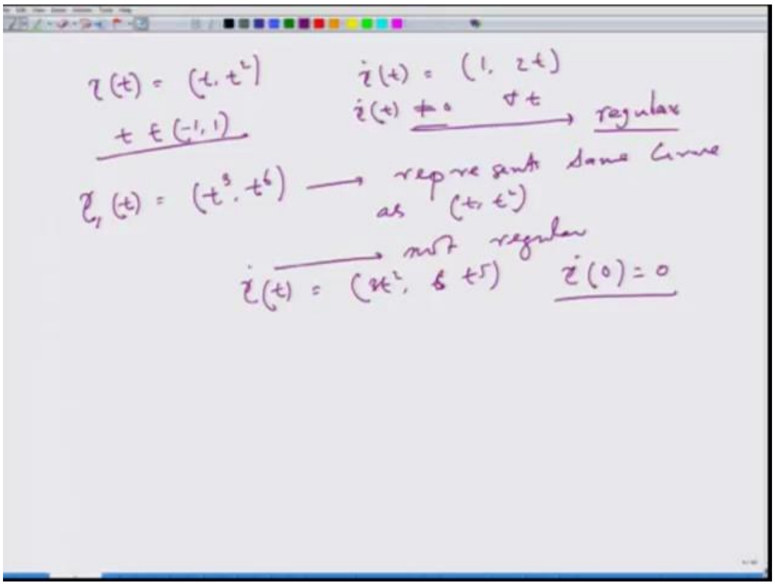

For example, if you look at this curve gamma T is equal to T T square. What is gamma dot T, this is 12T, so gamma dot T is not equal to 0 for all T. Let us say T in some interval -1 to 1. Now, if you look at this curve which is say let us say T cube T 6. This represents same curve as T T square, speed has been tripled but this is not regular. Why? Because gamma dot T is 3T square 6 T 5, so gamma dot 0 is 0, while T is in -1 to 1. Whereas this parameterisation is regular.

So, this curve, same curve can have parameterisation which is regular and nonregular. So, we should be concerned about which curve should have regular parameterisation and which does not have a regular parametric duration. That is one of our main point in the 1st lecture. But to state that as theorem or result, we will 1st talk about self invariance of a curve which is known as arc length.

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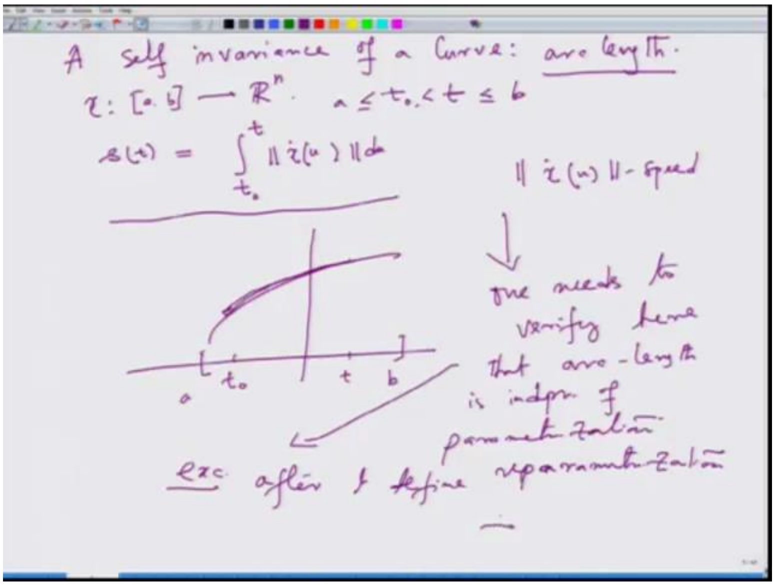

So, I have a curve gamma, let us say from a b to Rn, remember n is always 2 or 3. I have 2 points T nought, let us less than T, arc length is T is defined to be T nought to T, this way. What does it mean? Well, here is my curve, here is T nought and this is T. So, curve is like this. This is the point gamma T nought. What is the speed at this point gamma T nought? Gamma T is gamma dot T nought. And at this point it is gamma dot T. And looking at the absolute value of… Okay, I should, I am using both T, so maybe I should use gamma u d u, so this is the speed. And on integrating the speed from T nought T, that gives me the length.

That is from time T nought to T, if I travel with the speed of gamma dot u or Norm, how much I travel, this is given by X T and this is called the arc length. Now you see I can have different parameterisation of the same curve as we have seen. So, one needs to verify here that this quantity I am defining arc length is independent of parameterisation. Okay. That is easy to check actually. And I will put that as an exercise after I define reparameterisation, that in SN curve what does it mean by reparameterisation? So, what is reparameterisation of a smooth curve, because we are dealing with smooth curves only? Okay. Here is the definition.

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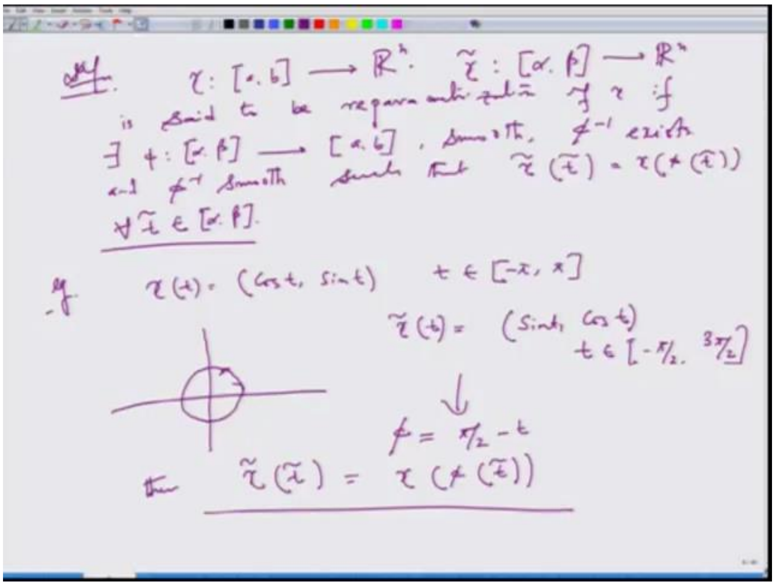

Suppose I have a curve gamma from an interval a b to Rn, then another curve gamma tilde, actually it is the same curve gamma tilde, from some other interval alpha, beta to Rn is said to be reparameterisation of gamma is there exists a phi from Alpha beta to a b which is smooth, phi inverse exist and phi is also smooth, such that gamma tilde T tilde equal to gamma phi T tilde for all T tilde belongs to…

So what I have done, let us see the example of the circle. I have gamma T equal to cos T sin T. T in let us say - pie to pie, that will give us a circle. Correct. And I look at the curve gamma T tilde equal to let say sin T cos T. And this time T in - pie by 2 to 3 pie by 2. If we immediately verify that if I take phi equal to pie by 2 - T then gamma tilde T tilde equal to gamma phi T tilde. This represents the same curve but the parameterisations are different, these are 2 different functions, right and defined on defined intervals.

So, to end the lecture and to motivate you into the course, I will write down some assignment problems, I mean it will be there in the assignment also, but you must solve them because unless you do not, you will not be able to follow the next lecture.

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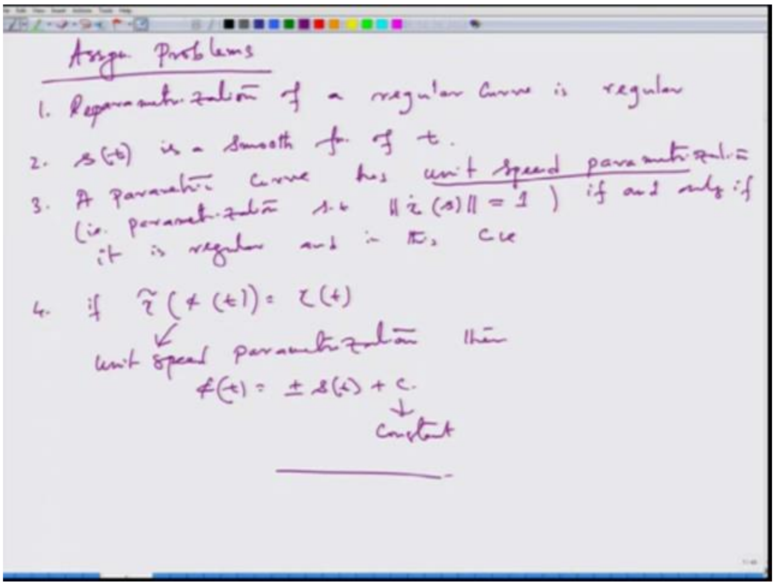

So, assignment is a part of this lecture, I want to emphasise this again and again. So, I assignment problem, I will just make some simple observations, that reparameterisation of a regular curve, we have defined regular curve, is always regular.

Next arc length is a smooth function of T. Third, a parameter curve has unit speed parameterisation, that is a parameterisation, that is parameterisation such that gamma dot S, the speed is always 1 if and only if it is regular and in this case, this is our fourth problem. If gamma tilde phi T equal to gamma T, this is unit speed parameterisation, then phi T equal to + - ST + some constant. This gives the unit speed parameterisation, though, sometimes the unit speed parameterisation is also referred as arc length parameterisation. That we travel according to arc length, then it is unit speed. We will continue in the next lecture.