Differential Calculus of Several Variables

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Module 02
Lecture No 09
Mean Value Theorem.

Okay, so today we will discuss the possible version of Mean Value theorem and application, there are many application, we will discuss one application, I will mention about it in my lecture of course.

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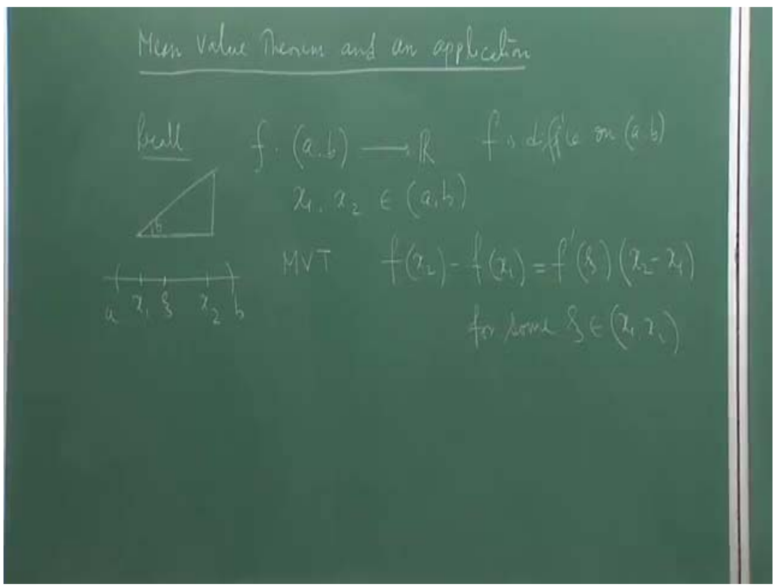

You see let us recall what the Mean Value theorem for function of one variable says, suppose I have F from interval A B to R, and F is differentiable on A B, entire A B, right. Now if I have take any two points X 1 and X 2 in A B then the mean value theorem says that the difference F X 2 minus F X 1 this is given by so value difference between this 2 is given by F prime Xi into X 2 minus X 1 for some Xi between X 1 and X 2, some Xi.

It just says, what is important here, you can write it in this form but this theorem doesn't give what is Xi, it just gives that this will be given by F prime Xi, or in the other way to say that F X 2 minus F X 1 divided by X 2 minus X 1, write. F X 1, F X 2, so this is the difference between F X 1 and F X 2, so this ratio is tan of this angle tan theta, so this ratio which is tan theta of this angel, that is given by, if I divide by this thing that is given my gradient at some point. Actually it is a (())(2:47) or Rolle's Theorem.

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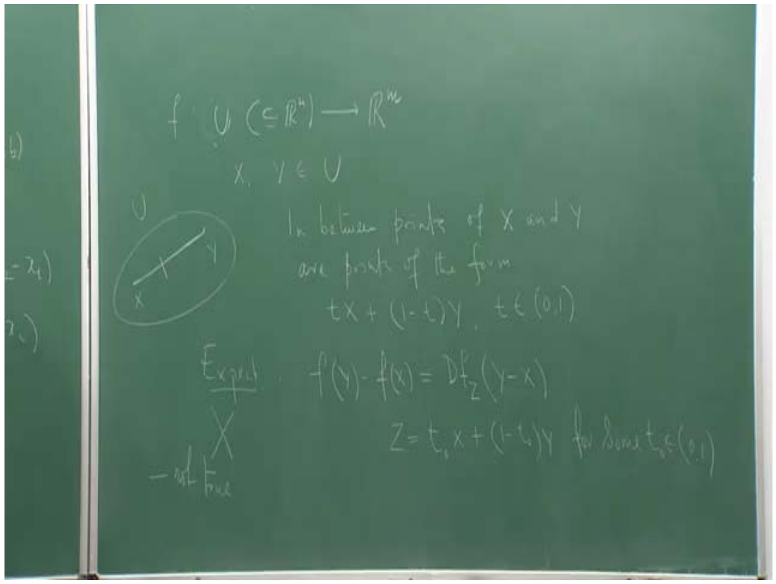

Now to generalize it from a function from say U in RN to RM first thing we encountered that I must and suppose X and Y are two points in U that is your U. X and Y two points, now to make sense of Xi, you see in between X 1 and X 2 in the interval, to make sense of a between point X and Y we need some notion of that, what is in between two points in an open set in RN.

Well so most natural is that you joint X and Y and you get a line over here, if the line entirely belongs to you which may not, we will see example, and Xi is some point here, JD some point here, so in between points of X 1 and X 2 are points of the form some all the points in the line that is T X plus 1 minus T Y T in 01. You see if you start at 0 you are at Y and end up at 1 you are at X and in between all these points can be expressed, by TX plus 1 minus TY for some T in 01.

Okay, so we should expect that a statement like this holds there MVT should read like F Y minus FX equal to DF at some Z Y minus X where Z is for some T not in 0 to 1. That will be the exact analog of this statement. This is what you should expect for MVT in several variables. Unfortunately this does not happen, this is not true. Why, let me show you an example.

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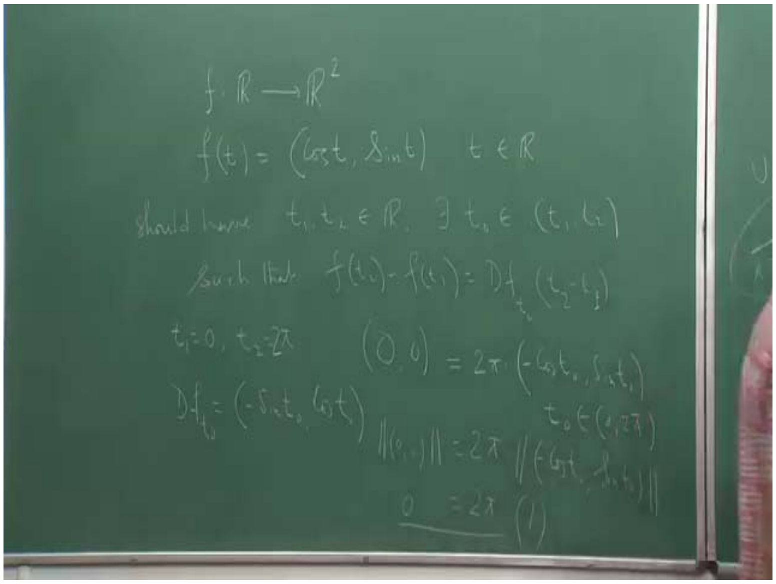

Okay, I take F from the entire R to R2 and I take a very simple function cos T, sin T, T in R. If such a statement is true, if what we expect is true then what it should be, should have give T1 T2 in R, there exists T not in between T1 and T2, such that F T2 minus F T1 equal to F prime at T not, F is from R to R2, so this is a 2 plus 1 vector, right, or should I write DF according to our notation, DF T not acting at T2 minus T1.

Okay, this should be true. Let us check, let us take T 1 equal to 0, T2 equal to pie, what is a T2, cos pie 1, sin pie 0, what is F T1 cos 01, sin 00, so left hand side is 0, equal to 0 is 00 of R2. What is DF at T not, this is very easy, minus sin T not cos T not, right. And T2 minus T1 is 2 pie minus 0, sorry not pie 2 pie, cos 2 pie is 1 and 2 pie minus 0 is 2 pie, so this will be equal to 2 pie into this vector cost T not sin T not. So T not is in between 0 to 2 pie some number if I follow this.

Now then there is an immediate problem, what, let us take norm on both sides, this is a norm of 0 vector which is 0 and here is 2 pie into norm of this vector is cos square T not plus sin square T not 1 root over that is 1. So I will get this, so which is a straight forwards contradiction. So this statement does not hold. To make it hold what we need that MVT Actually depends how do you look the function and when a look at the function the graph of the function or in R3 the region of the function where it is define, you look it from a direction and once you fix a direction that is going to be true.

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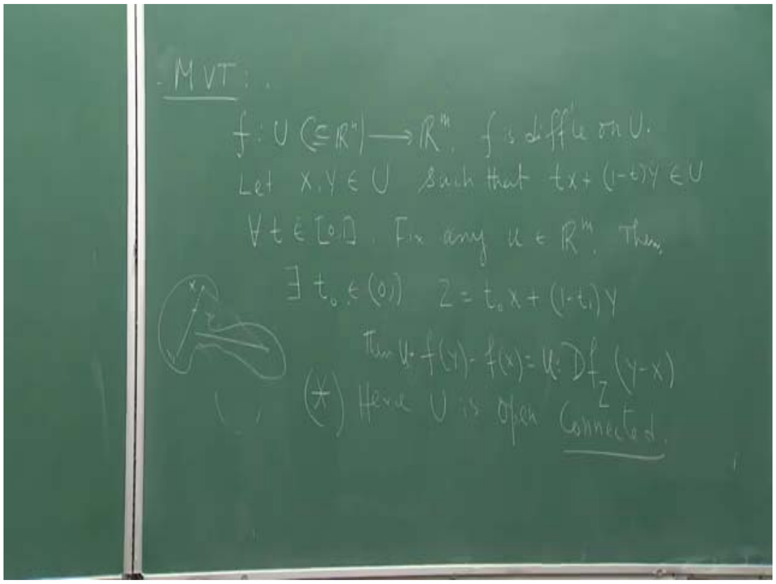

So what I mean here is the statement of MVT in functions of several variables, you have to fix a direction. So here goes a statement of MVT for function of several variables. F function from U in RN to RM, F is differentiable on the entire U and let X and Y be two points in U such that the line joining X and Y that is T X plus 1 minus T Y belongs to U for all T in 0 to 1, okay. So situation is like this U may be element like this or maybe a open connected state like this, I have X here, and Y here, not here. But line joining X and Y.

So I should have X and Y like this or X and Y like this, I cannot take X and Y like this because the line joining X and Y is not in U, so I have to take X and Y such a way that the entire line joining X and Y is in U, so fix any U in U in RM, you can choose arbitrary RM, so this is kind of fixing a direction, fix any U but once you fix, fix for the statement of the theorem, okay. Fix any U in RM.

Then there exist a T not, so something in between the line joining X and Y, such that if I take Z equal to T not X plus 1 minus T not Y then FY minus FX equal to, okay I have to fix look at the direction, okay let me write it later what is the direction means. FZ at Y acting at Y minus X which is not true so you have to look at from the direction of U left hand side and right hand side, this is the correct statement for MVT for function of several variables.

And the proof is extremely easy, how. So X and Y is in U and, and we have always taken U remember for the beginning of this course U open connected, many statements are true whatever we have proved just for open but here I emphasize I need connectedness so maybe for that in the statement I include this part that here U is open, very good, open I need for differentiability and connected and emphasized, so I need U connected.

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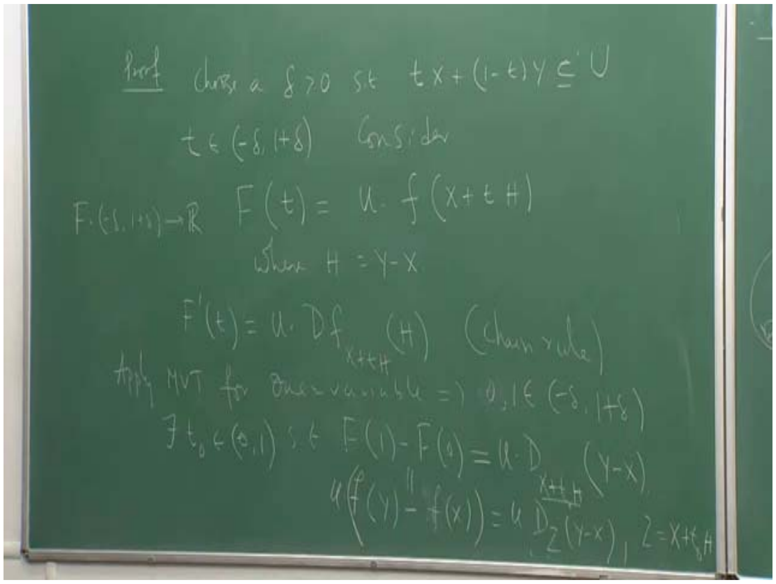

Okay, so the proof is here, choose a delta greater than 0 such that T X plus 1 minus TY belongs to, this line is a sub set of U but is already my assumption but here I have chosen this line little bit extended, this is because I want to apply differentiability and I don't want to apply differentiability at any point, but I can do this, why, because you have chosen open. So if X is there, there is a ball around this so I can always extend this line little inside U and here always there is a ball around Y so I can always extend this line, it will be beyond Y.

So I choose (15:00) 1 minus delta to 1 minus delta and I will apply for the point 0 and 1 in between. So consider the function F T equal to U dot F of X plus T H where H equal to Y minus X. What is F prime at T? F is differentiable by chain rule? This is F, D F at X plus TH at H. This is Chain rule. Okay. Apply MVT for one variable, F is from minus delta to 1 plus delta this interval to R. MVT for one variable function will imply their exist, T not MVT one variable function with 0 and 1 in minus delta to 1 plus delta so there exists a T not in 0 to 1 such that F of 1 minus F of 0 equal to U dot D X plus T not H at H, H is Y minus X.

What is F on minus F0, F1 is Y, F0 is X, D, call this Z not, Z at Y minus X, so your Z is X plus T not H and if you write Y equal to Y minus X you get Z in this form, maybe 1 minus T not and T not but that does not matter, so this is a proof, plus look at it carefully you will work it out. So this is a proof of MVT.

Now as per the application which is very fundamental and all of us use it randomly, whenever we see a function, differentiable function which is 0 derivative 0 it says F is constant. So that is what we want to prove now.

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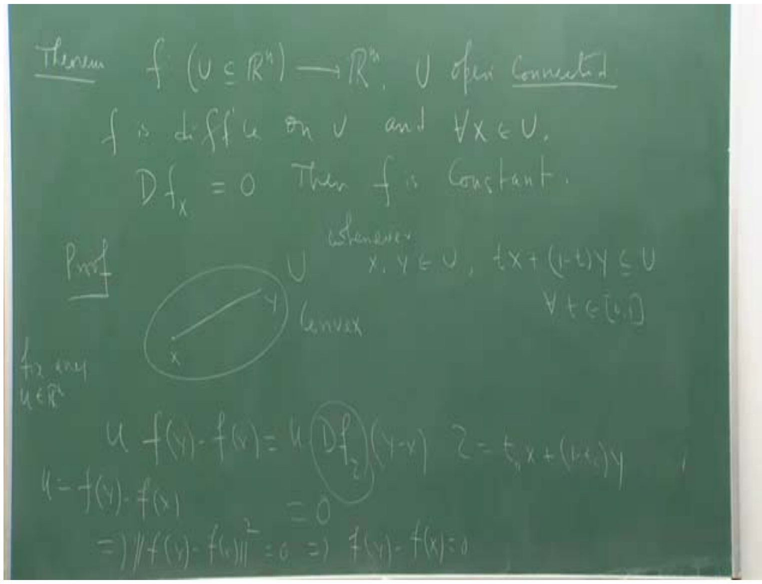

So then application I have this theorem. So again F from U to R, U in RN to RM, let me emphasize U is open, connected, F is differentiable on U and for all X in U D F X is equal to 0. So for linear operator this is 0. DFX is a linear operator from RN to RM, so as a matrix M cross N is 0. Then F is constant. There is a constant in (19:02) vector in RN and this is a direction application of MVT. How?

Proof, see just think about it, I will not write the proof, suppose U is this kind of set that whenever X and Y in U the line joining 1 X and Y is also in U for all T that means such a state is called convex state. Any two points is there and a line joining is there is always there. Now you can apply MVT directly, how?

MVT will tell you F minus F X, you fix any direction U correct so fix any U in RM, this is going to be equal to U dot DFZ Y minus X so Z is of the form, some T not X plus 1 minus T not Y, just not while we prove it. But this fellow is 0 matrix operator, so this side is you know 0 in the product U which is still 0, so U dot F Y at that is 0 whenever I fix U.

Now you can take U to be FY minus FX, X and Y are fixed, so FY minus FX dot FY minus FX will give norm of FY minus FX, norm of square equal to 0, norm of square of something equal to 0 will imply FY minus FX equal to 0. So FY equal to FX. And you can apply for any two points as long as U is convex; any convex set is open connected. But one may immediately recognize that there is a problem. What is the problem? U maybe open and connected but it may not be true that for any FXY they can be joined by a line.

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For instance if I have my U like this and here is Y and here is my X, if you try to join it by a line this line, this part of the line will go outside U. Or for that matter U is like this, here is my x, here is my Y, I cannot join X and Y by a line it will go outside. But what I do always you see I can see do a simple trick, I do not join X and Y directly, I put a point in between Z, join X and Z and join Z and Y. Here also I can choose not 1 Z but may be many Z1, Z2, Z3, Z4, and then finally Y. So I can join them by broken lines, polygonal lines.

So I can join X and Y by polygonal lines meaning, choose Z not maybe equal to X, Z1, Z2, ZK finally which is Y, all ZI in U such that ZI plus 1 to ZI this line lies in U. Now you do what, you do simple thing. You apply MVT on the edge broken part. So YU minus FX, first I apply MVT to FZ minus FX, here I will get Z1 equal to FX. Then I will apply to Z1 Z2, here I will get FZ equal to FZ1, so one and finally FY equal to FZ4, FZ3, FZ2, FZ1 FX, so FY equal to FX. So write down the proof now.

But here remains a catch, what is that, that given any open connected set I will be able to do it. There is a very nice exercise I will put it in the assignment, but let me just for 1 minute give you the hint how to prove it. So this proof is okay. Modulo is proof this statement…

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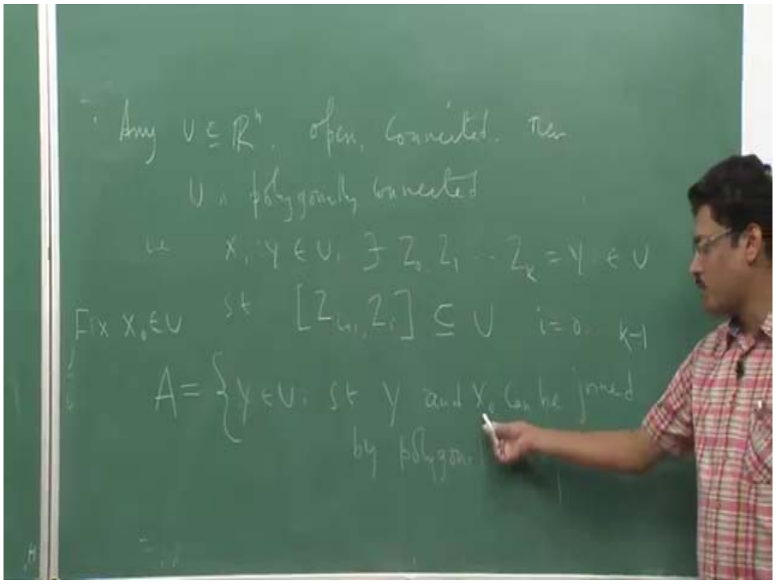

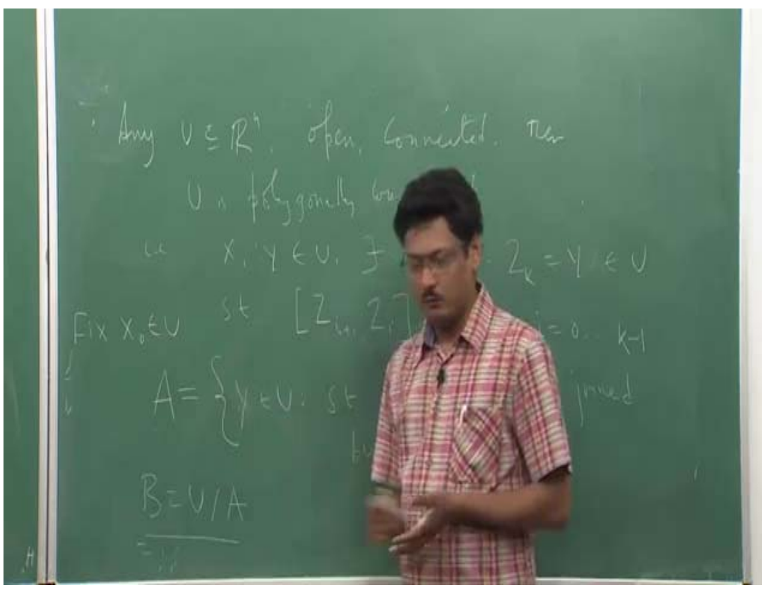

Any U in RN which is open connected then U is polygonally connected, that means precisely this that, that is given X and Y in U there exists point Z not, Z 1, Z K for some K all in U, Z K equal to Y all in U, such that this lines ZI plus 1 ZI this line entirely belongs to U, I equal to 0, K minus 1. If I proof this, this proof will work fine and here I use connectedness and for that what we do, I will put in the assignment so you do all yourself. You put a sub set A of U.

So fix any point X not, consider all point Y in U such that Y and X not can be joined by polygonal lines that is this broken lines. A is a non empty set, and you show A is open and you take B equal to A compliment and you show B is also open. So you can write U as A union B to his joint set, both are open, U is connected, then obviously it will follow that U is either A or B, one of them has empty, so A is non empty so B is empty so A is entire U. So do this exercise it is in the assignment 2. Okay, that will end.