Differential Calculus of Several Variables
Professor: Sudipta Dutta
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Module 02
Lecture No 06
Examples of Differentiable functions.

So we start the second module of this course. This week we will be mainly concerned with the properties of derivatives. So in the last lecture we have seen given a linear map operators from RN to RM, how it is to be realized as a matrix ()(0:36).

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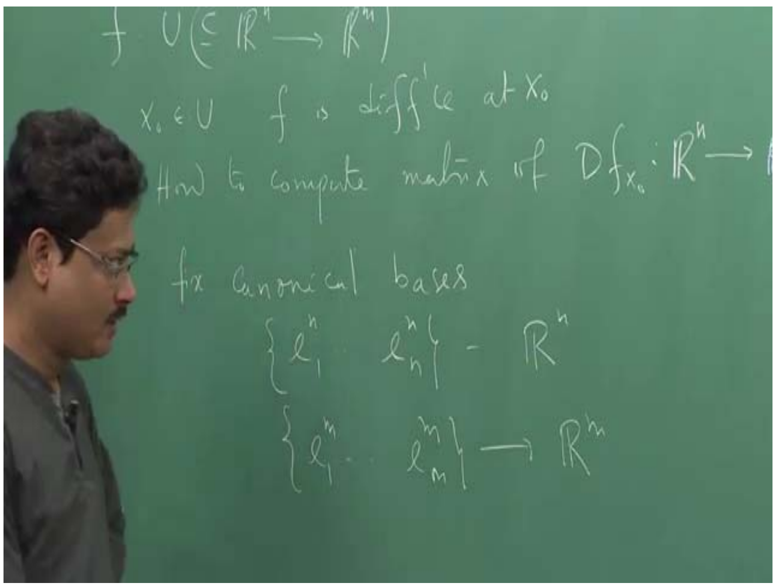

So we continue with that that suppose I have F again U in RN to RM and X not in U, is differentiable according to our definition at X not. So, how to compute matrix of this derivative? Remember derivative is a linear map of RN to RM. Okay we follow the same pattern as we did in the case of any linear map from RN to RN, We observe the Matrix will be a M cross N matrix.

What will be the first column? If I fix basis, fix Canonical bases, E1N, ENN, this is for RN. And E1M, E1ENM this is for RM, correct.

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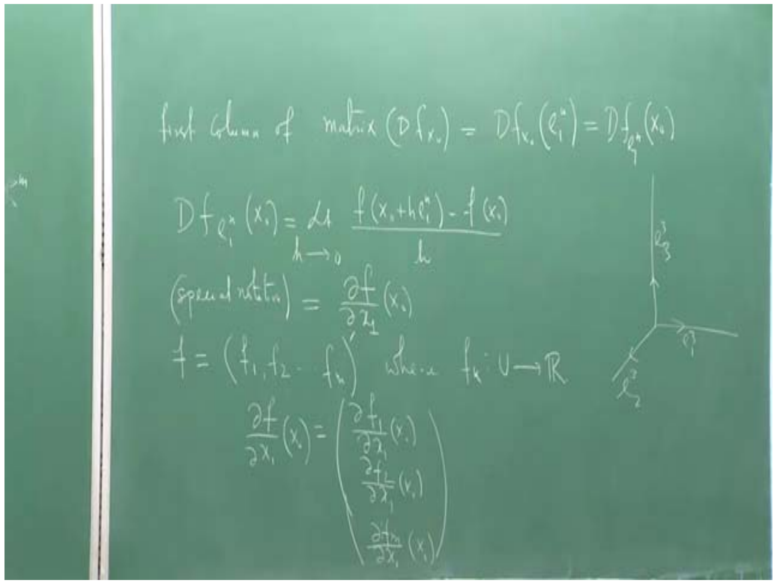

With that from the previous calculation we know the first column of DF, of matrix of DF X not this will be DF X not acting at E1N. If you recall the calculation this is what we have. But we have already proved something that E1N is a direction in RN, right.

For instance if I having R3, this is the direction E11, E2, E13, E33, E23, so this is the direction E13. And you know this fellow will be equal to the directional derivative who is in the direction EN at the point X not. Remember we have proved in the last lecture. So what is this, now I apply a definition, this is Limit H going to 0, H, F of X not plus HE1N minus F of X not. We denote it by some special notation, so this is a special notation for this particular direction.

If I fix the Canonical bases, we make the special notation for this Limit and we call it dell F, dell X1 at the point X not. Now F is from RN to RM so I know F can be written as F1, F2, FN where each FI or FK is from U to R. So F is actually the vector, so maybe I should write in the column vector. So if I write dell F, dell X1 at X not, this will be a column vector, dell F1 dell X1 at X not, dell F2 dell X1 at X not and so on up to dell FM, dell X1 at X not.

So I got the first column of the matrix of DF X not. Now there is nothing special about first column. If I change 1 to any variable XK then I will get the Jth column.

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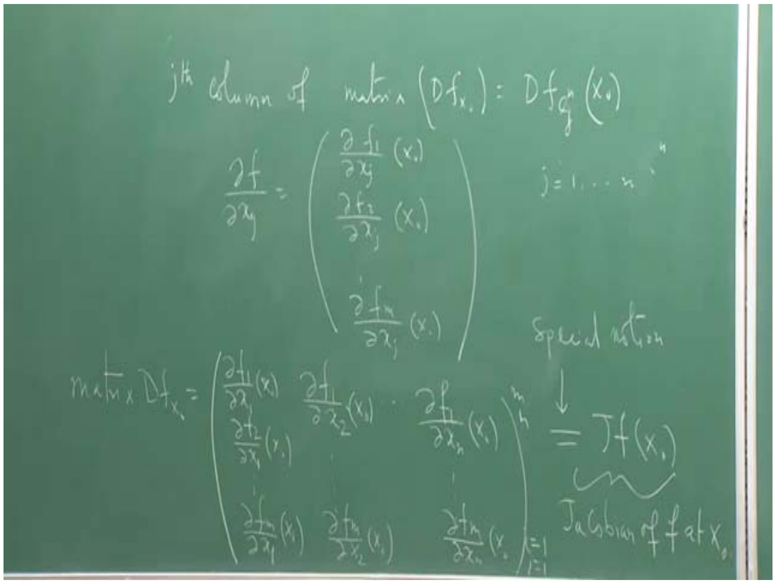

So what I mean here that, Jth column let us say of matrix of DF X not will be DF E J N at X not and according to this description I will write at dell F dell X J which is dell F1, dell F1 dell X J at X not, dell F2 dell X J at X not, dell FM dell X J at X not. Now I got all the columns. Now I put J equal to 1 to N, I got all the columns. So finally I get matrix of DF X not is equal to first column, sorry, second column and finally the Nth column. This is I equal to 1 to M, J equal to 1 to M.

So this is M cross N matrix which is the matrix of DF X not and again if you see in books or notes they have again a special notation for this 1, usually the book do not write in this form, matrix DF X not, they write it as JF at X not and it has a name, it is called Jacobian of F at X not. So look at it very carefully and then I want to make a comment.

So what we have done here, ()(9:14) we compute the matrix of DF X not assuming if it is differentiable. F differentiable then this will be, I look at at, I wanted to first confine the first column or Kth column, I evaluated at EKN and I know by previous theorem that this is directional derivative in the direction of EKL and I calculate that, put the columns together, I get it.

But there is a point here. Thing is that, suppose I get any function, if I give you any function F from RN to RM, or some sub set of RN to RM, you know in the first example, one of the examples the directional derivatives in all direction may exists but still the function may not be differentiable in our sense. Because it may not fail to be continuous, so given any function F, maybe all this fellows exist, dell F, I dell XJ all of them exist.

But if we still fail to be differentiable so given any F actually after computing this I have to check that this is a derivative that is it satisfies our definition. I will explain it through an example.

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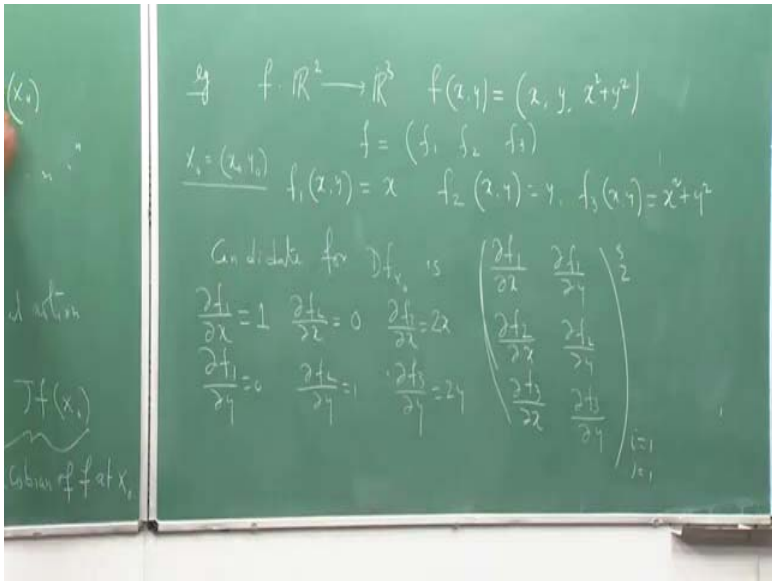

Let us take a very simple example, let us say F X from entire R2 to R3 and F of XY, two variables, let us say, let us take something simple, okay. So according to ()(10:56), first I will write as F equal to F1, F2, F3 where F1 is from R2 to R equal to X, F2 XY is Y, F3 XY is equal to X square plus Y square, correct. Now I know so I am checking at a point X not equal to X not Y not, some arbitrary point.

So candidate, I write candidate because I still do not know, let me write it, let me say it I still do not know given arbitrary point at X not Y not F is differentiable or not because I have not checked the definition. But if it is differentiable then candidate for D F X not is, is from this formal is dell F1 dell X, then dell X2 dell Y and then dell, sorry, dell F2 dell X, dell F3 dell X then dell F1 dell Y, dell F2 dell Y, dell F3 dell Y.

Correct, this will be a I equal to 1 to 3, J equal to 1 to 2 because R2 to R3, so let us calculate the entries. Dell F1 dell X which is how much is that, this is simply 1 and dell F 1 dell Y, this is 0, dell F2 dell X, this is F2 is XY, dell X is 0. Dell F2 dell Y, this is 1, dell F3 dell X, this is equal to 2X and dell F3 dell Y is equal to 2Y.

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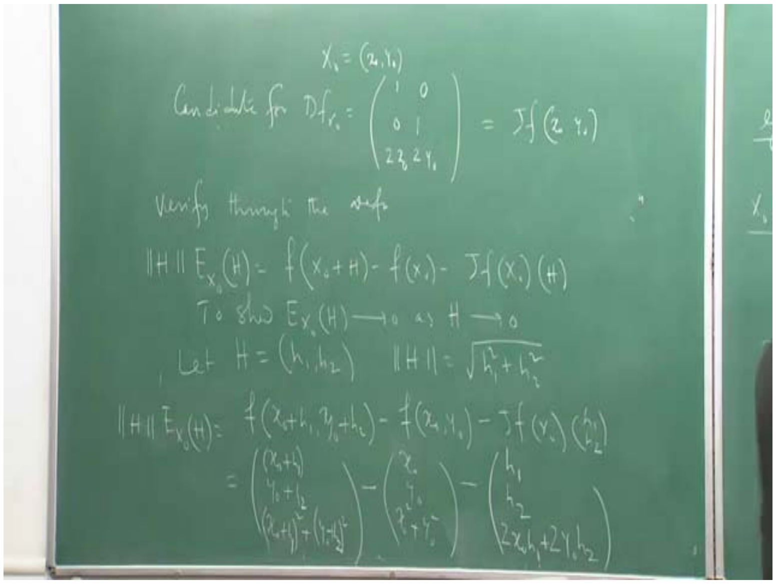

So again candidate for DF X not is how much, 1001, 2 X not 2 Y not. I have to evaluate at X not Y not. But this is only candidate, now I have to verify definition.

Verify through the definition, that is I have to put H EX not H equal to F of X not plus H minus F of X not minus, so this is according to all notation, JF X not Y not, J F X not acting at H and to show E X not H goes to 0 as H goes to 0. Well let us check, let H, H is in R2, H1 H2, so norm of H is H2 square. Let us calculate this fellow, F of X not plus H1, Y not plus H2 minus F of X not Y not minus J F X not at H, at H is H1 and H2, which is equal to if I write it in column form, what is it?

X not plus H, Y not plus H, Y not plus H, X not plus H whole square plus Y not plus H whole square minus X not Y not X not square Y not square, minus JF X not, this matrix acting at H1, H2, H1, H2. This matrix acting at H1 H2 you can easily compute that will be how much H1, then H2 and then 2 X not H1 plus 2 Y not H2, okay! Continue the calculation here.

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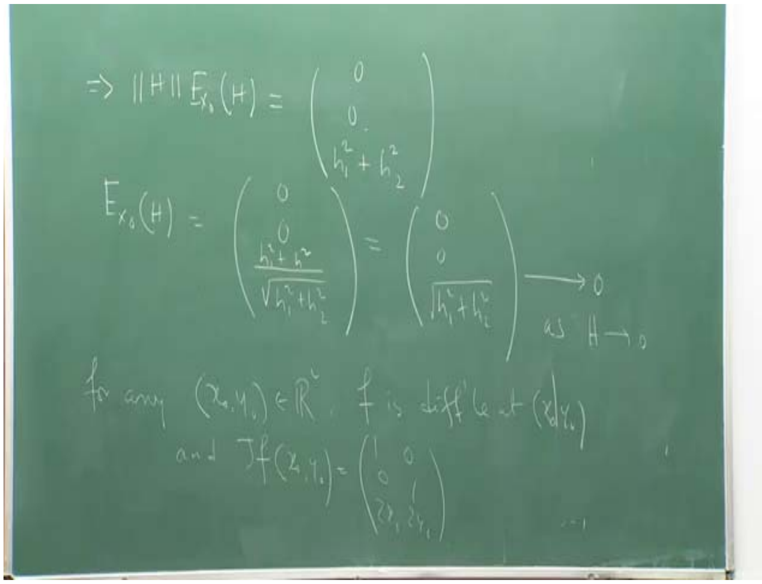

This implies equal to, how much, this will get cancel, second column will get cancel, third columns remains, 0 0 H 1 square plus H2 square, so E X not H is simply 0 0 H 1 square divided by H 2 square divided by norm of H, but norm of H is this, which is 00 root over, of course this goes to 0, as H goes to 0 because H goes to 0 implies norm of H goes to 0 this is precisely this quantity.

So after this calculation we can now surely declare for any X not Y not in R2 F is differentiable at X not Y not and derivative at A Jacobian matrix is 1001 2 X not 21. So you see there is some bit of work done, it has to be done. Even if you calculate this matrix J F X not Jacobian matrix you have to actually check that this thing, this candidate is actually the derivative, it may not be. We have examples.

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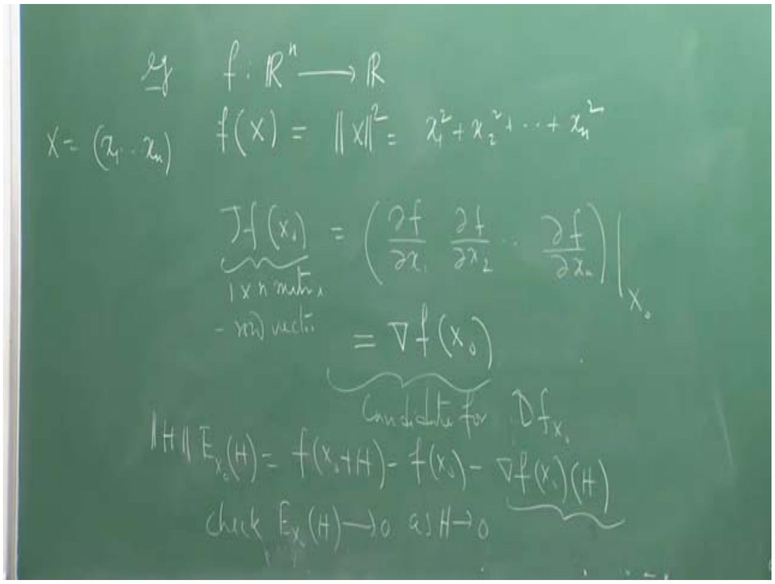

Let us look at another example, I will let you to do it, very easy example. F is from let us say RN to R, so real value given by F of X, let us say X equal to X1, X2 XN, equal to just norm of X square, which is by delimitation is X1 square, X2 square, XN square. Now there is M equal to 1 so there is only one F, so Jacobian matrix will be according to our description, this will be a matrix 1 cross N matrix that is row vector.

And what is that row vector will be, this is dell F dell X1, dell F dell X2, dell F dell X N, evaluated at the point X. Again we will see in the book people do not write JF X not for function from RN to R, one usually write it as Grad F X at X not. So this is again candidate for DF at X not and we have to check that H E X not H equal to F X not plus H minus F of X not minus Grad F X not at H and I have to check, E X not H goes to 0 as H goes to 0.

I leave the verification on you, so write down everything and check it actually works. And with one work I will finish the lecture today that is you can easily see this. Okay, first of all what is this fellow? A 1 cross N matrix acting on a 1, N cross 1 vector that is actually inner product.

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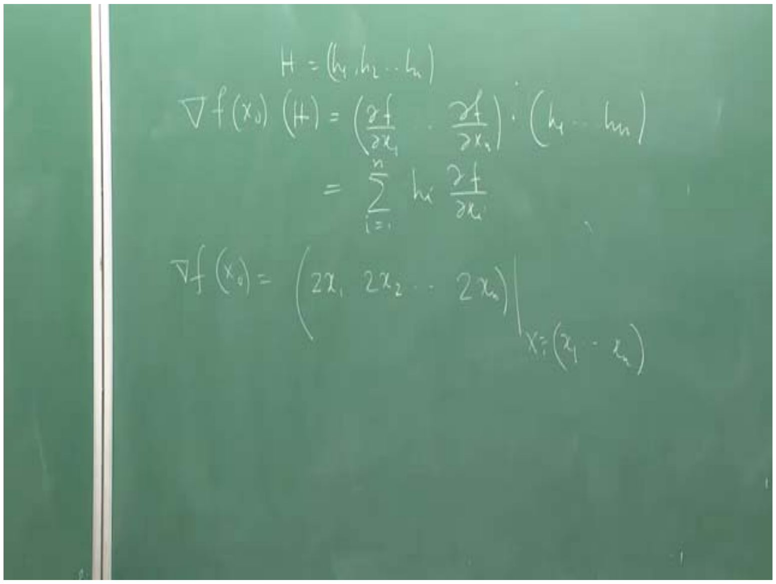

How does this vector X on H, if H is H1, H2, HN, then this is actually dell F dell X1, all the way to dell F dell XN dot H1, H2, HN which is summation HI dell F dell XI, I equal to 1 to N, so you use that and of course we can calculate the dell F at X not is simply dell F X1 is 2 X1, 2X2, 2XN evaluated at X equal to X1, X2, XN. Okay. So do the exercise and we will continue from here.