Differential Calculus of Several Variables
Professor: Sudipta Dutta
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Module 1
Lecture No 4
Derivatives: Possible Definition.

Okay, today in the fourth lecture we will start with a derivative of function of several variables.

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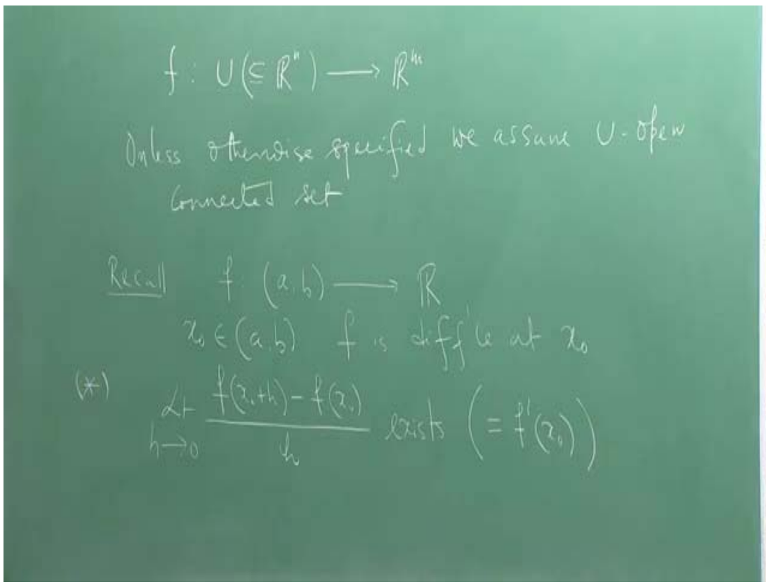

Just once again I have a function F define from sub set U in RN to RM, M and N as before greater than and equal to 1. We'll talk about derivatives, unless otherwise specified, you know it happens in for function of one variable also. So unless otherwise specified, we assume U to be an open connected set. Because there is always a problem in defining derivative at boundary, it becomes one sided.

Okay, so before defining or what should be the definition of derivative here, let us try to see or recall back what happens in function of one variable. So recall back, suppose I have a function from an open interval AB to R, X not is a point in AB and we say F is differentiable at X not, something like this happens that limit H goes to 0, correct. This exist and we call it F prime at X not.

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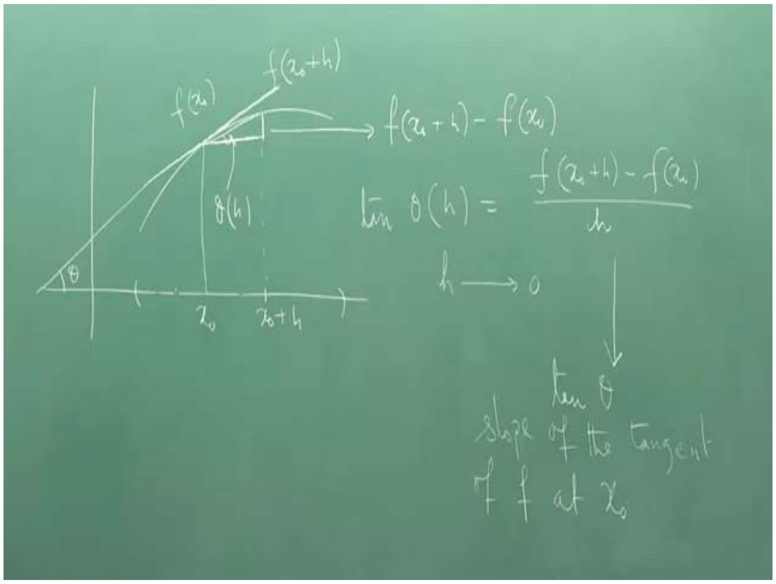

So what is the meaning of this definition? Let's put a star here, what is the meaning of this definition start, well that is here. You have defined here, now you have a point X not and here is your FX not, X not plus H, H goes to 0, H can go to 0 from either side, negative side and positive side. If X not. If you take negative then X not minus H here and if you take positive then, okay let us do it little bigger, so this is the interval H, length of the interval H, this is X not plus H and here is your FX not plus H.

Look at the numerator, this is precise the difference this, correct. I am dividing by H which is this length. So you see if I join this two points by a straight line if I have this angle this numerator, if I call this angle depending on this theta H, theta H is this ratio, tan of theta H. And what is idea of defining this, well as H goes to 0 this, as H goes to 0 this ratio actually gives you, this ratio goes to tan theta which is the slope of tangent at FX not. This is the geometry behind this definition.

If you do it from X minus H, X not minus H, similarly I have to approximate this side and this will also the, demands both way it should converse to the slope of the tangent at FX not, at X not. Sorry, slope of the tangent of F at X not, yes, okay. So simple geometrically geometric interpretation of derivative.

Now if I try to just generalize this notion directly to function of several variables, how should I think. Well, look at this ratio, this ration actually measure the difference between the value of FX not plus H and FX not, divide by the length of the interval, correct. So maybe our first instinct will be that, well, let us try this.

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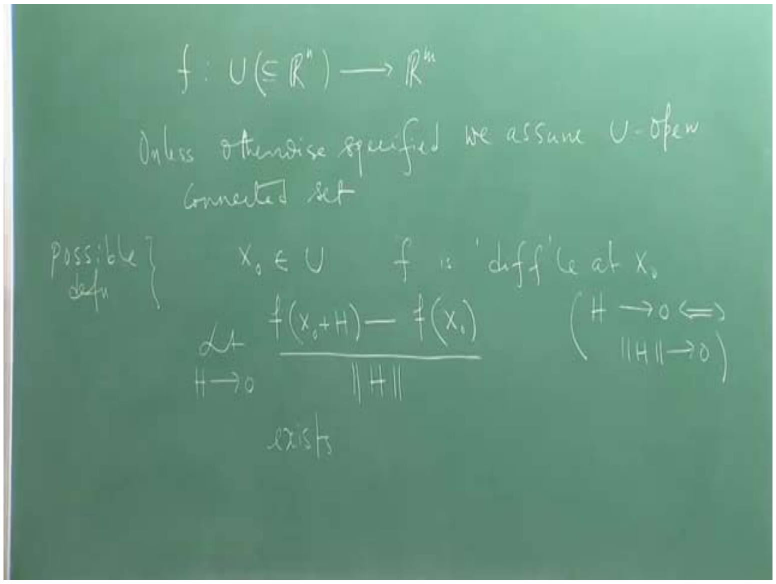

I have a function an open set U, open connected set U in RN to RM so given X not in U I should say F is, so this is a possible definition, okay. X not in F is differentiable at X not while what I do I look at X not plus H minus, so this is RM value, so it is a vector in RM minus this vector FX not and H is increment so I divide by the length of the increment and here the concept of length is norm, and then try to put limit H goes to 0, we call that H goes to 0, in this is same as saying norm of H goes to 0, thus follows for the definition of the norm, right.

And if this exists then I will call it differentiable and I will denote it by F primes. Well, what is wrong, I mean fine, we just lifted the idea from one variable to several variables, but whatever we lift, if we go back when N equal to 1, U is an interval and M equal to 1 that is the case of single variable, this should match to our original definition of concept of notion of differentiable function for one variable. And looking at this definition you can see that immediately there is a problem.

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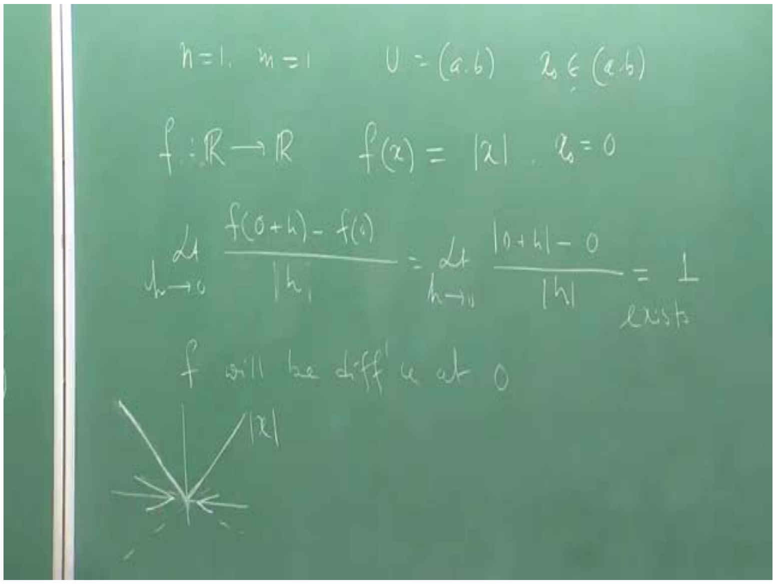

What is the problem? So what I am trying to do here, I have come up with a possible definition of derivative function of several variables. Now I see it is, if it matches with my original notion when N equal to 1, M equal to 1, U equal to AB and X not is a point in AB, look at this function, FX R to , the famous 1, mod X and X not equal to 0, just put this definition. Limit, now single variable H goes to 0, 0 plus H minus F of 0, divide by length of H which is Limit, H goes to 0, by definition 0 plus H minus 0 divide by H which is equal to 1, so exist.

That means according to my definition this F will be differentiable at 0. But all of you knew it is not, it is the first example you do of a differentiable function, that what happens here, this is mod X, graph of mod X at 0. This is like this, and as I said that existence of derivative means it gives the slope of the tangent at a point but at this point of course you cannot do any unique tangent. There is no unique tangent exist, F is not differentiable. So what is going wrong?

So this definition, I have to discard. What went wrong? Well, if I look back then I see the problem is that I have put the length here and while actually derivative I don’t put a length here, I just divide by H, so what is happening, why mod X is not differentiable that all of you know, if I approach 0 from this side, this limit is plus 1 and if I approach 0 from this side then the limit is minus 1. So limit doesn’t exists such 0, therefore it is not differentiable so I cannot put this mod here.

So now I reflect once again, maybe is because that I have to approach through direction, this direction matters, positive direction or negative directions, and in real line there is only direction, this one, and its negative one, so basically one. So maybe whilst defining derivative for several variables we should take care of directions. That we should be able to approach for all possible directions and what we can come up with a possible solution …

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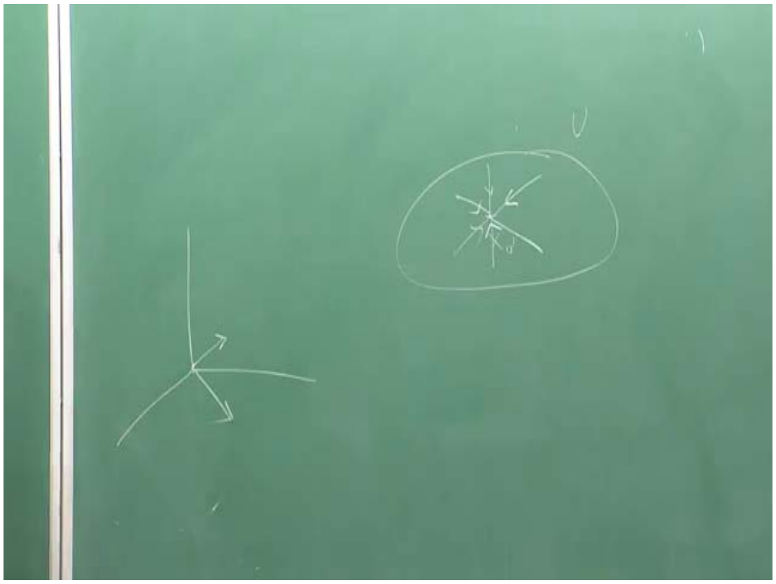

Well here is my U, I want the derivative at X not, I should be able to approach X not from all possible direction and then we will see that if the limit exists, that is what I mean here. There is another possible definition, okay, X not is in U and I take U in RN a norm vector that will define the direction, right. Any normal vector defines a direction. Direction is precisely given by all normal vector.

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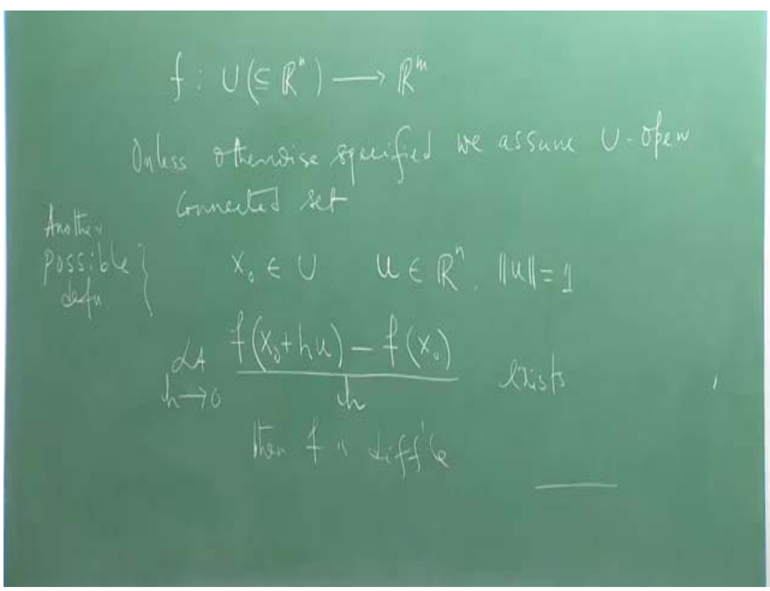

And I should define and I should look at F of X not plus I should approach through these directions, then F of X not divided by H (I cannot put sign here because I have to approach from negative as well as positive of U), and then put H goes to 0, H is real. And if this exists then F is differentiable. I am not saying that this is the definition. I said it is another possible definition which we after reflection about the drawbacks of the previous definition we make come up with this definition. Okay, not bad.

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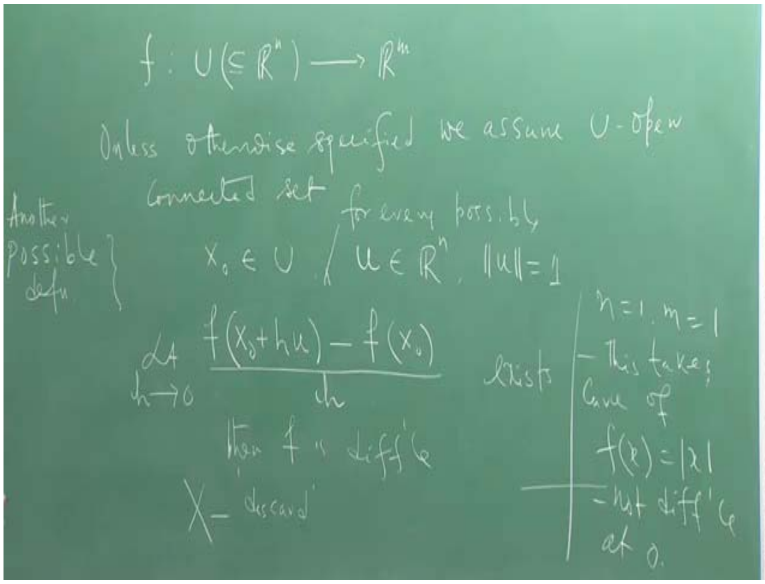

This will take care for N equal to 1, M equal to M, this takes care of FX equal to mod X, it is not differentiable at 0, if I put this definition. That is the modification I have done, not bad. And I want for every possible, I should write every possible U this thing happens. Okay, but let me put another example here.

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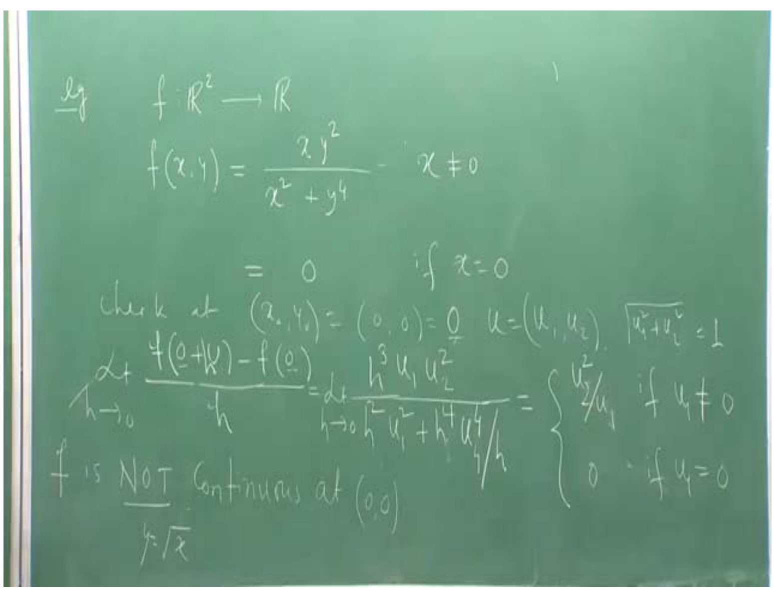

Let us say X from simple one R2 To R, let me recall F of XY equal to XY square, X square plus Y power 4, where X not equal to 0, and 0 if X equal to 0. You can just see that this function according to this definition, if X not equal to 0, Y not equal to 0, so at 00 except 00, everywhere it will satisfy this definition, everything is very nice. But let us check it, check at X not Y not equal to 00.

I put this definition, Limit H going to 0, F of 00 plus – So it if is a U with norm of U1 U2 equal to 1, plus U1 (sorry, let me write it more clearly), F of 0 plus U minus F at 0, 0 means vector 00 divided by H that will give me what my definition U1 U2 square, oh sorry there is a H. That will give me H cube U1 U2 square divided by H1 square U1 square plus H24, sorry H square H4, U power 4, correct, everything divided by H. Check that is correct.

And now if I put, so I have a Limit H going to 0, if I put it will go to 0 you can calculate this will turn out to be, U1 by, sorry U1 square by U2, sorry U2 square by U1, if U1 not equal to 0 and 0 otherwise. This is the Limit. Very good, Limit exists at 00, so according to my, this definition F is differentiable at 00. But you can easily check if F is not continuous at 00, why you just approach through the line Y equal to root X, you will see that at 0 it doesn't match. If you check F is not continuous at 00, so then I have a serious problem.

Because I do not, I want the differentiable function to be at least continuous, if I am able to draw tangent to a graph or tangent to a point, tangent at a point to the graph and the function is not continuous at all there is a jam then what do you mean by a tangent, this defies our geometry. So this derivative again I have to discard, but possible you will not discard as, it is not too bad so we will discard it with a little bit reservation and we will come back to it.

We will call it later on directional derivative and it will be useful for us, but for time being this cannot be the definition of differentiable function of several variables because if it is differentiable function at least we want continuous and this function is not continuous but turns out to be differentiable according to this definition.

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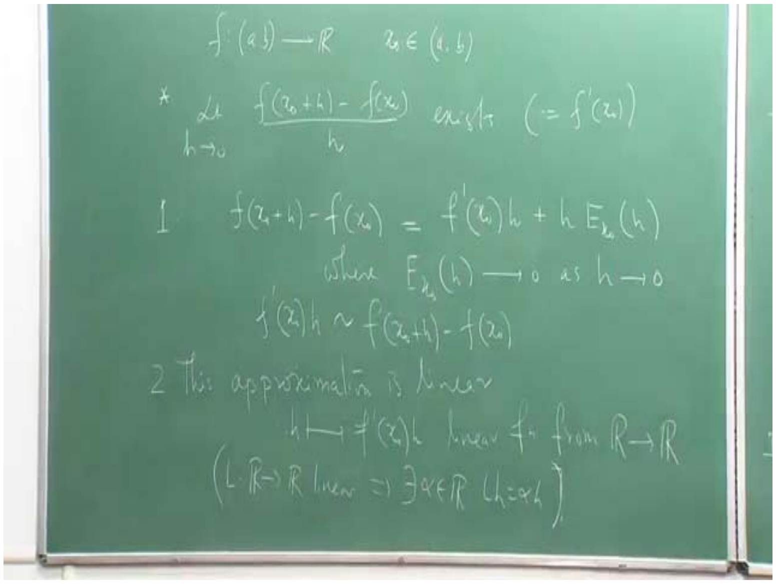

So I have to look back my star a little bit carefully, what was star, F AB to R. Limit H going to 0, F of X not plus H minus F of X not, divided by H exists and we call it equal to F prime X not. We observe what is happening is actually that this F of X not plus H minus F of X not there is an increment that is equal to from this I can write, F prime of X not into H plus H into to some (())(21:28) where goes to 0 as H goes to 0.

So that means to say F prime X not into H approximate this increment F of X not minus F X not, correct in this sense. And second this approximation is linear. What does it mean that H going to F prime X not into H, this is a linear function from R to R because and R to R any linear function, I am not writing it, listen to me carefully. Any linear function from R to R that is given by alpha in R such that L of H equal alpha H by multiplication.

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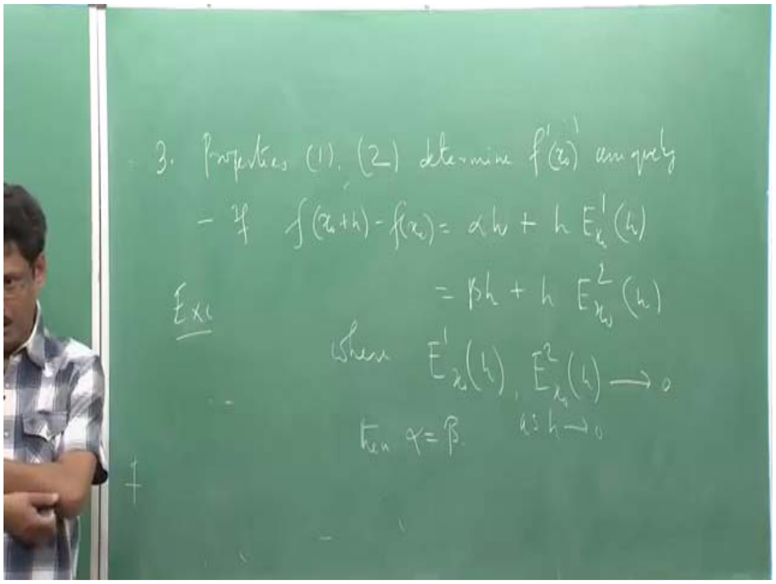

And the last thing property 1 and 2, I have written on that board, determine F prime X not uniquely. I will leave it as an assignment exercise, and what it mean by this is if F X not plus H minus F of X not equal to some alpha H plus H into EX not 1 H and also equal to beta H plus H into some EX not 2 H where E1 X not H and E2 X not H both goes to 0 as H goes to 0 then alpha equal to beta. So if F prime X not is uniquely determine based 2 properties, this I put an exercise and the next class will start from here and come up with its right definition of differentiability.

Thank you!