Differential Calculus of Several Variables

Professor: Sudipta Dutta
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Module 1
Lecture No 3
Continuity and Connectedness.

Welcome to the third lecture of this course. So today we will talk about another property you learned for function of single variable that is intermediate value property.

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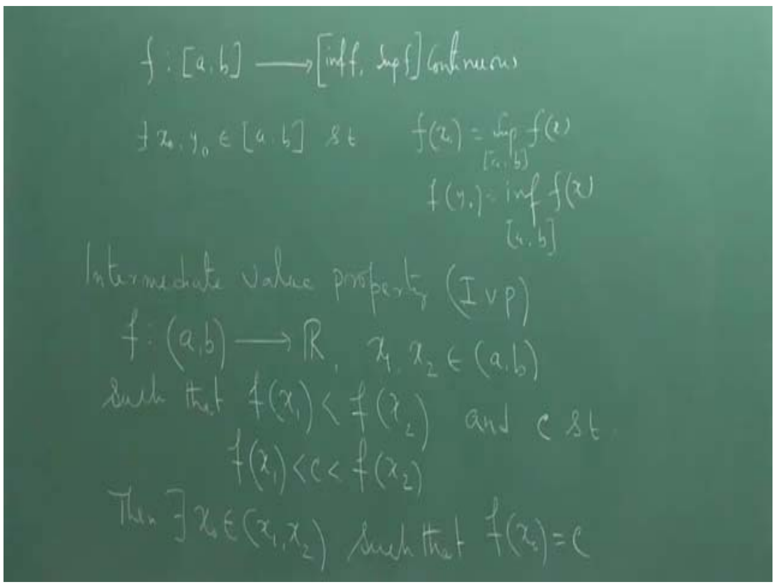

Just recall yesterday we said that if I have F from AB to R then their exist point X not such that, so there exist X not and Y not in AB such that F of X not is surepmum of F on AB and F of Y not is infimum of F on AB. This is the consequence of so called Bolzano Weierstrass Theorem. But actually something more is true and all of you know actually if I have a close interval and this is continuous of course. We are talking about continuous function, then I also know the range precise.

The range is actually infimum of F to supremum of F. That is range is also closed and all the values between infimum and suprimum is attained. This is so called IVP or Intermediate Value Proportion. What does it say? In general, suppose I define F from any interval, I may not be closed interval, I can be closed doesn't matter to R, suppose I have F1 F2 in AB, such that let's say F of X1 doesn't matter which way, F of X1 is less than F of X2, and C is a number which is in between.

Let me write line below, then there exist X not in the interval X1 to X2, sub interval X1 to X2 which is sub interval of AB such that F of X not equal to C. So if attains, if continuous function attains two values, two different values it attains all intermediate values. Proof of this fact if you look at the picture of the continuous function is very (())(3:57), I mean very easy to understand.

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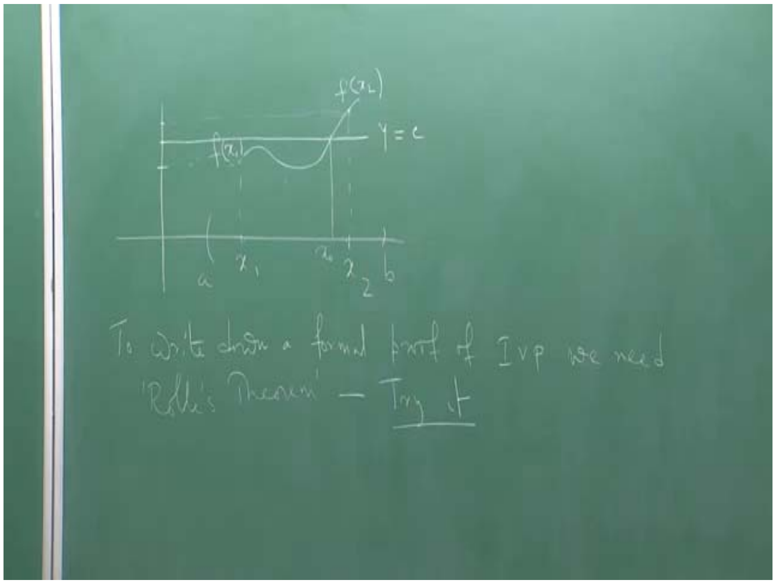

What is happening here like this? Suppose I have AB here, I have X1 here, I have X2 here, this is my full, maybe my F is like this, so here is my F of X1 and here is my F of X2, this point, okay. What this statement says? The C is a point in between FX1 and FX2, this is FX1 on the Y axis, this is FX2 on the Y axis and it says that if you look at the line Y equal to C then it must intersect the graph.

That continuous function going from FX1 to FX2 cannot pass without crossing all the lines in between that's what it says. So looking at the picture the truth is, I mean the proof is very obvious but actually to write down a formal proof. Formal proof or IVP we need so called 'Rolle's Theorem'. This is a, it is there in any standard book on One Variable Calculus, for (())(6:04) or Principle of Mathematical Analysis by Ruddin.

So I will suggest all of you to get your hand at calculus, try it. We have done it in our first course, still try it yourself. Recall Rolle's Theorem and try to prove this statement which is actually there in the picture. You have to see how to use, prefer Rolle's Theorem what function actually you have to choose. You have look at X minus C that is not important for us. What you want is actually the version of IVP for function of several variables.

Well before actually going to the version of IVP in several variables let me sight an example. This is fine, F is continuous on an interval two values are there, in between every value is taken. My emphasis is here that this interval is playing a big role. Now look at an example.

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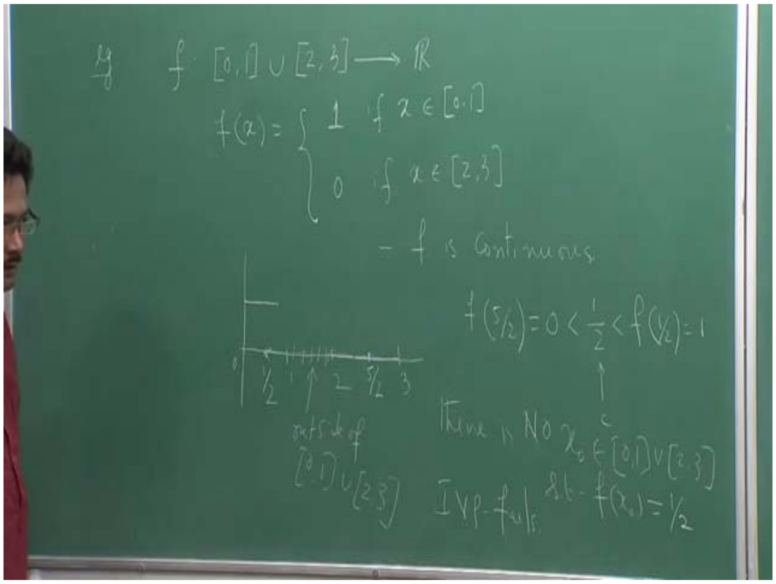

Let us F, define down this interval, union… I will give you the definition of X, 1, 0. So F is like, so here 0 to 1, F is 1 and here it is just 0, this is % (())(6:04). You go by the continuity of F that is given epsilon, use any epsilon there is this delta such that X minus Y less than delta will imply FX minus FY less than epsilon and using definition you can easily check F is continuous.

Now F of this point say is 1, F of this point is 0, okay. So let us say half and let us say this is 5 by 2, so F of 5 by 2 is 0, so I take a number half, this is my C, which is less than F of half which is 1. So it is an intermediate value this half. And there is no X not in 01 in the domain such that there is NO (write in big capital) equal to half. So IVP fails.

So what is happening here? Earlier F was define from A to B, the difference is between these two intervals, this two domains A to B which is an interval and let us say 01 union 23, which is union of 2 disjoint intervals is, if you look at any two points in this interval that can be joined by a path, in this case just a straight line, whereas here, if I take a point here and point here this cannot be joined by a path or a line which lies entirely in the domain.

Whatever path you try to, here you have only straight line so if you want to joint by straight line, this line, this part of the line is outside of the set A, outside of the domain, okay. So this is something which plays an important role in IVP. We formalize this notion in mathematics by so called the notion of connected.

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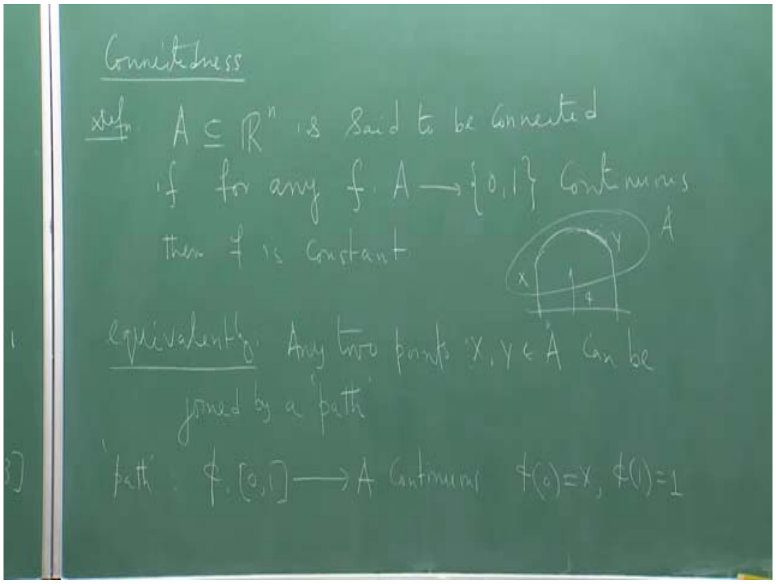

Here is the definition, said to be connected. Look at the definition carefully, said to be connected. I am not just mentioning whilst defining the (pa) connectedness I am not just mentioning the path. I will write it down, don't worry. Said to be connected if for any two valued continuous function, that you consider any continuous function which takes two values just 0 and 1 to 1 on 2 does not matter 0 and 1, two values and continuous then f must be constant, that is F takes just one value, it cannot take both the values, okay.

And I say this is equivalently any two points X and Y and A can be joined by a path. What is a path? Path is a continuous function from some interval 01 to A continuous such that (())(13:16) of 0 is X and (())(13:21) of 1 is 1. So if here is my A here is X and Y, I can have a function C such that (())(13:38) of 0 is X and (())(13:42) is Y, such a thing is called path.

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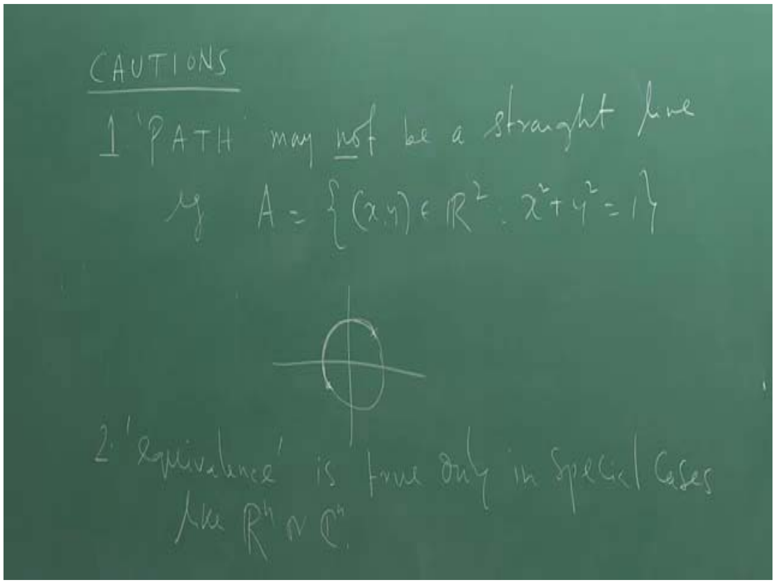

So there are two words of caution here that this equivalence. I will leave it an exercise for this equivalent. I will put it in the assignment also. But there is a word of caution, two words of caution, is path as in the case of interval may not be a straight line, it can be any continuous function, any any curve. For example if you take this set A XY in R2, X square plus Y square equal to 1, so this circle, if I take 2 point on the circle I cannot join them by straight line, which lies entirely in A.

But I can join by this arc, so this A is connected but not any point joined by straight line but that can be joined by an arc. My second caution is that this equivalent condition is equivalence, this equivalence is true only special, only in special cases like RN or CN are for instance non-linear space, don't worry about it. But there can be spaces which are connected for this definition but may not satisfy this equivalent condition so this stronger than this definition and this is sometime refer as path connected (())(15:47) notion in topology and we don't have to bother about, as far as the calculus is concerned we don't have to bother about this. This two conditions is equivalent for us.

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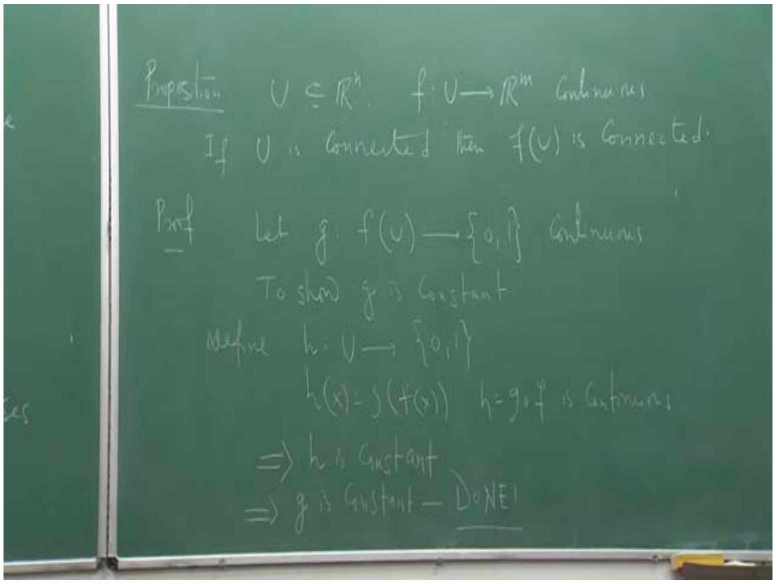

Now the mystery about IVP lies here and then it can be easily generalized to several variables. So let me put is as a proposition. Let us say U or A doesn't matter in RN, F from U to RN continuous. The statement says if U is connected then FU is connected. Look at the statement, continuous function stay connected to connected states. Okay. And therefore if you think about it connected states in real line if you have IVP in the one variable case, connected states of real line has to be interval. I will emphasize it again. Just think about for a moment any connected state of real line has to be an interval, no other choice.

Proof, well let go by the definition, let us have G from FU to 01 continuous to show G is constant; G is either 0 or 1. It is very easy because G from FU to 01 is continuous so define or consider this function H from U to 01, H of X equal to G of FX, okay. We have seen that if G is continuous, F is continuous then G compose F is continuous. So H is a continuous function from U to 01, U is connected so according to definition H is constant. Now H is constant, so H is either 0 or 1, but that all implied G is constant, right. I am done.

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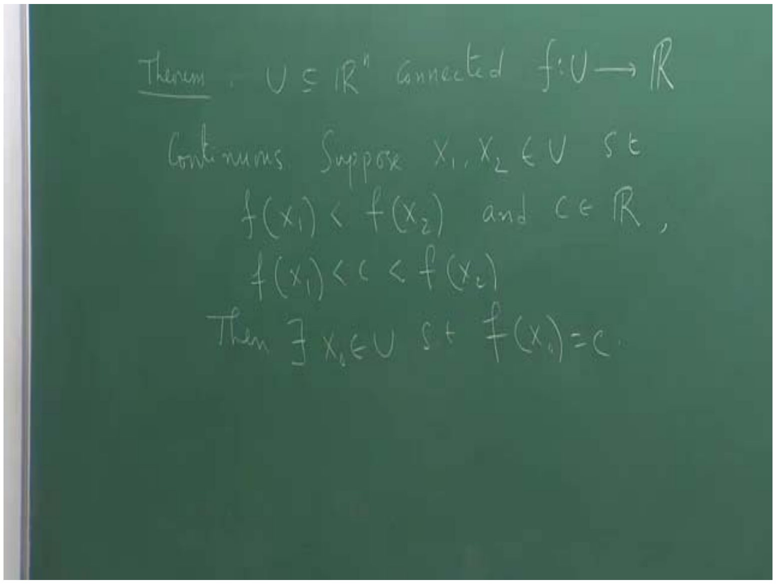

So is the proof and this will give us a version for IVP Theorem? F from U to R, not RM because I have to say X1 is less than X2, F of X1 is less than X12, so intermediate value so this relation of greater than, less than is only valid in R.

Connected this is F from U to is continuous, okay. Suppose X1 and X2 in U such that F of X1 is less than F of X2 and C is a real number, in between then there exist X not in U such that F of X not equal to 0. This is exact analog of intermediate value property for function of several variables taking value in R. R is necessary because I have to compare F of X1 and F of X2.

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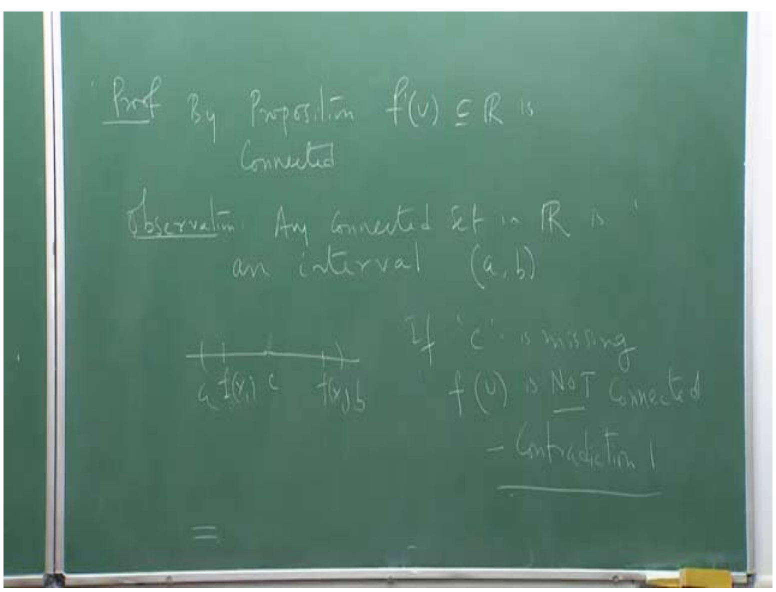

Quick proof, very easy, by proposition FU in R, sub set of R is connected. Observation or just reflection, any connected set in R is an interval, let us say AB, so FX1 and FX2 are in the interval, this is the range AB, FX1, FX2, C is in between. Now if C is missing then this is no longer an interval, if I take out C this no longer remains an interval, if C is missing F of U is not an interval so not connected, this is a contradiction, that ends the proof.

Okay, that is about continuous function, next class we will start derivatives.