Differential Calculus of Several Variables
Professor Sudipta Dutta
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Mod 04 Lecture Number 18

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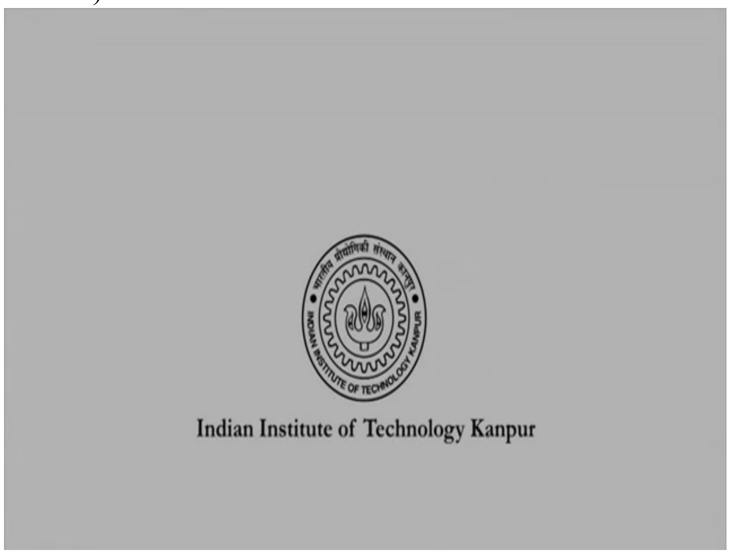

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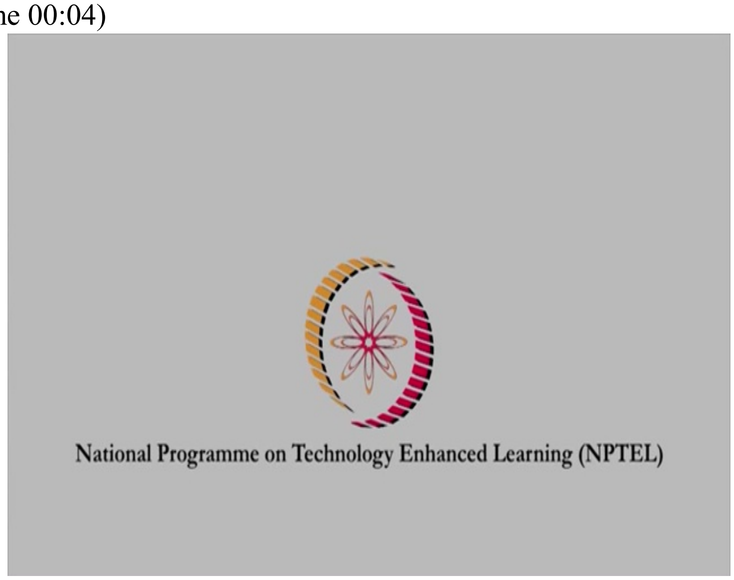

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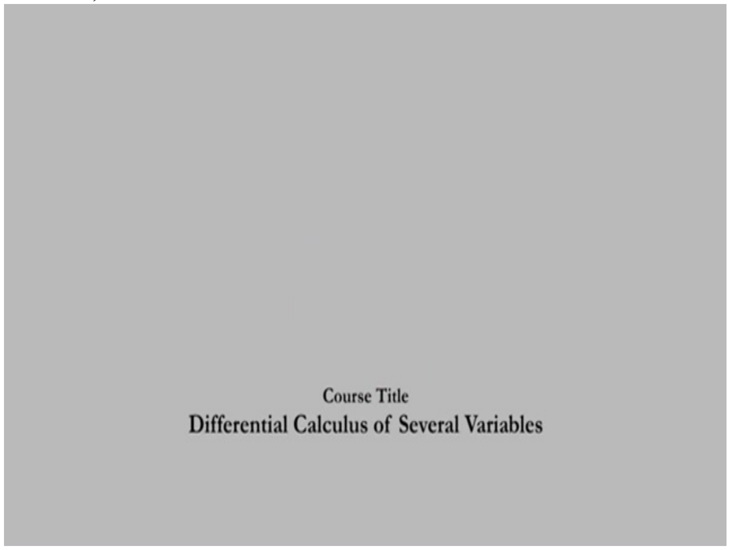

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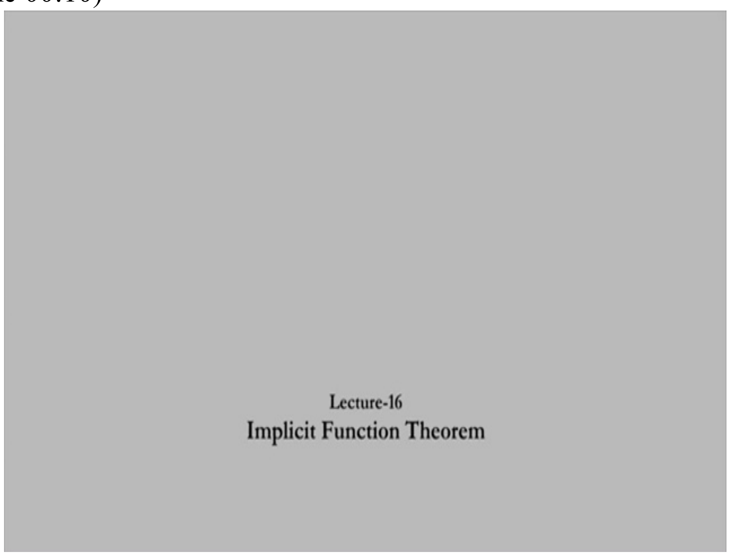

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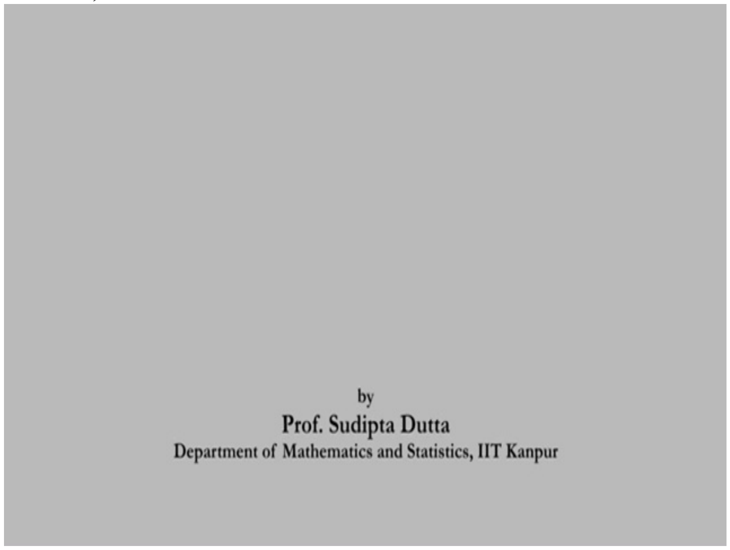

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So, Ok let us come back and continue with the implicit function theorem.

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Implicit function theorem again, what we stated and proved yesterday.

So this was the theorem

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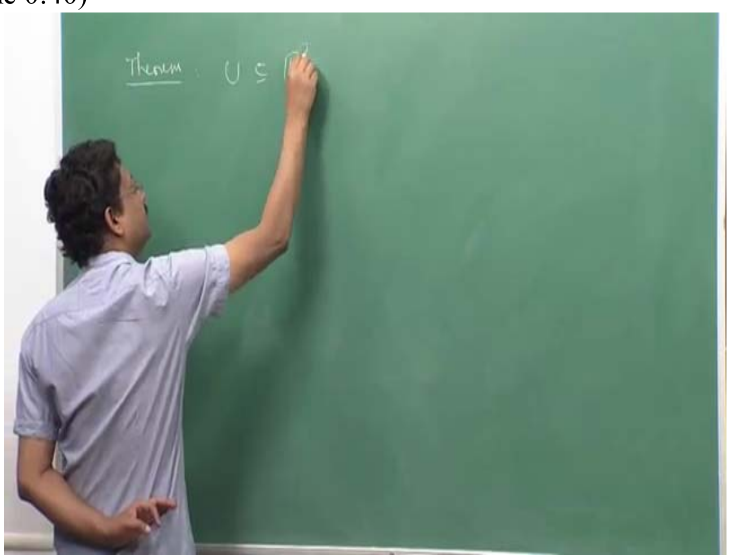

R2 Open set and f runs from R2 to R

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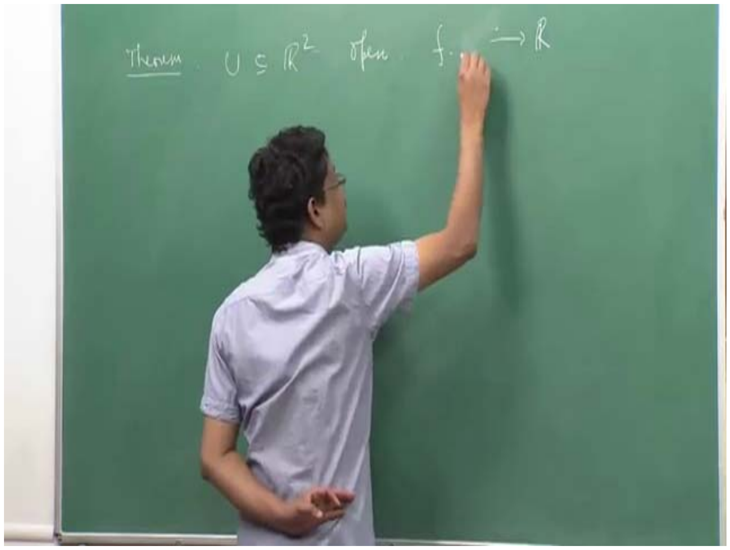

sorry U to R   with continuous partial derivative

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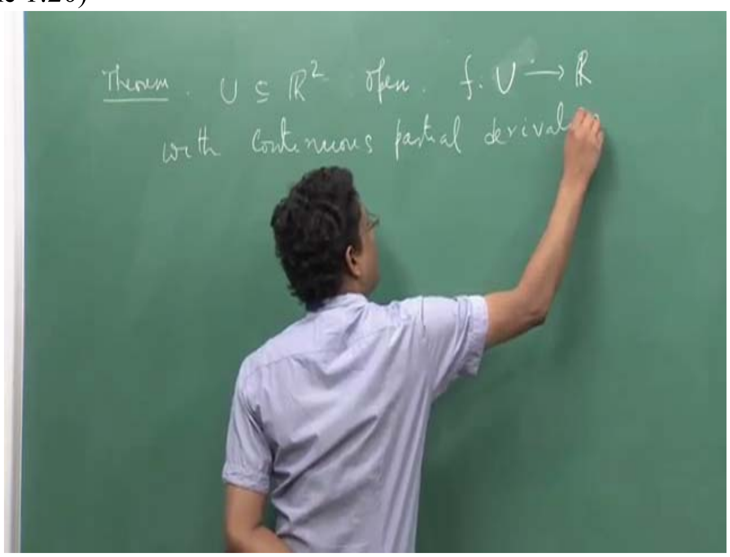

We had a point “x naught” “y naught” in U, “del f” “del y” we assume,

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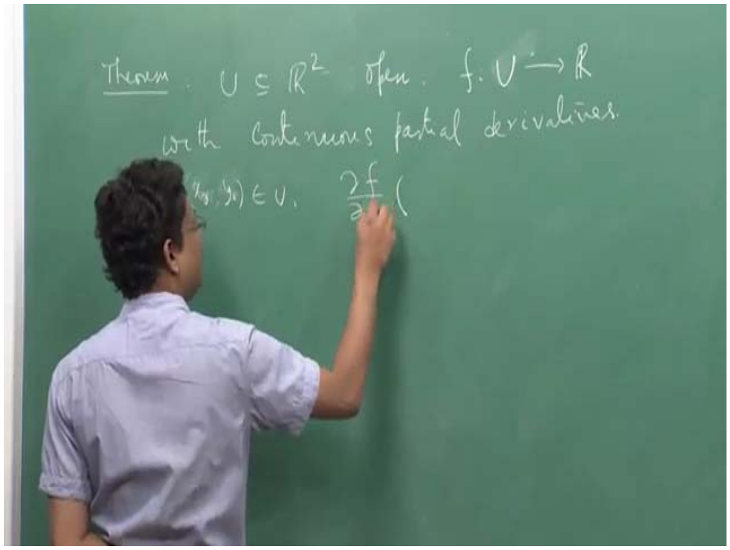

does not matter, we can assume “del f” “del x” as well

“x naught” “y naught” greater than zero or less than zero

and f of “x naught” “y naught” is the level curve, some constant we will put it 0.

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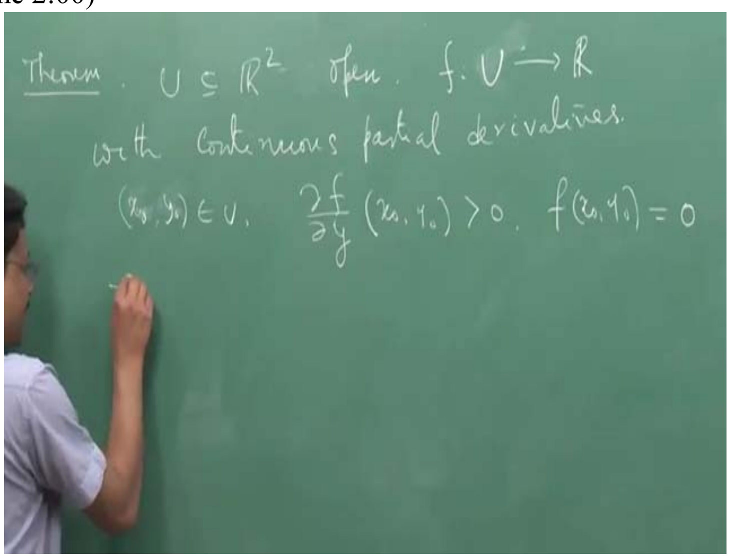

The statement says,

then there exists “epsilon” greater than 0 and g from “x naught” minus “epsilon”, “x naught” plus “epsilon” to U actually

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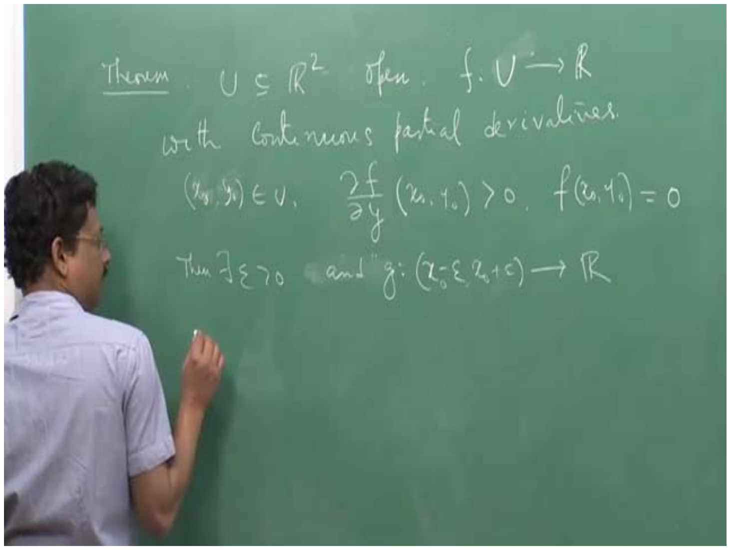

how did I state it yesterday g “x naught” minus “epsilon” such that

1 f of “x, g x “equal to 0

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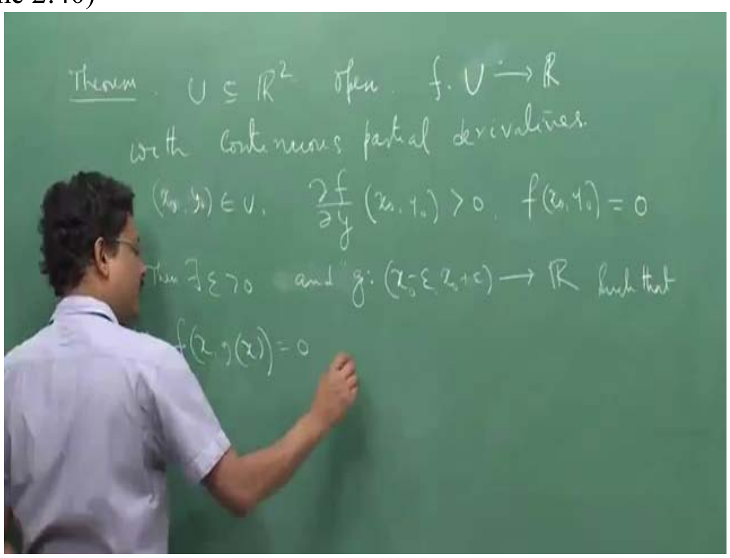

for all x in this interval and of course, g of “x naught” is “y naught”

And second was, we get, if f is smooth enough, we also get...

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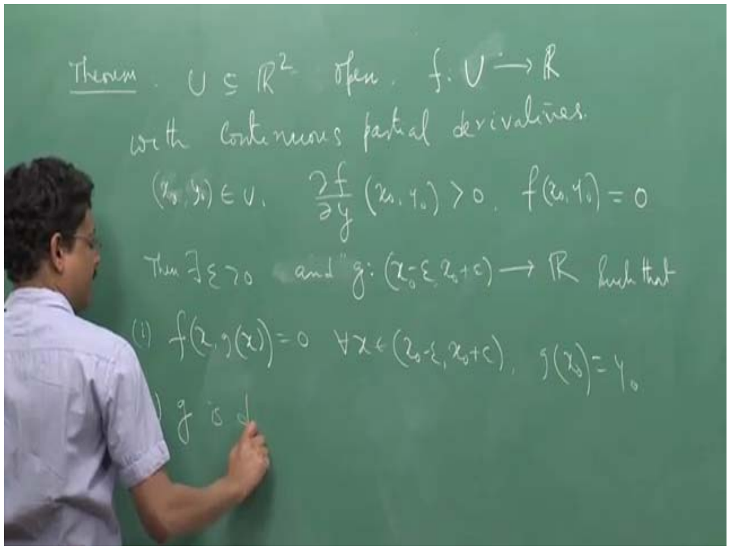

g is smooth, g is differentiable....

Well, so we have proved it,

So if you look at the proof, then actually nothing changes

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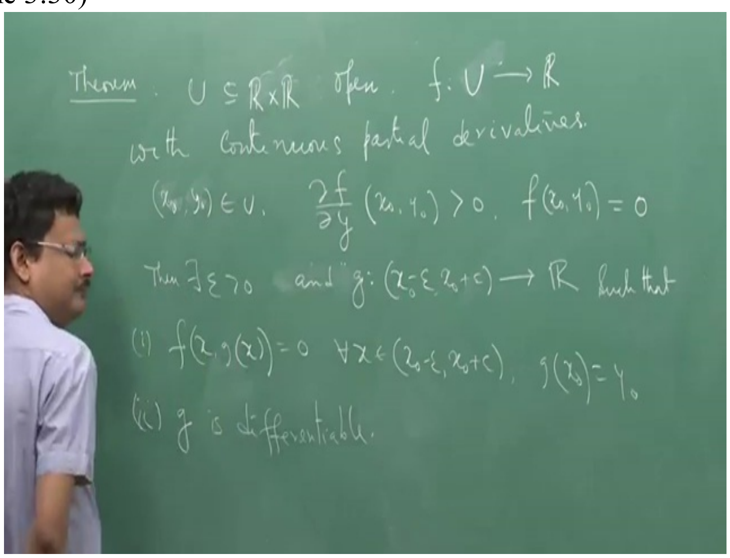

Ok, let me write, this R2 as R X (cross) R correct?

I can write R2 as R X (cross) R. Say it is a function from R X (cross) R to R

And if I have

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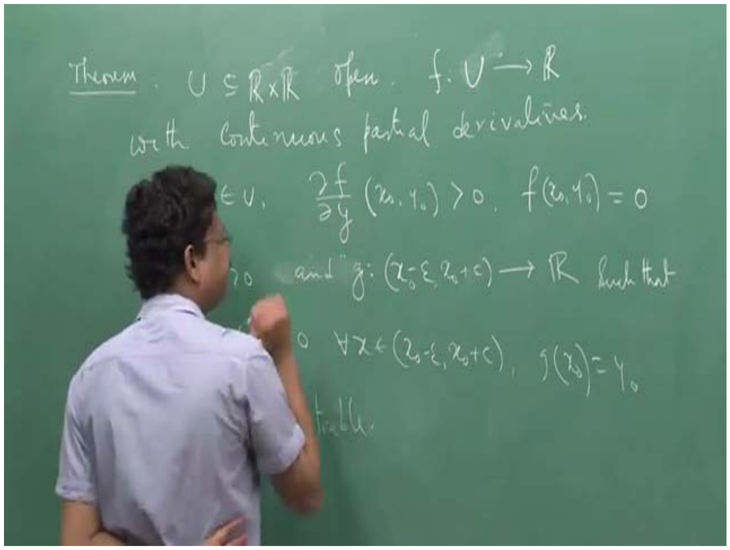

“del f”, del ...so we have emphasized it yesterday and we could actually have del...Ok

For time being, let it remain “del f” “del y”

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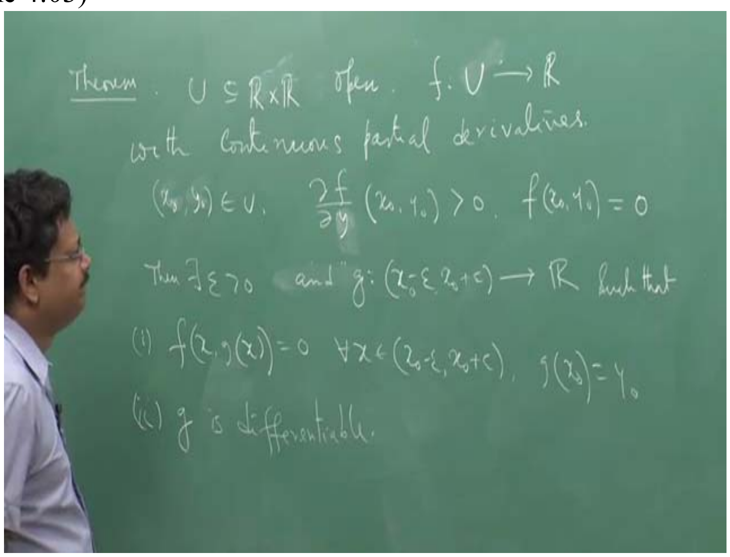

And I will have a interval around “x naught”

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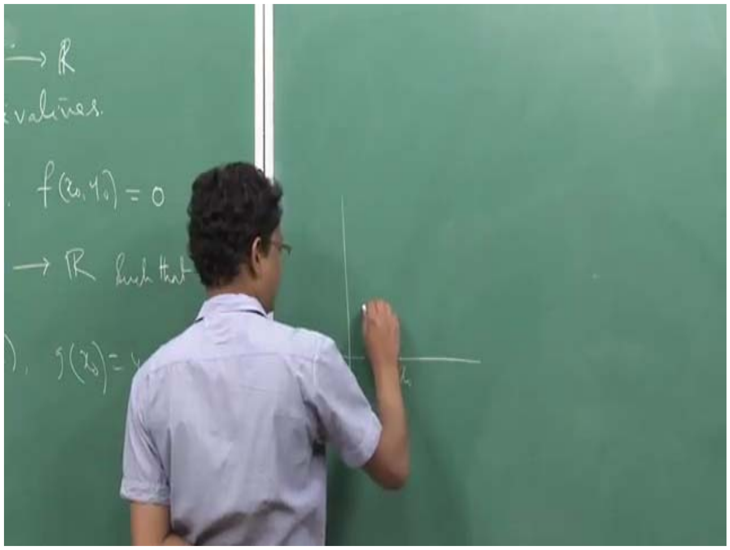

so this was the picture

where I can make g defined and for each g,

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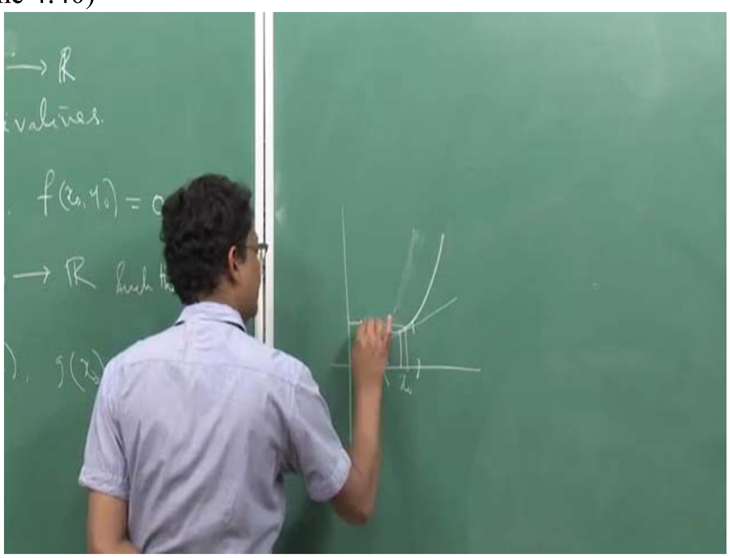

as there is an unique tangent, for each g, I can have unique y, and I can prove g is differentiable

If you look at the proof yourself,

if I, instead of R X (cross) R, if I put “R n” X (cross) R and in that case, I will have

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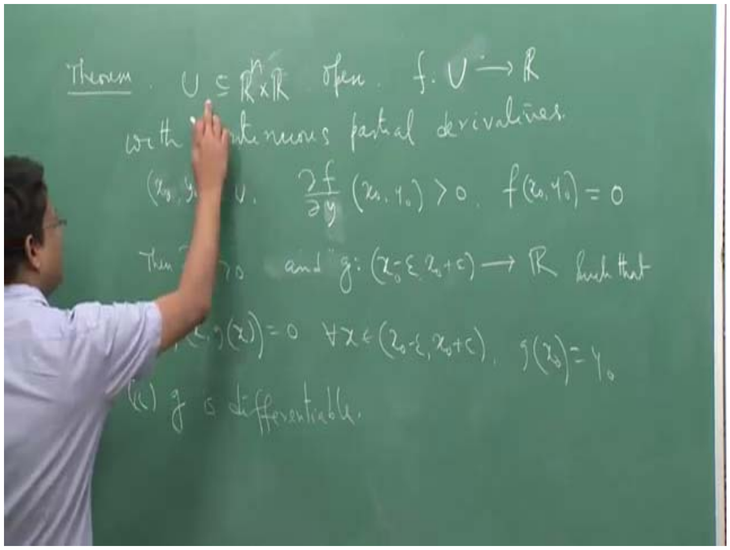

U here and “x naught” is a vector in “R n”.

“y naught” still a point in R and I have, instead of “del f”, “del y” let me say

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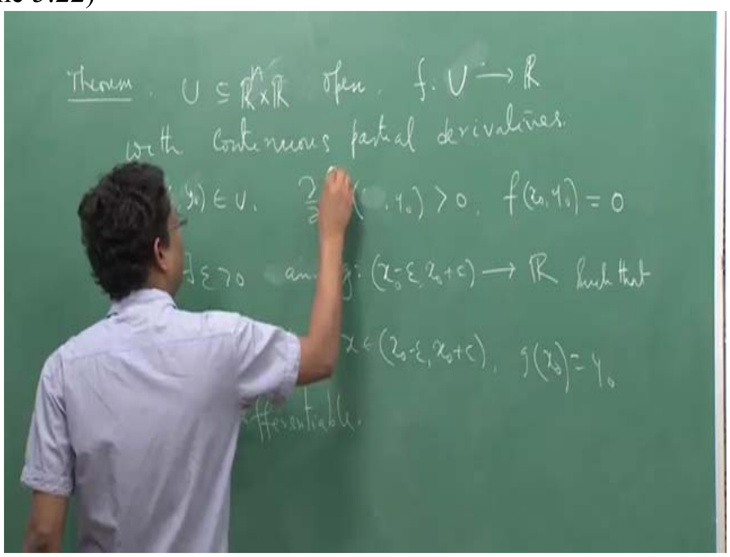

“del x i”, “x naught” “y naught”, each of them positive and

f of “x naught”, “y naught” 0

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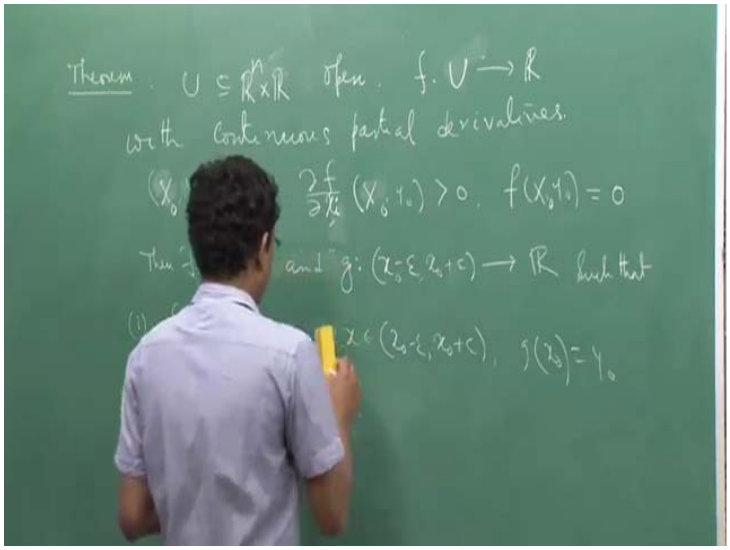

the proof verbatim

but here I have to change interval to a neighborhood of “x naught”

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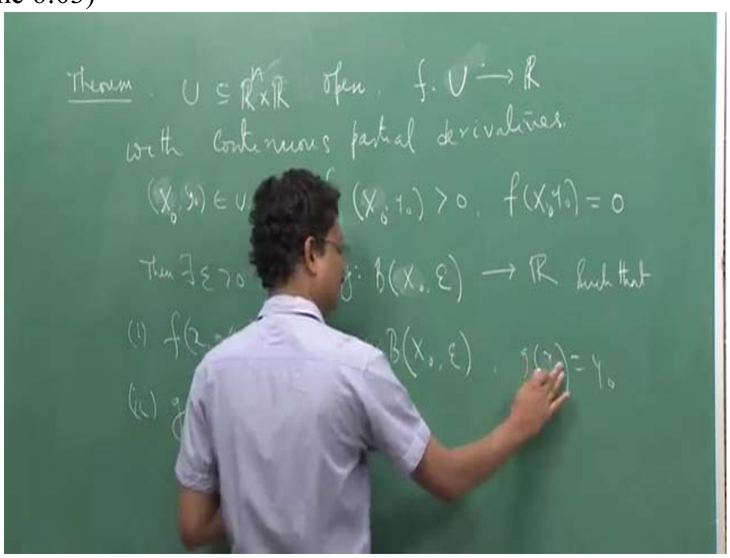

And that's it.

Proof will go exactly the same.

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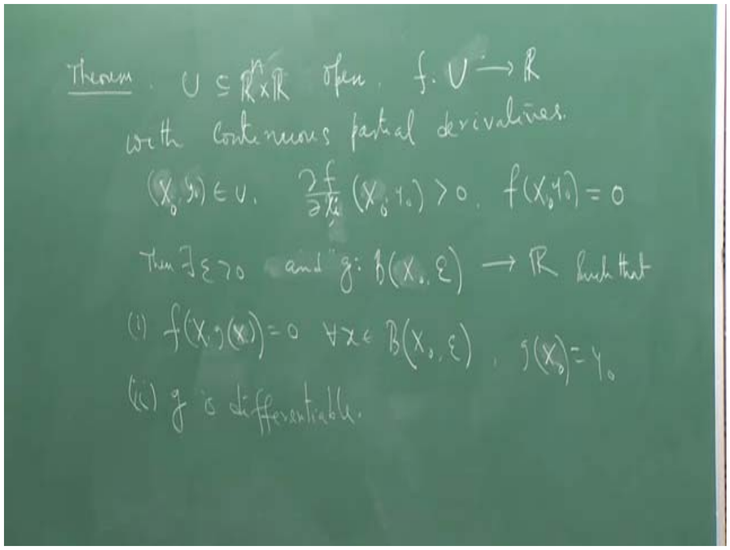

If you work it out, this is the statement for “R n” X (cross) R.

Well, the general implicit theorem is not from “R n” to R, “R n” X (cross) R to R; it is actually “R n” X (cross) R k to

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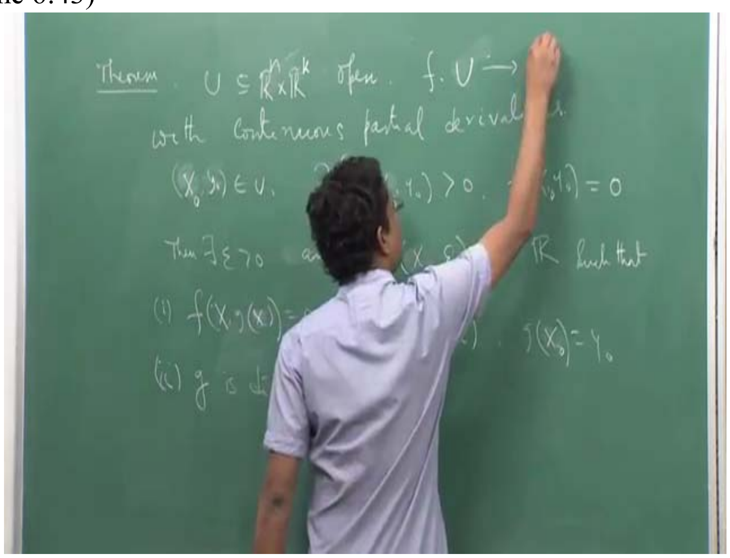

“R n”, so n and k, you can take arbitrary

and then “y naught”, where “x naught” belongs to “R n” and “y naught” belongs to “R k”.

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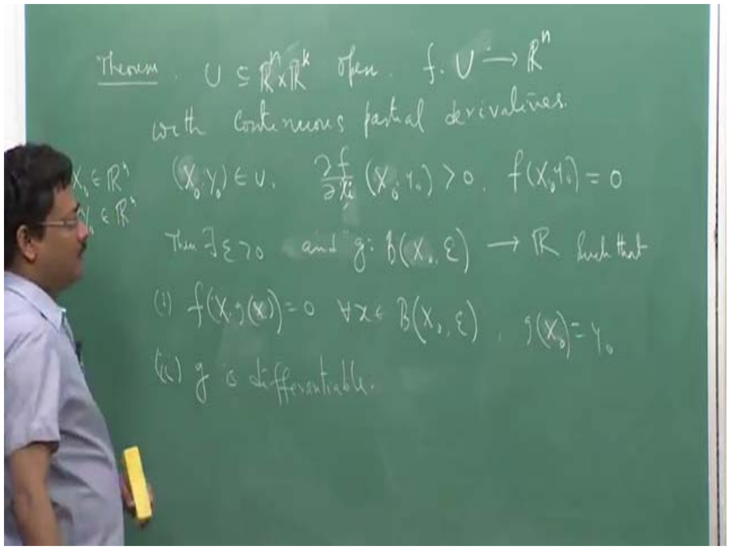

So in this case this part, I will have f, the function of...

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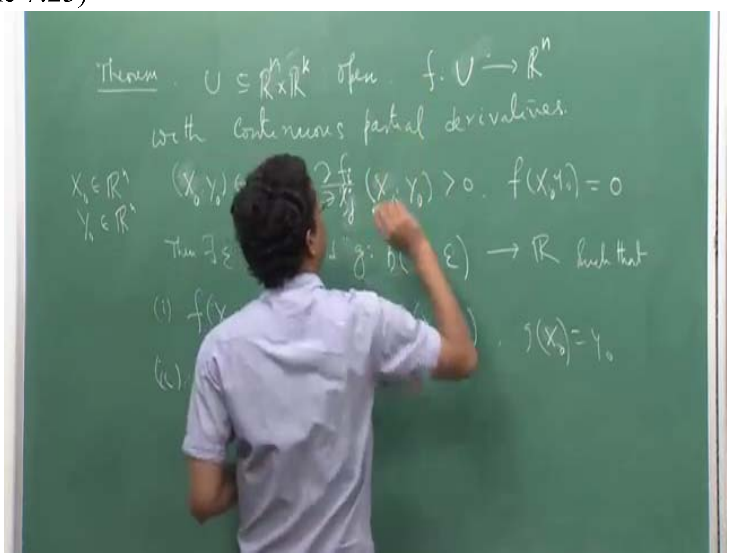

So f you write as...f 1, f 2, f n, then

f n plus 1 to n plus k

and here you take

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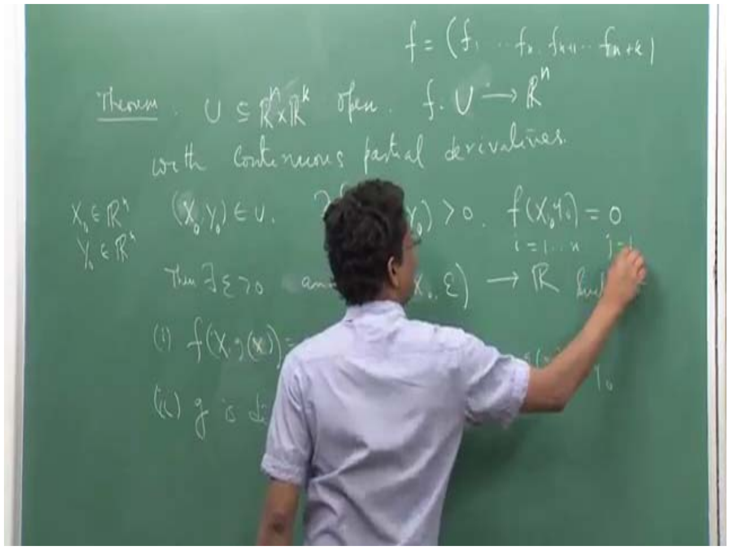

, i equal to 1 to n, j equal to 1 to n, the first component-wise partial derivatives are continuous and this fellow, “x naught” instead of greater than 0,

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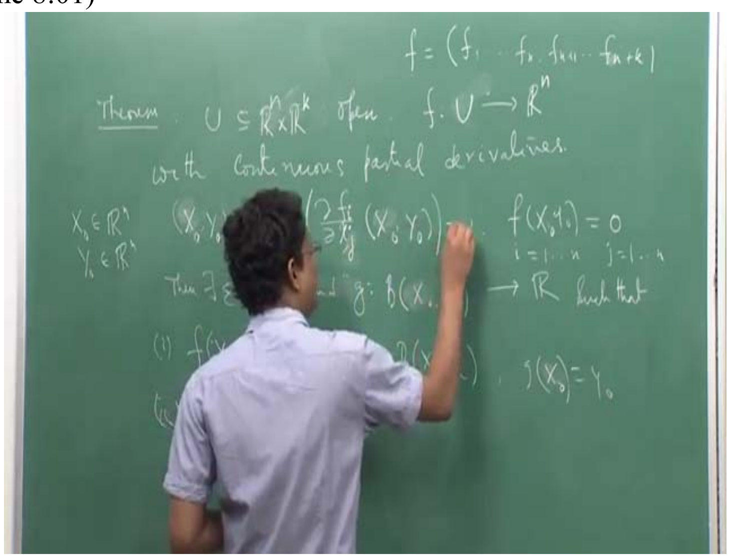

I will say, determinant is greater than 0 or non-zero

Then I will have the same, and now it will take values in “R n” such that the same thing happens

and g “i x naught”, g of “x naught” is equal to “y naught”.

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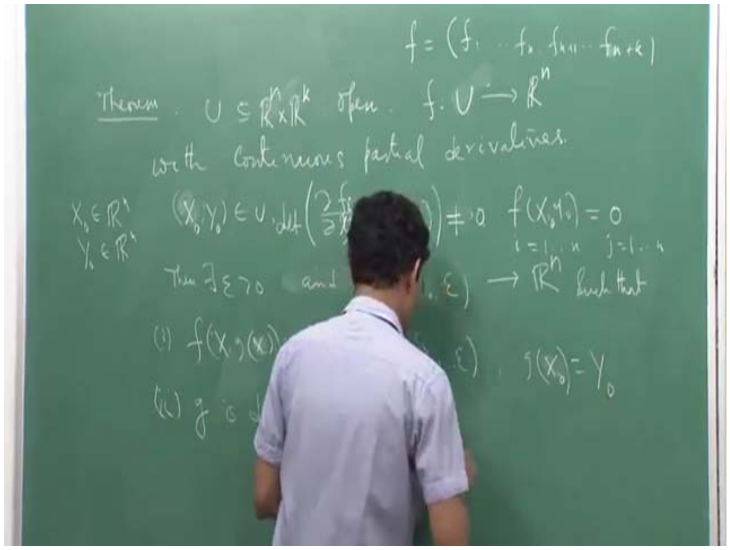

g is differentiable, with j g at “x naught” is “del f i” “del x j”

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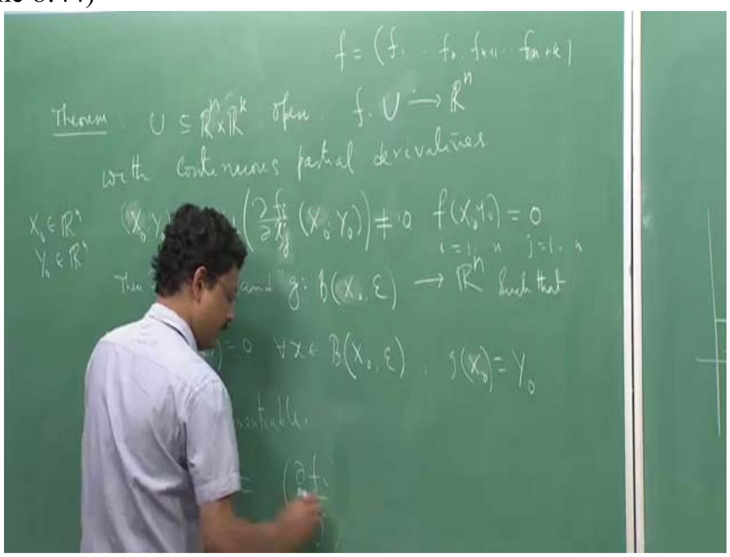

at x, y inverse, this is m X (cross) n.

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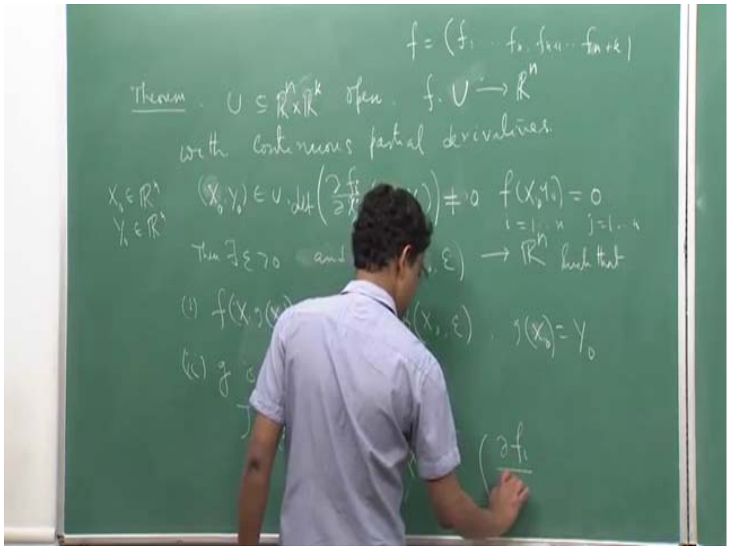

"del f i" "del x j", i equal to 1 to n,

let me put k here

k equal to 1 to …

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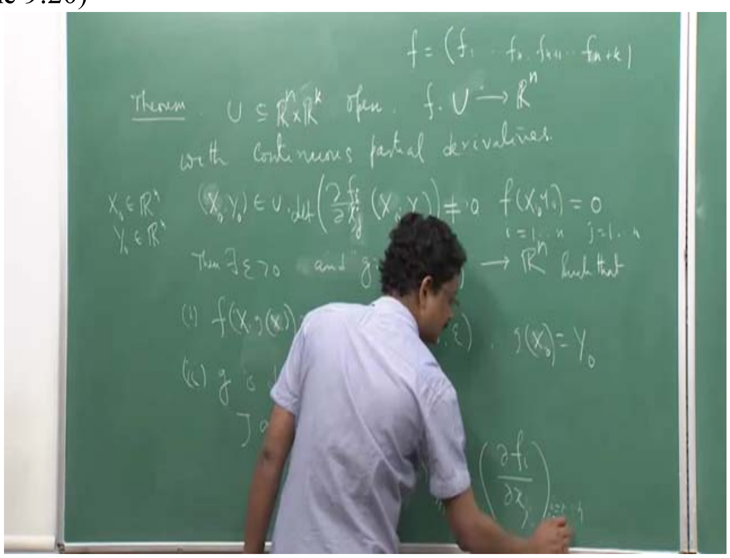

j equal to 1 to k

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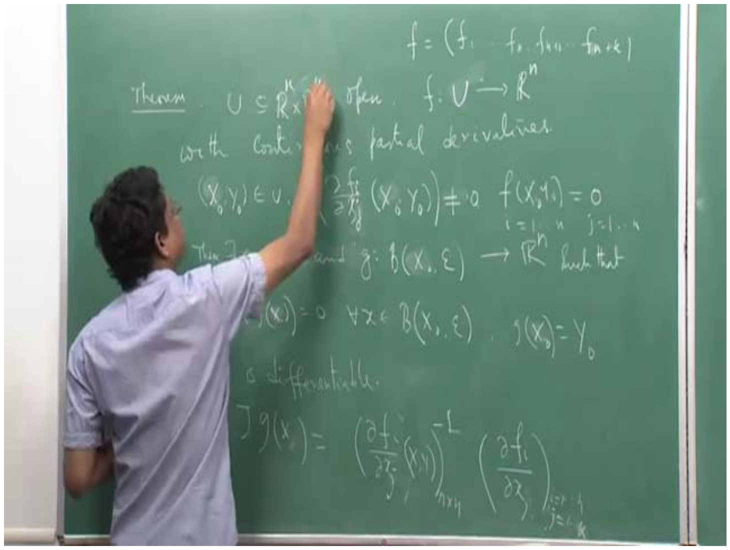

Well, this is “R n”

So this is the question of implicit function theorem, the most general form of implicit function theorem.

I think I have messed up with some n and k

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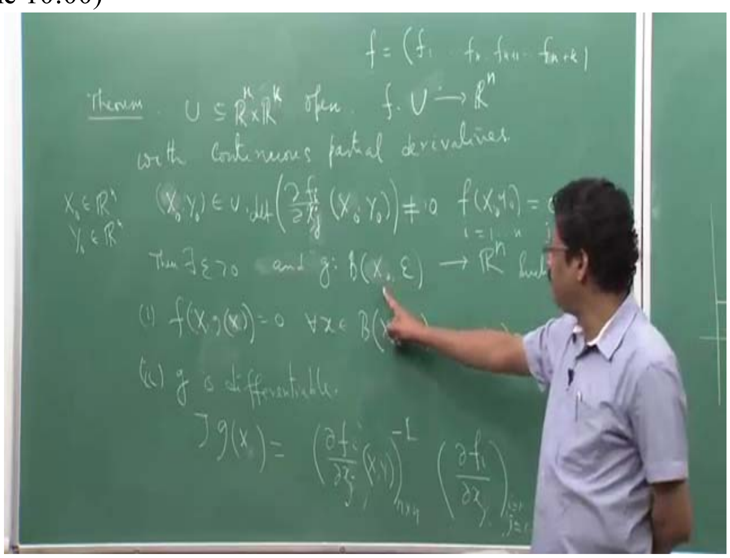

g will be from “x naught” “epsilon” to “R n”.

So I should take...Oh...Ok

So instead of, in the first version, if you assume this, instead of “del f” “del y”,

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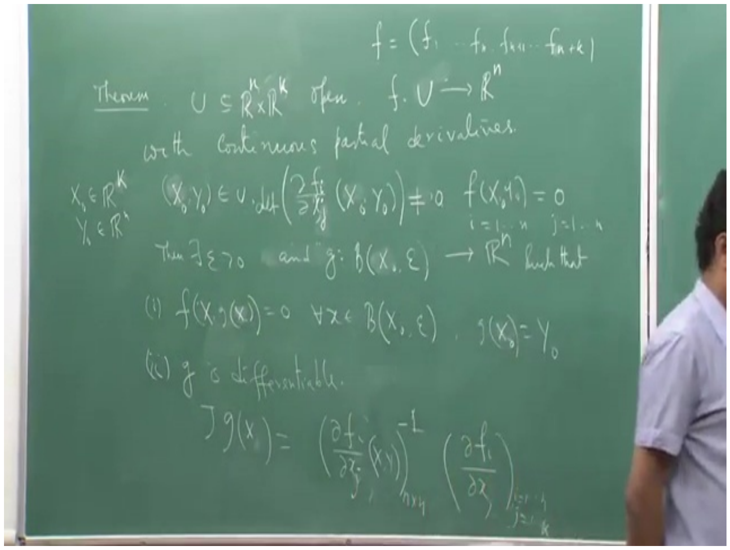

if you prove it for “del f” “del x”, you will get this.

Because the final ......matches will be n X (cross) k and that will be the derivative of g.

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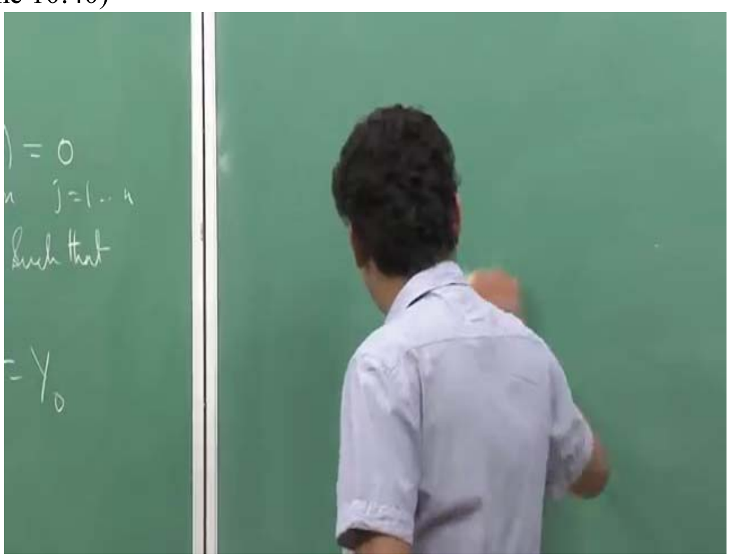

And this proof actually does not follow what you wrote down earlier but this proof follows from the earlier version,

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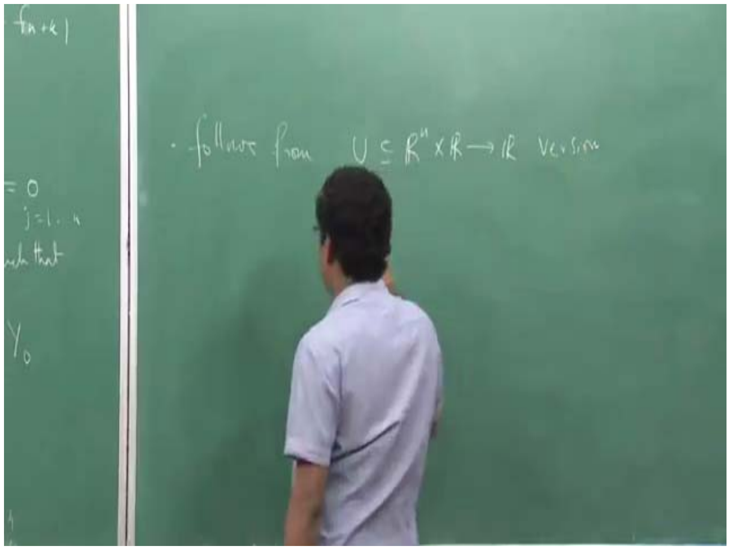

follows from U from “R n” to R version by induction

So we have to do induction on k and n both.

So I will not prove that theorem because this is just mere formality, the induction argument

But if you are really interested, it is available, very nicely available

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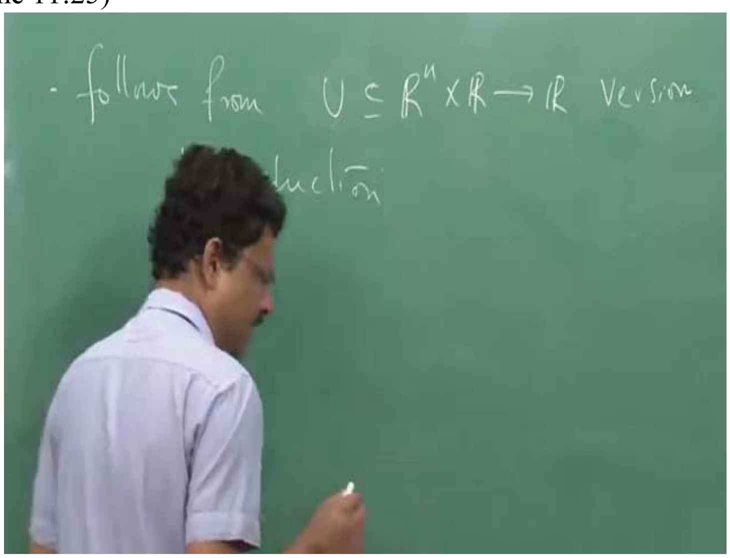

in this source, which is available in…

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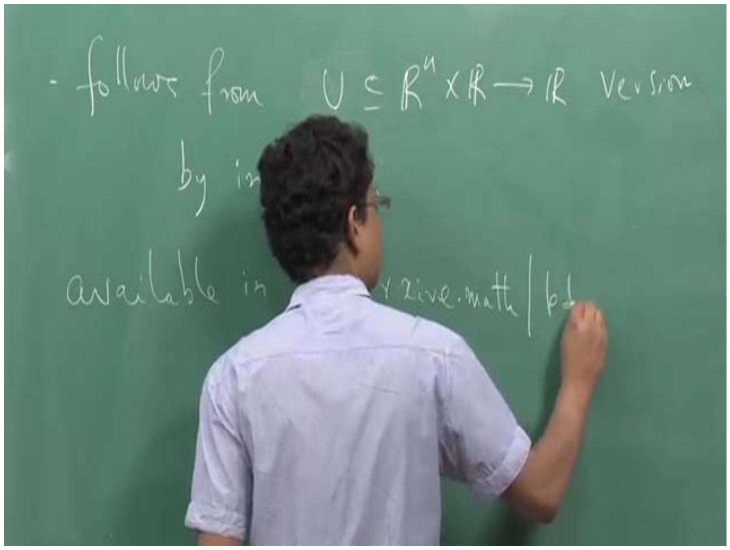

arXive.math/pdf and the number is 1212.2066

So if you check this link, you will find the proof by induction from the previous version,

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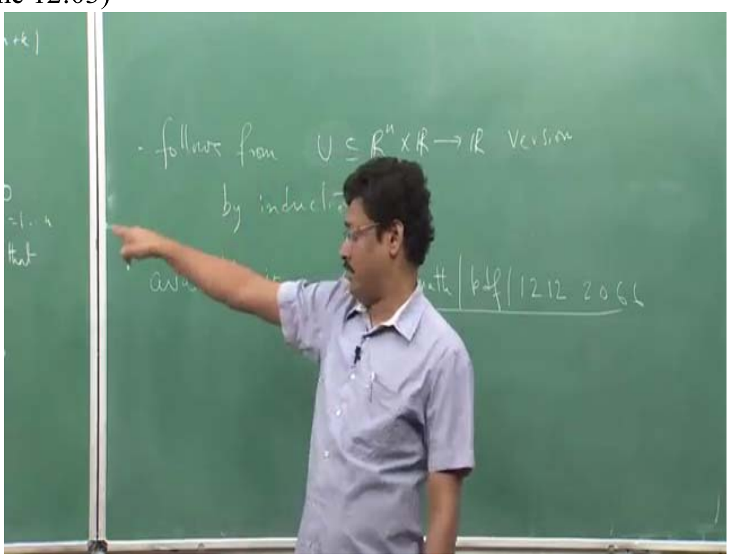

this proof will be based on the induction and it is written very nicely here.

And there are some other comments about implicit function theorem, and I want all of you to check that Source

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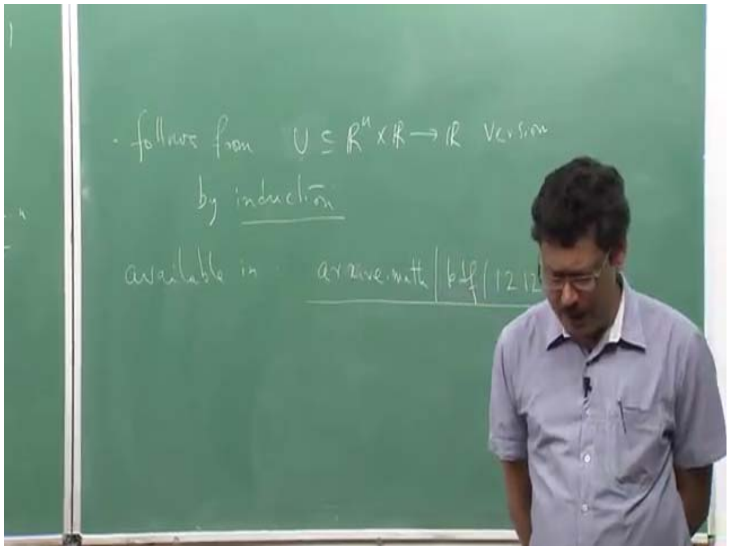

So this is all about the statement and proof of implicit function theorem.

So what you want to do now, you want to do some applications.

And if you talk about the applications of implicit function theorem, I mean

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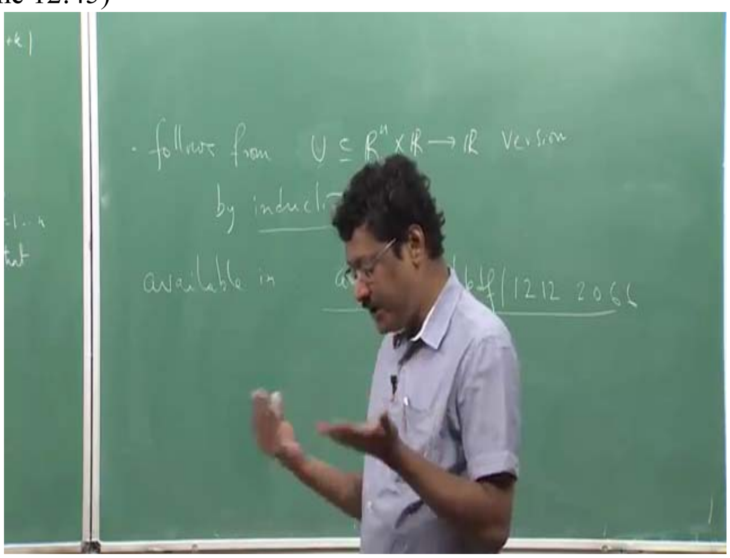

the entire subject of differential geometry that thrives on the statement of implicit function theorem So, we will need... we could talk about in a full course in implicit function theorem and in different areas.

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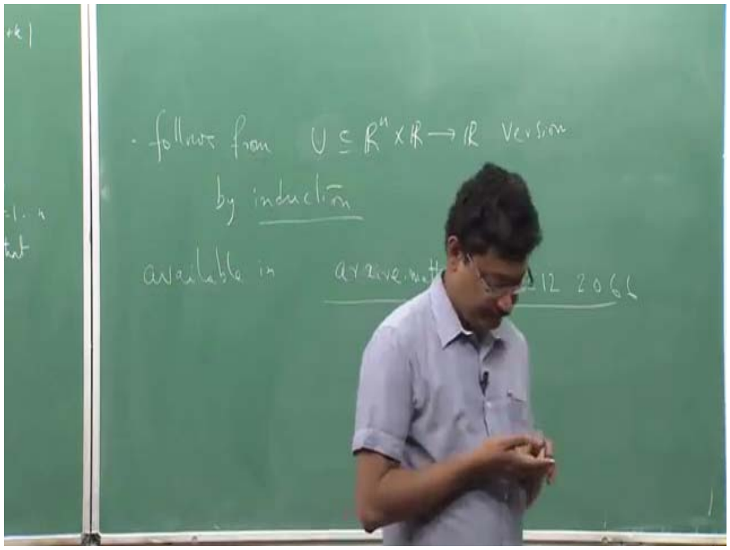

But for this course, we will take up only 2 very important ones in part of this syllabus of differential calculus course

first of them is Lagrange's Multiplier Methods and second is Inverse Function Theorem.

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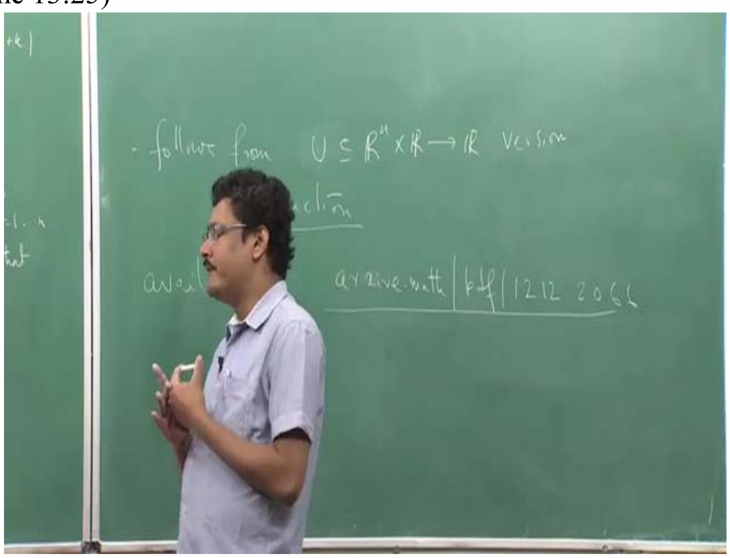

Inverse function Theorem, I will talk about it later, may be next to next lecture

Uh

But for today, we will restrict an application,

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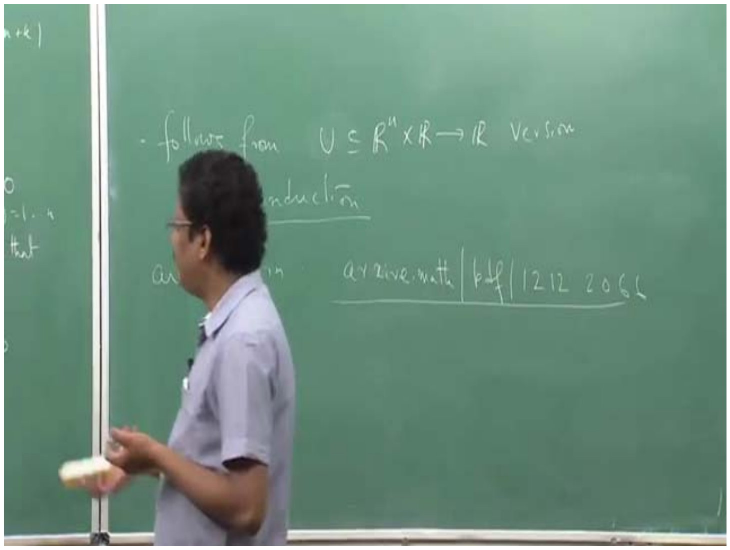

a direct application of implicit function theorem in so-called Lagrange Multiplier Method

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So, let's talk about Lagrange Multiplier Method?

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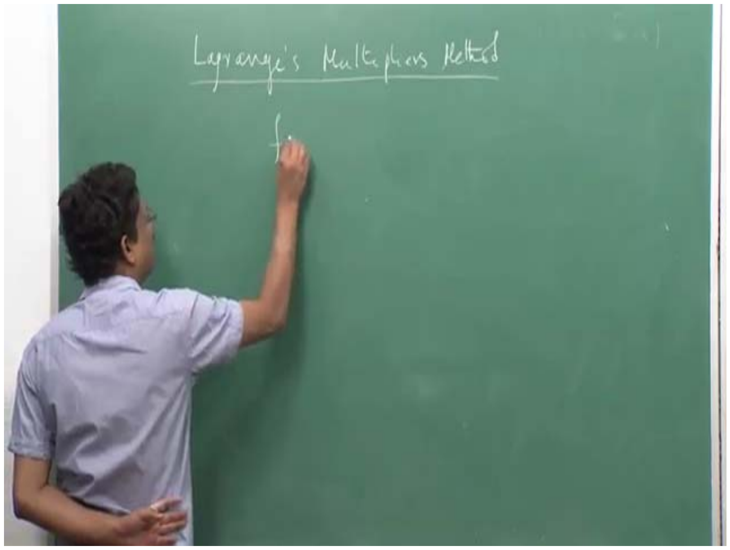

What it says, it says to find extremum, both maximum value or minimum value of a function f from "R n" to R, or say f from U to R, U in "R n" open set

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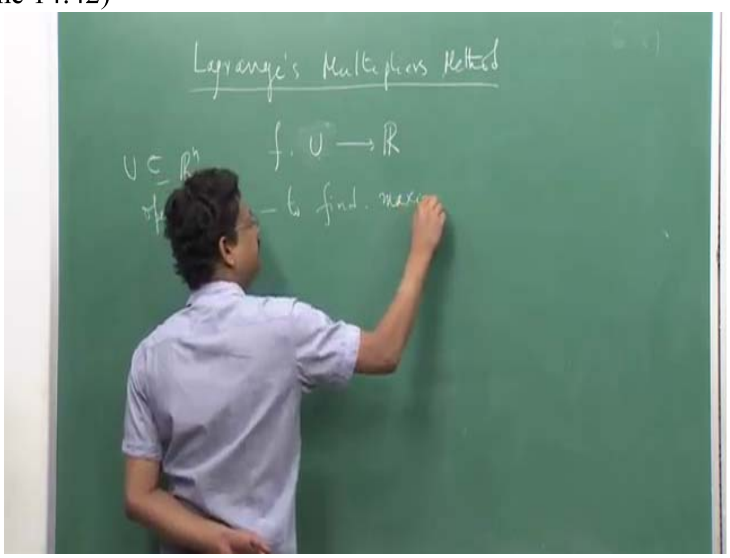

To find maximum or minimum, in general extremum of f, we have done this problem

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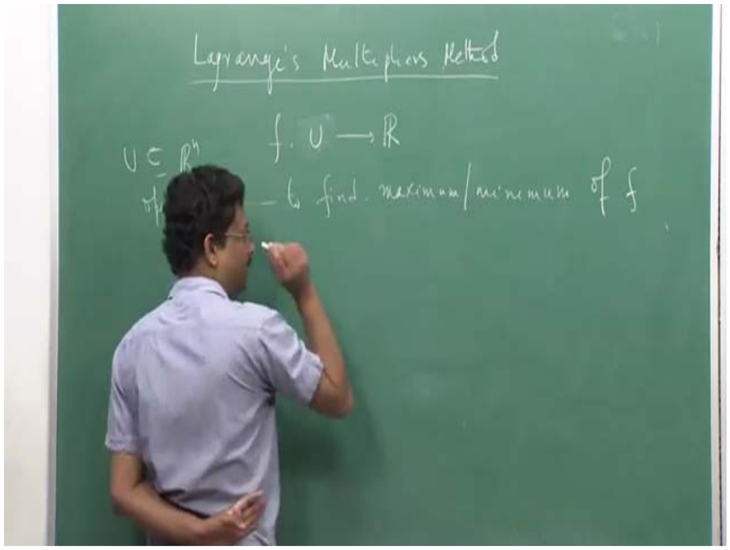

But now there is one constraint that is subject it to...

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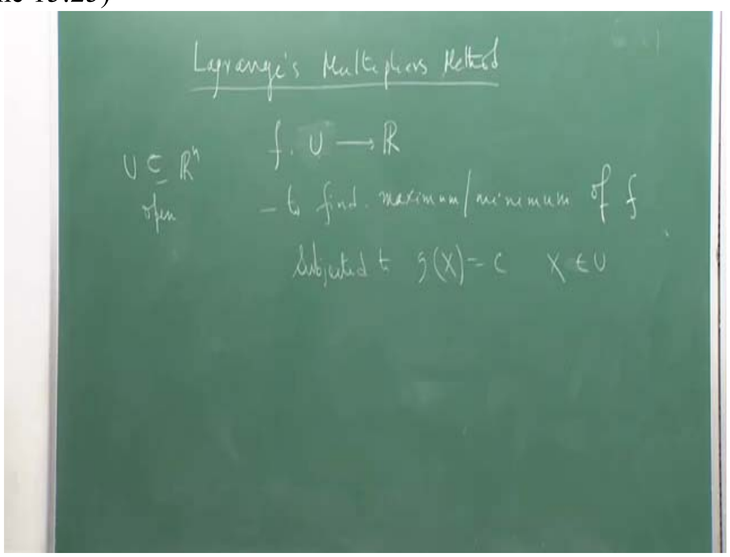

So you can think of it like this, that on a surface or curve whatever of g, f is defined on entire "R 2" but you have to find the extremum of f with respect to this constraint

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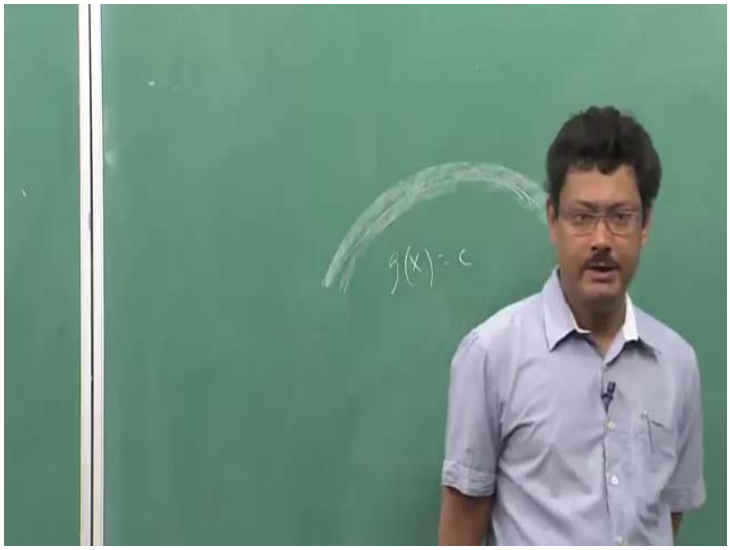

g x equal to c.

For example, in the assignment you will find many.

So we solve this example after we develop,

let us say,

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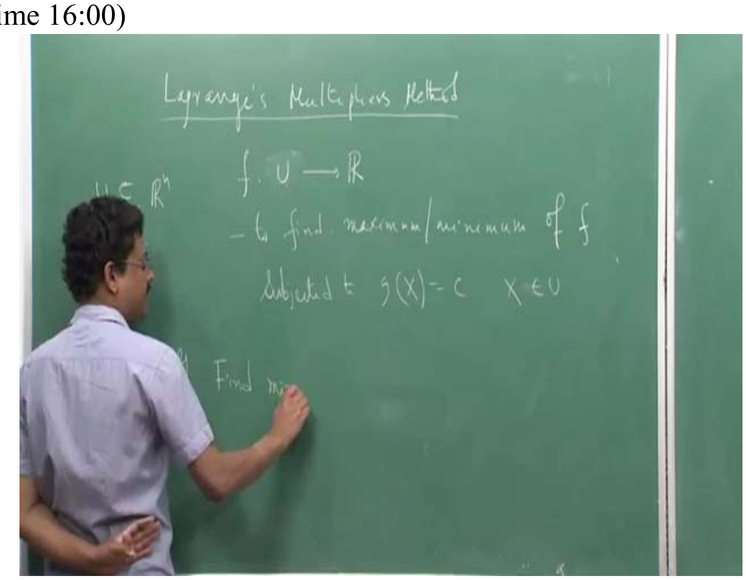

find minimum distance from point let us say, something

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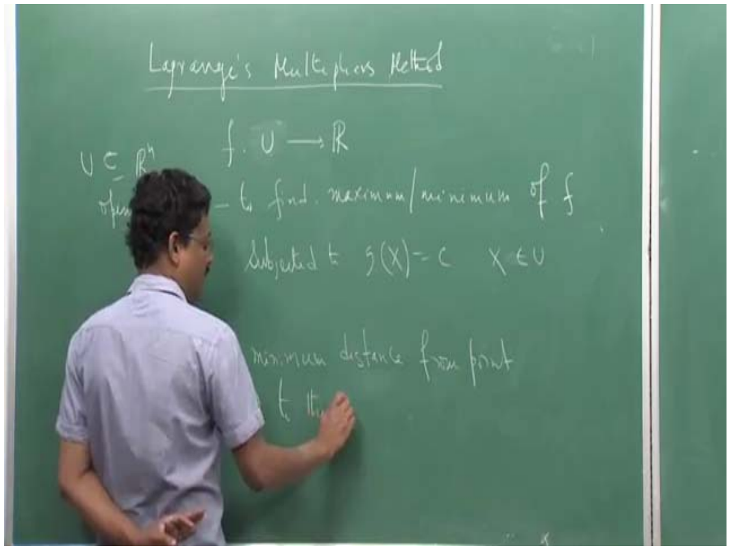

..-5, 0 to the curve or to the parabola y square plus 4 x equal to 0

So, what is the problem?

So, two dimensional problem so I have this parabola,

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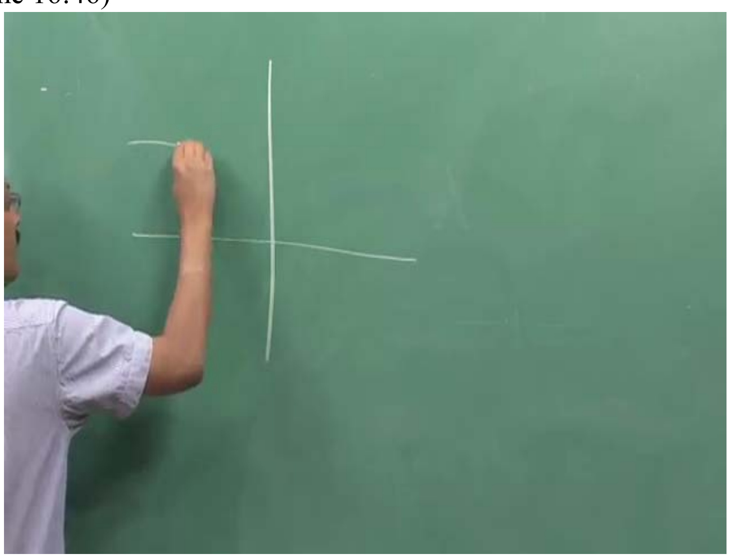

y square plus 4 x equal to 0.

So this is the parabola and I have the point minus 5, 0 here

I have to find the minimum distance

Say, from this point you can draw these straight lines, different straight lines to parabola

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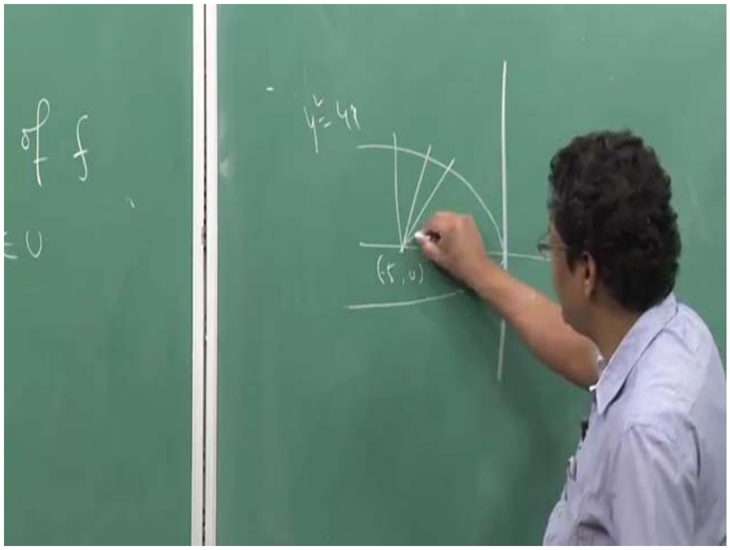

, I have to find the shortest one

So this is particularly, find the...so your f x will be f of x y will be ...

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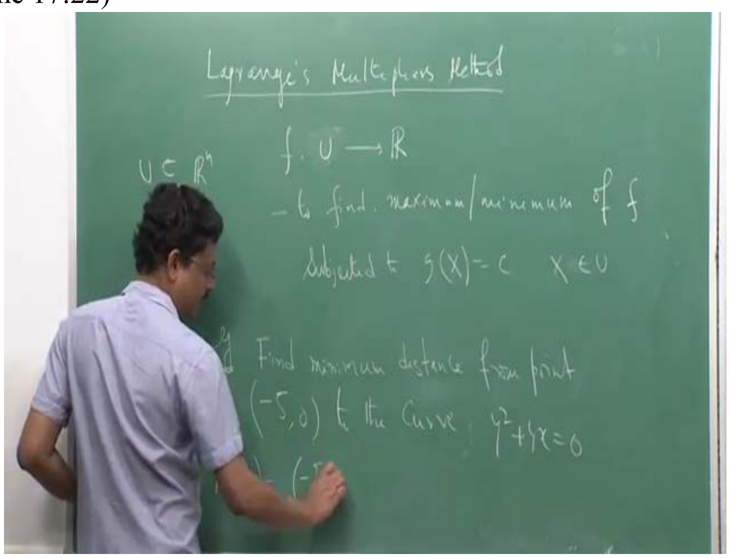

minus 5 minus x square y minus 0, y square

So this function has to be minimized subjected to y square plus 4 x equal to 0.

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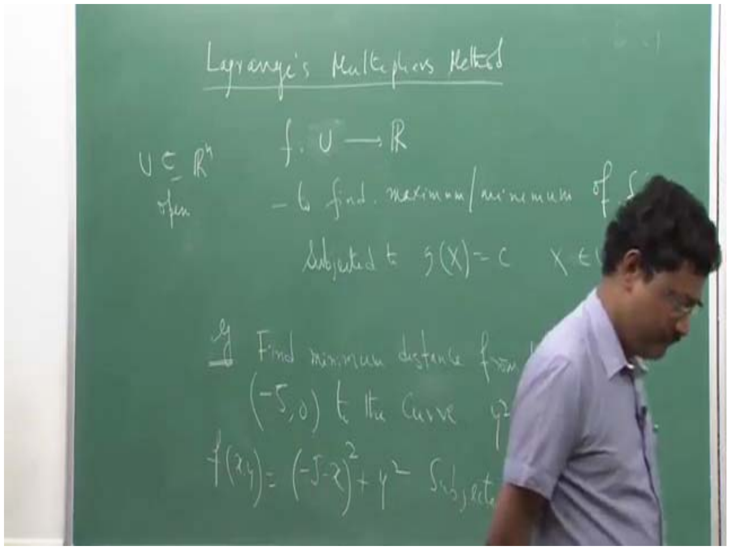

It is a typical example of Lagrange Multiplier method.

And it can be done...Ok, this is from “R 2” to R, function from “R 2” to R but again as I said, if you understand from “R 2” to R,

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you know everything from “R n” to R.

Of course I will take functions taking values in R, otherwise maximum, minimum does not make sense

Like implicit function theorem, if “R 2” to R concept is clear, from “R n” to R is just formalities.

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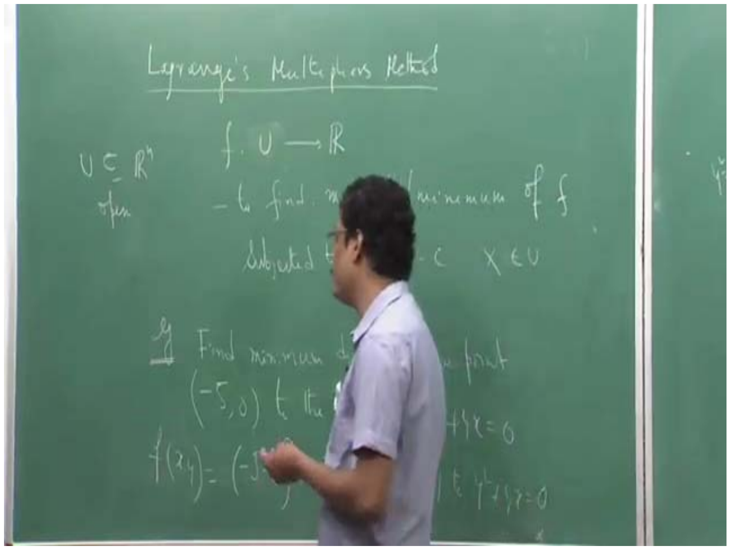

So I will illustrate both but as far the illustration of the method is concerned, I will take U in "R 2".

Like this example.

So let us see,

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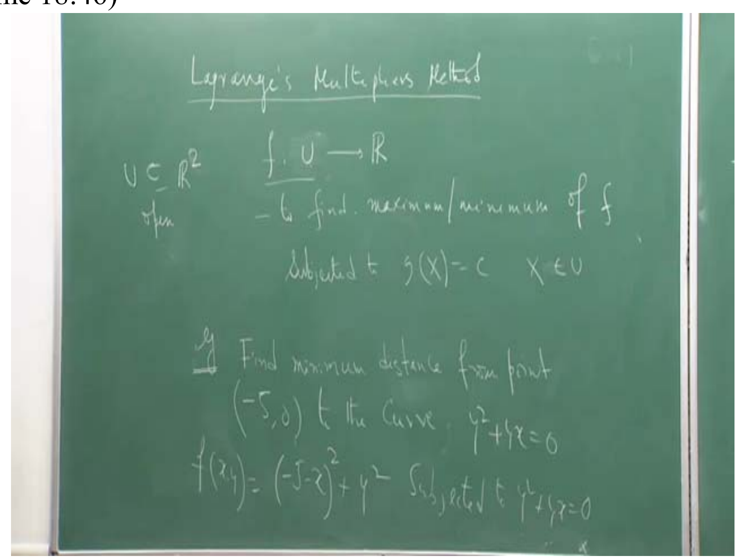

how do you solve this general problem?

SO how do we go about it?

We have f and we have some constant and as I said you are able to use implicit function theorem.

How?

Well

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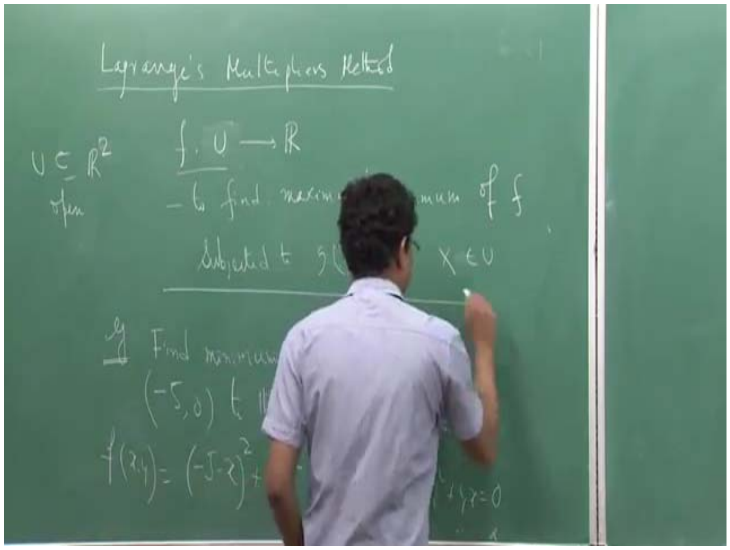

Suppose “x naught” “y naught”, this is an extremum.

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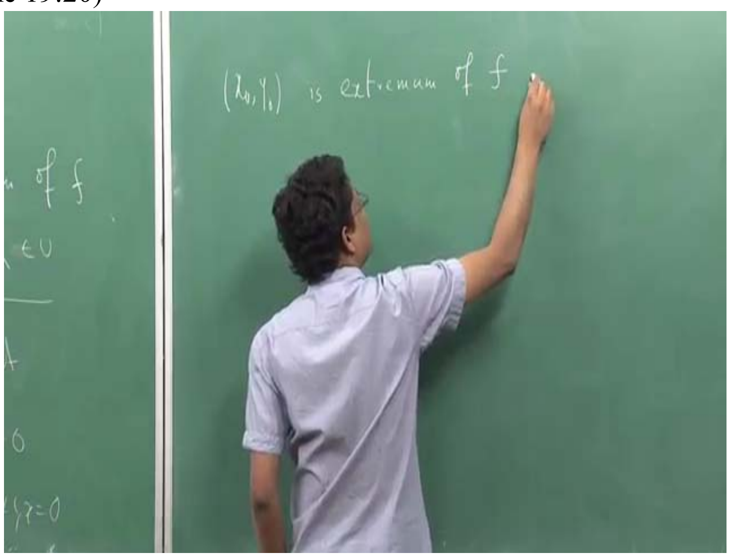

That is maximum or minimum of f subjected to g “x naught” “y naught” equal c.

Suppose we had a extremum, we may not have.

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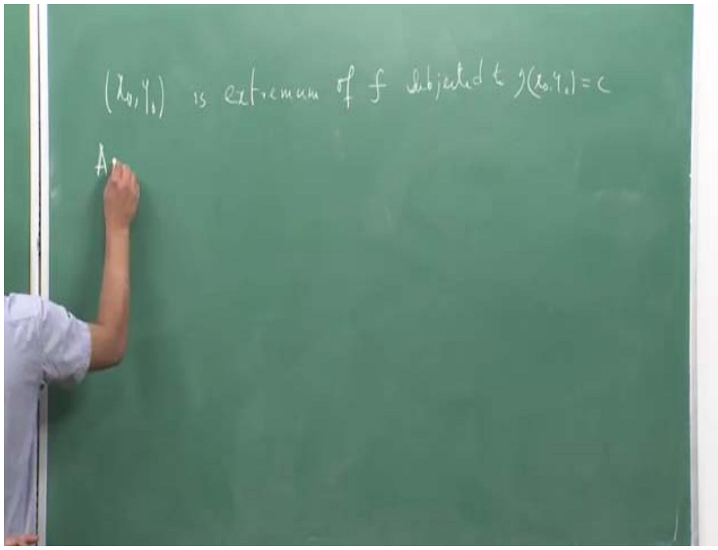

So we will assume nice property of f and g

Both f and g has continuous

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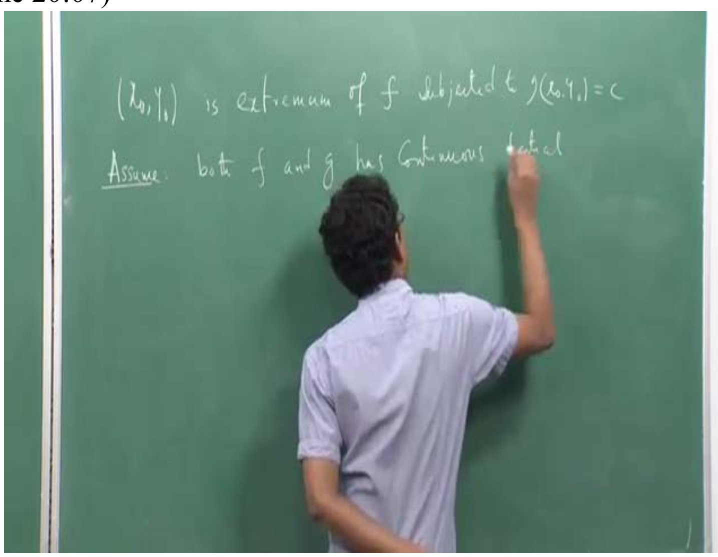

partial derivatives

This I need to apply Implicit Function Theorem.

Ok

Let me get back to the example.

What will we do?

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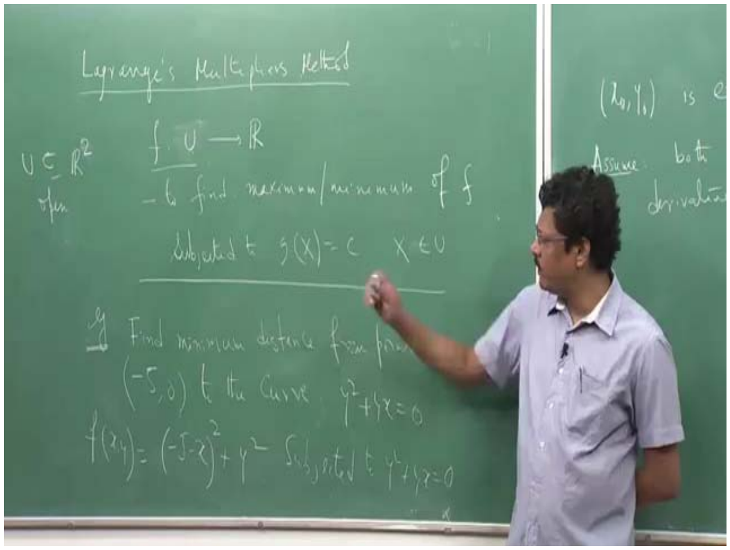

If I don't know the Lagrange Multiplier, what I will do?

What I will do is simply this

That I am looking for the curve y square equal to 4 x

So, on this curve y square equal to minus 4 x, x has to be negative

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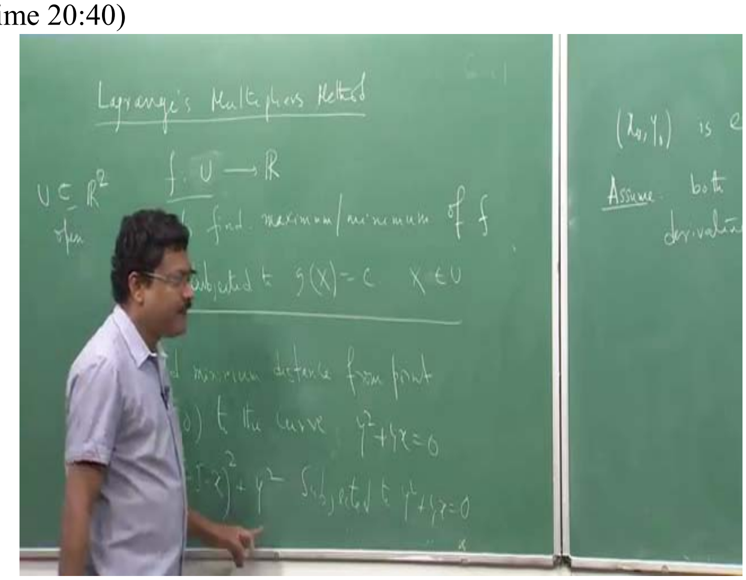

So actually the problem boils down to, for this example, that you look at this function, f x minus 5 minus x, that is, minus I can consume in the square

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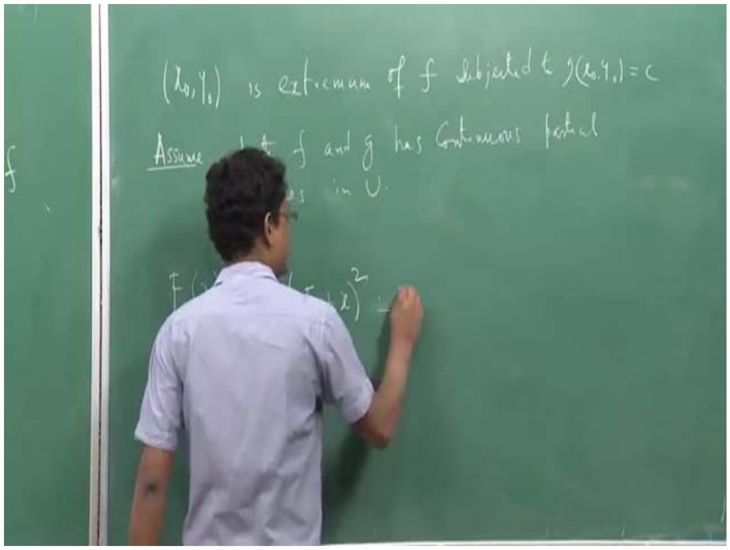

and then y square equal to minus 4 x.

So basically, this y square part, this y part I can avoid by replacing by a function of x only, which is 4 x here

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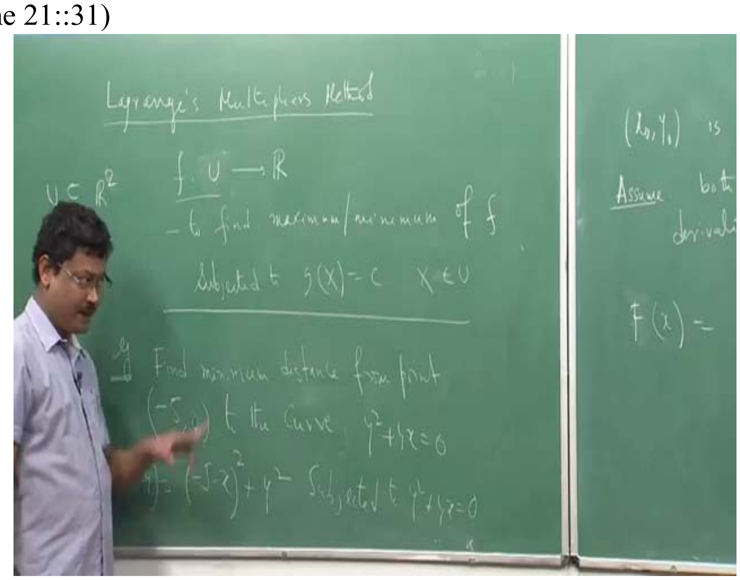

, minus 4 x here.

Now what does implicit function theorem says?

That Ok this is a level curve. Every level curve, if it is a nice function, then it may not be a graph,

So this level curve is a graph of the function

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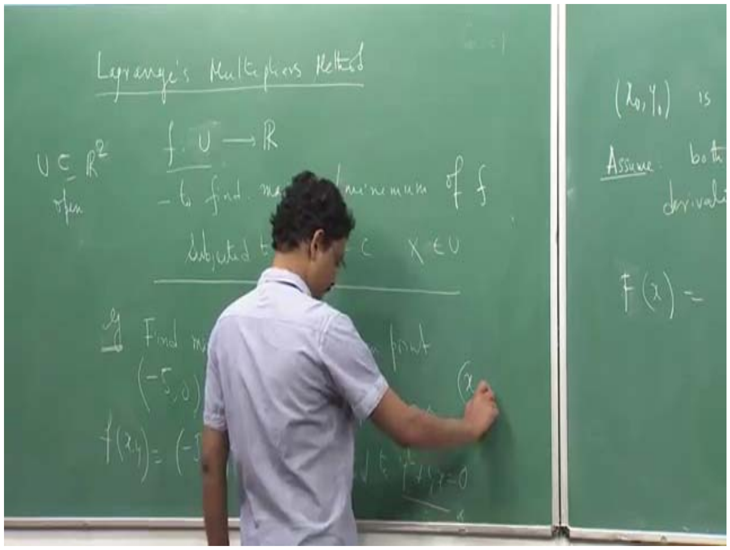

, x y square equal to minus 4 x.

So y equal to root over

So 2 minus x,

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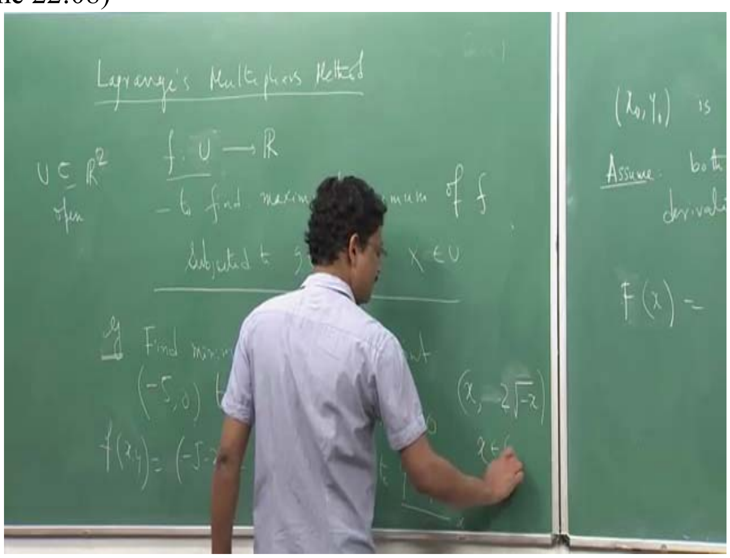

x in...x has to be negative to make this...

So this is graph of this function, locally and plus/minus

Two functions are there, one this and another is

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x minus this thing...

Implicit function theorem says that Ok, this level curve, I can locally look at as a graph of some function

Because if I look at the upper function

This is the parabola,

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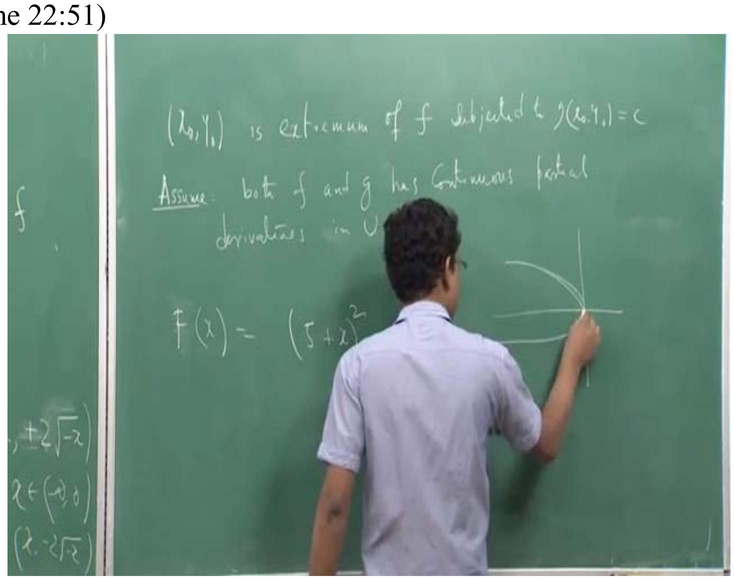

If I look at some portion here, then I can define the function g

For this portion I can define the function g which is

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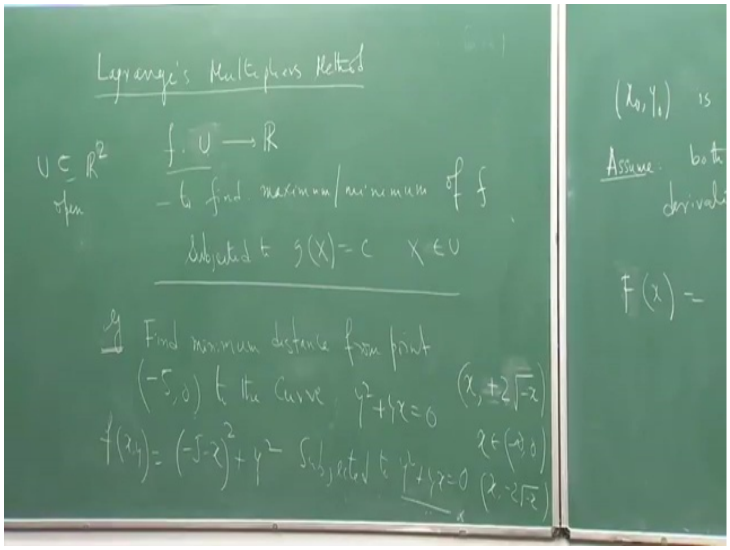

x plus 2 root 2 minus x

And for this portion, the second one, this one

So, I can replace this condition f x y and g x equal to c by a function x only

So that is the entire idea and implicit function theorem I can do it because I have assumed

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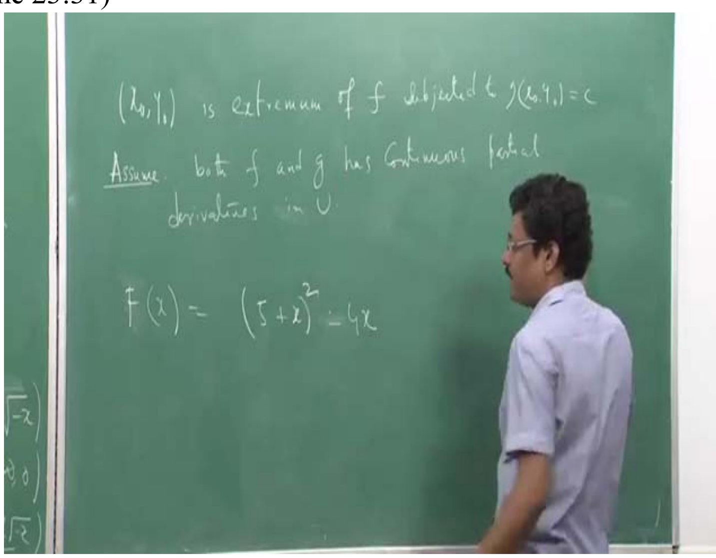

f and g has continuous partial derivatives

So the problem by implicit function theorem

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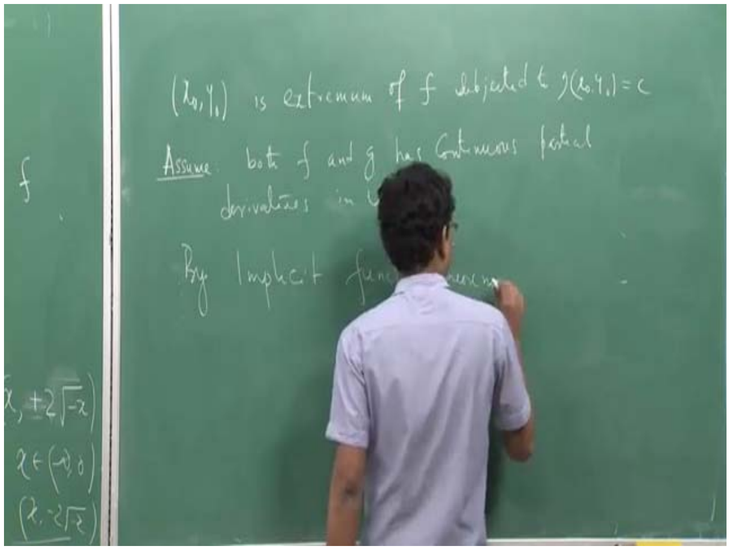

In a neighborhood of “x naught” “y naught”, let us say theta

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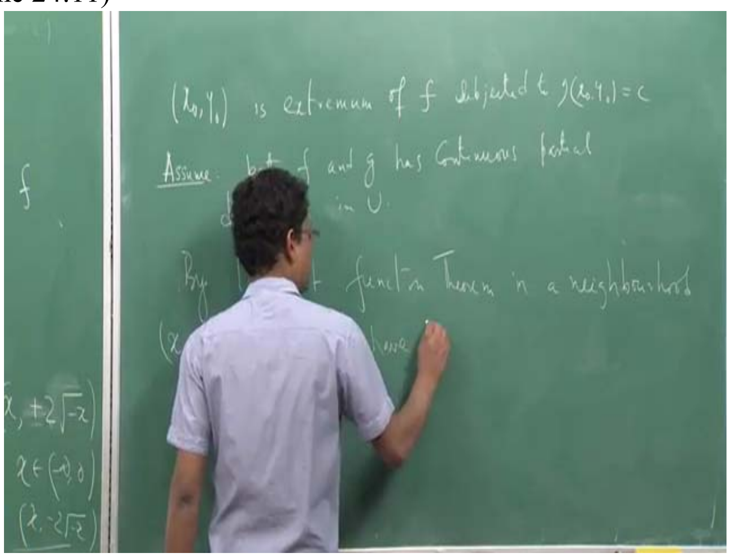

we have y equal to some function g x, from theta to R, right?

So the problem reduces to

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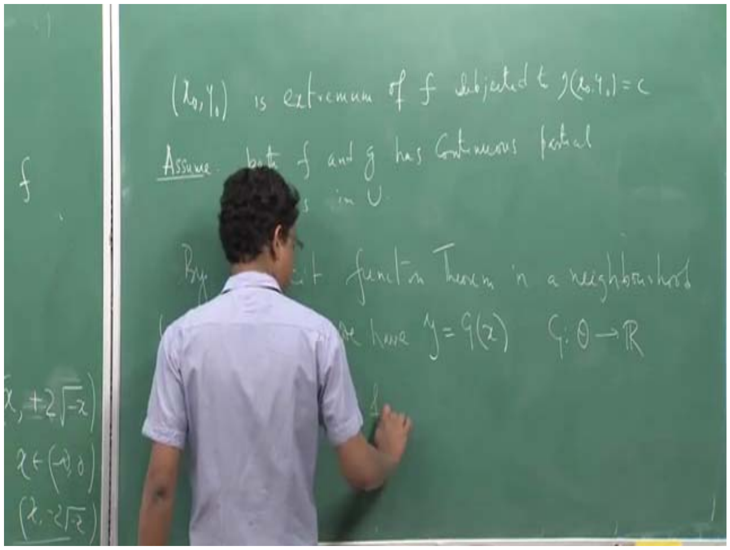

finding extremum of, here we are only interested

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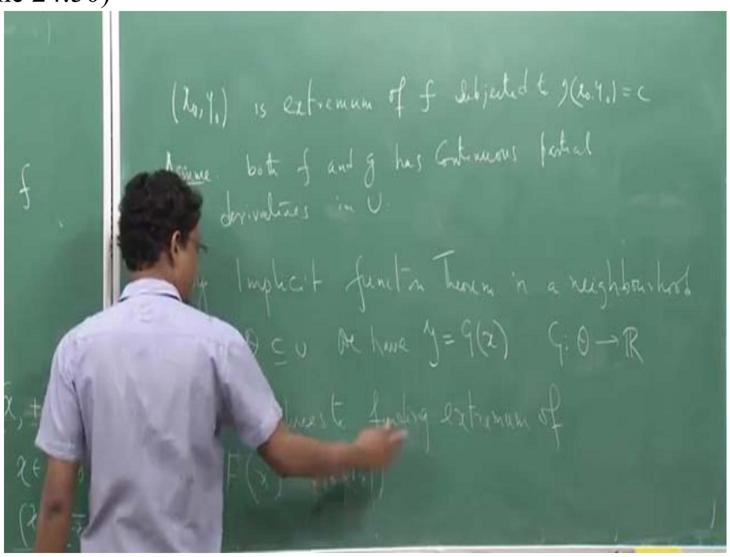

in local maximum, local extremum g x subjected to which one

g “x, g x “equal to constant

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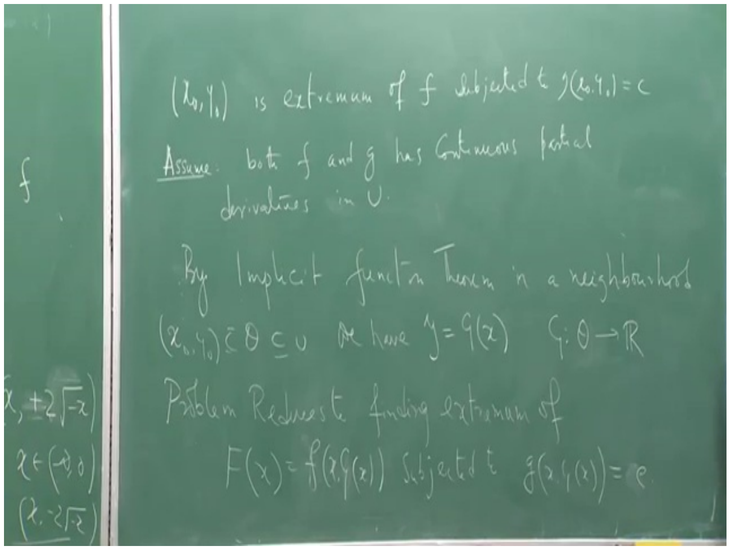

This is the function of x only

So here I have used Implicit Function Theorem in the neighborhood of “x naught” “y naught” as I have assumed f and g both as continuous partial derivatives

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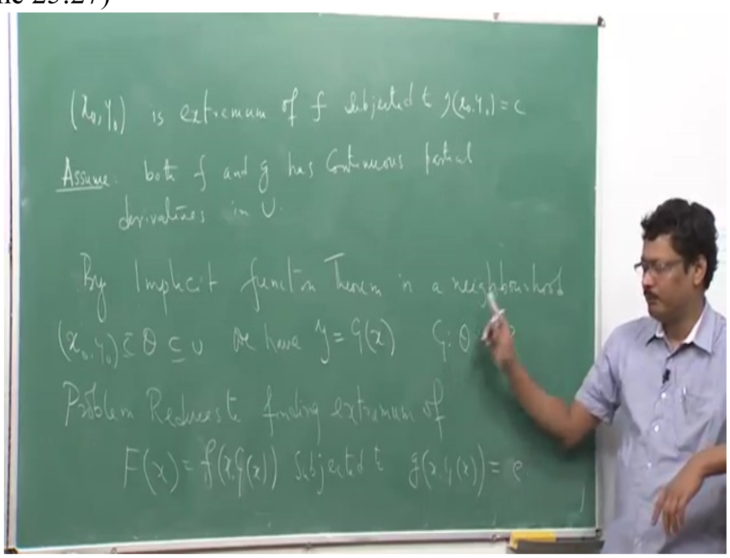

So if you understand this part, then next thing is then, again

because I have made it function of x only, so it is like again finding maxima, minima for real valued function

For two-valued function, partial derivative will be involved.

And then I have to put condition that

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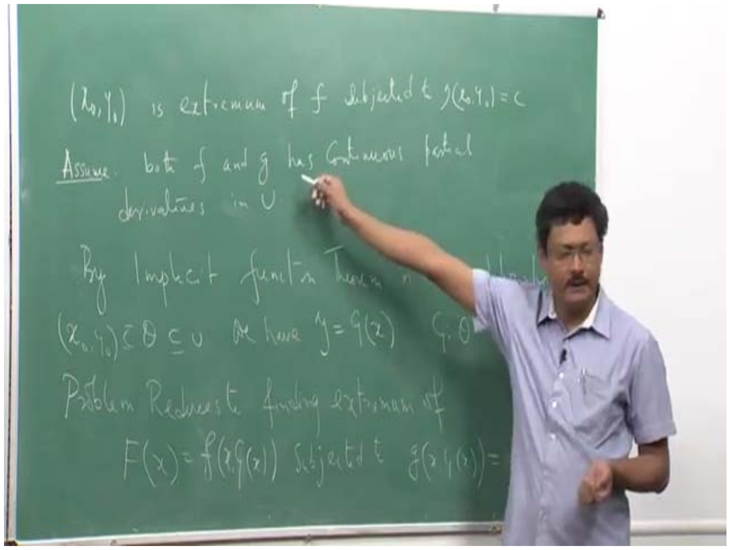

g does not vanish, the partial derivative of g does not vanish at “x naught” “y naught”, so I should have

del g “del y” “x naught” “y naught” not equal to 0

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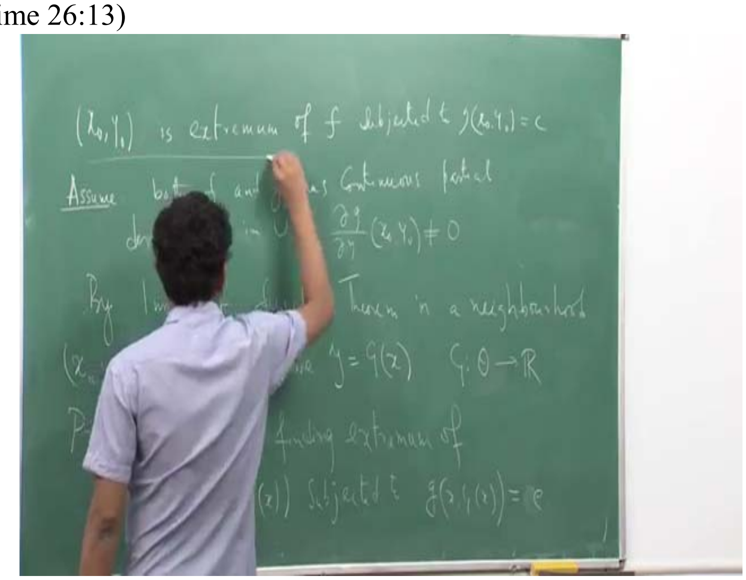

supposing this is negative.

So, why bother too much?

Let us say, this is not equal to 0, or let us say greater than 0, without loss of generality

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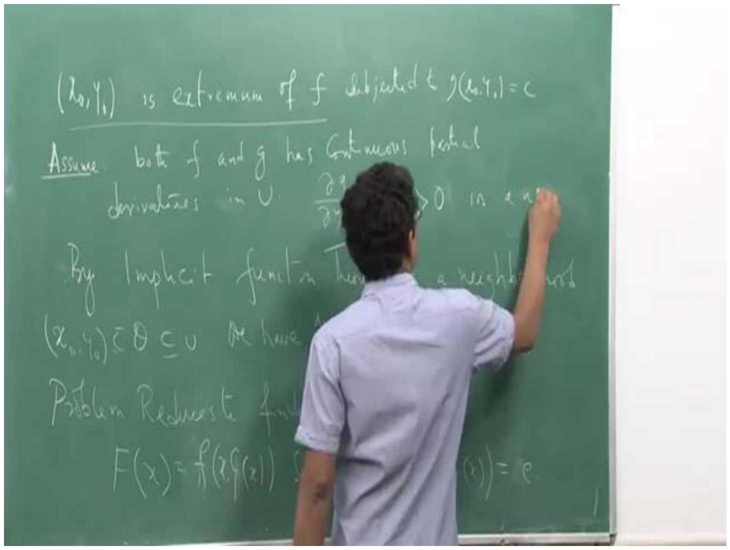

in a neighborhood of x

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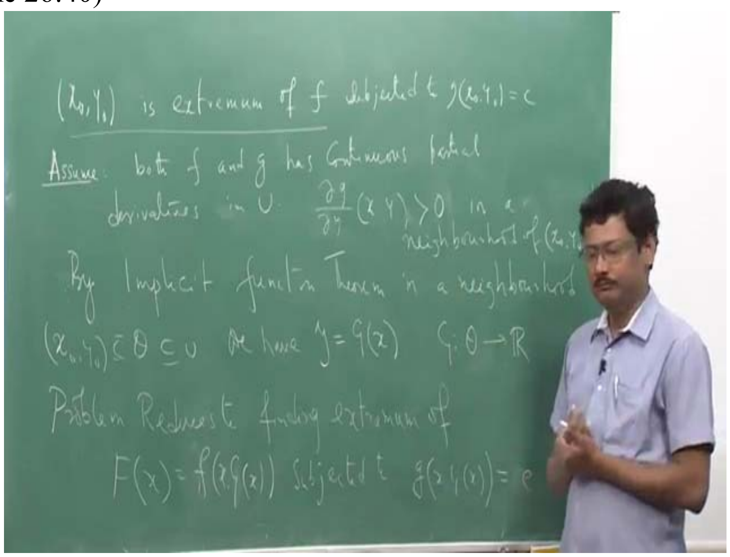

And then I can express y as g x and the problem reduces to function of one variable and we know that

I will illustrate in the next lecture.

We will continue from here,

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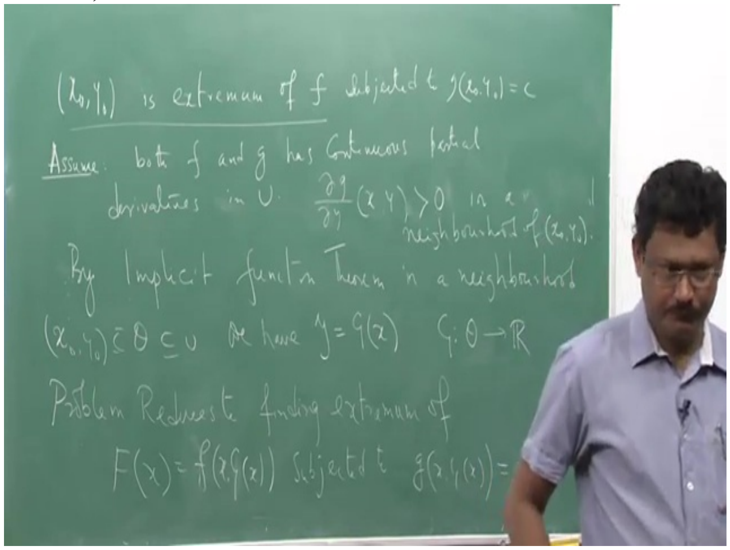

Thank you

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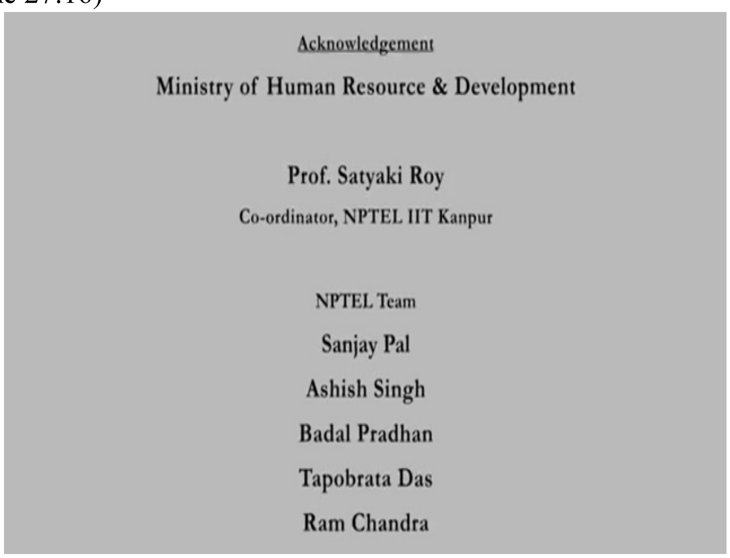

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Dilip Tripathi

Manoj Shrivastava

Padam Shukla

Sanjay Mishra

Shubham Rawat

Shikha Gupta

K. K. Mishra

Aradhana Singh

Sweta

Ashutosh Gairola

Dilip Katiyar

Sharwan

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Bhadra Rao

Puneet Kumar Bajpai

Lalty Dutta

Ajay Kanaujia

Shivendra Kumar Tiwari

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