Differential Calculus of Several Variables
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Mod 04 Lecture Number 16

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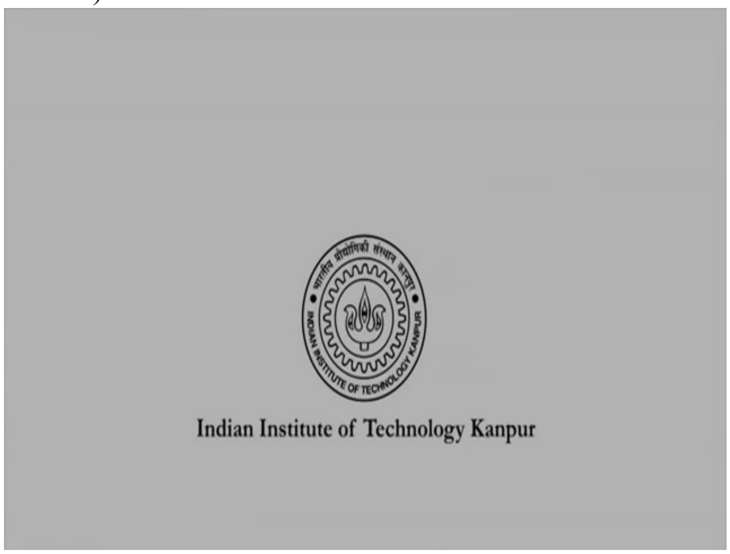

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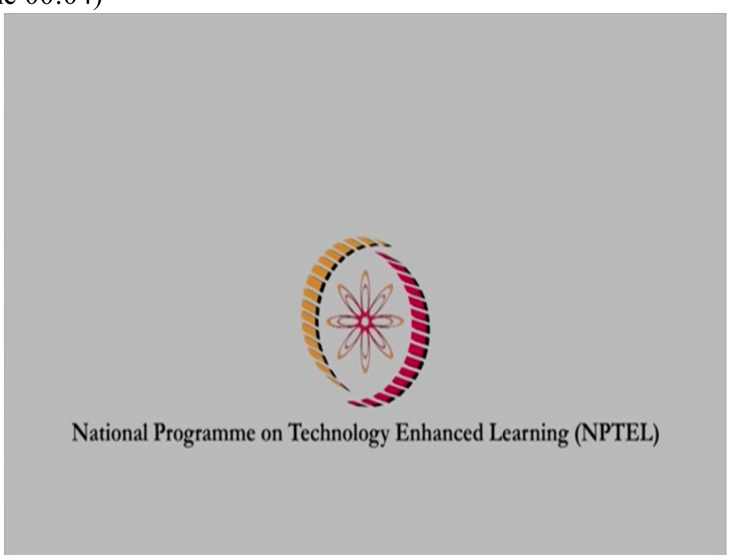

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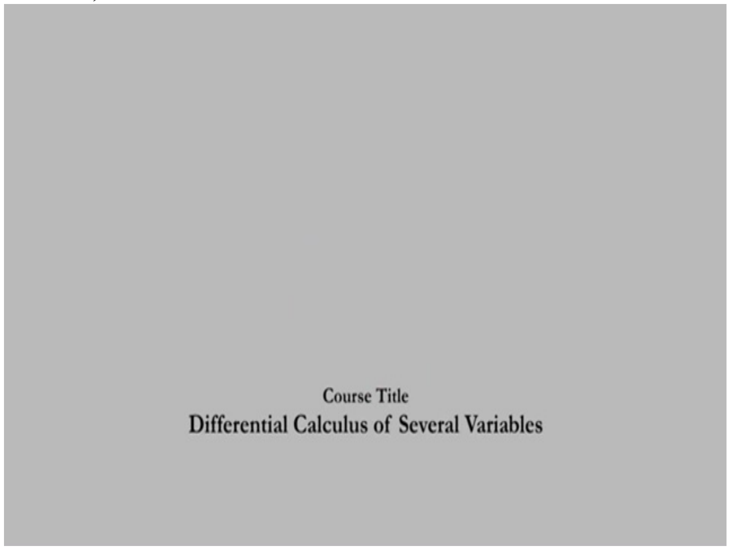

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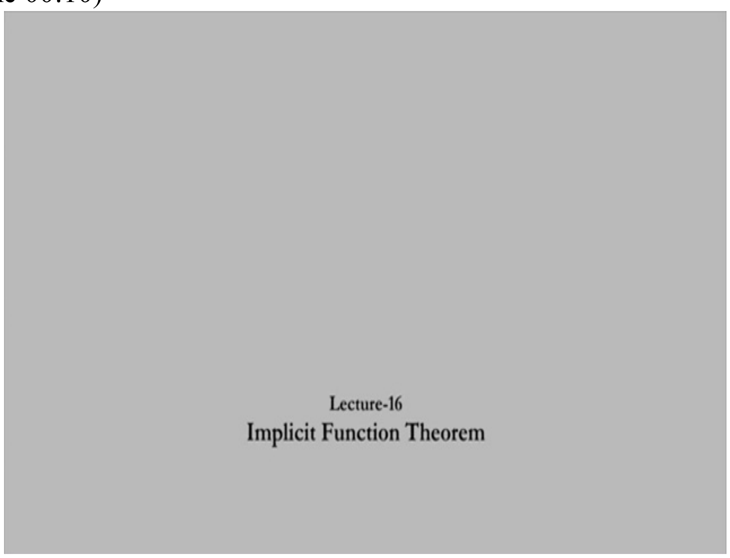

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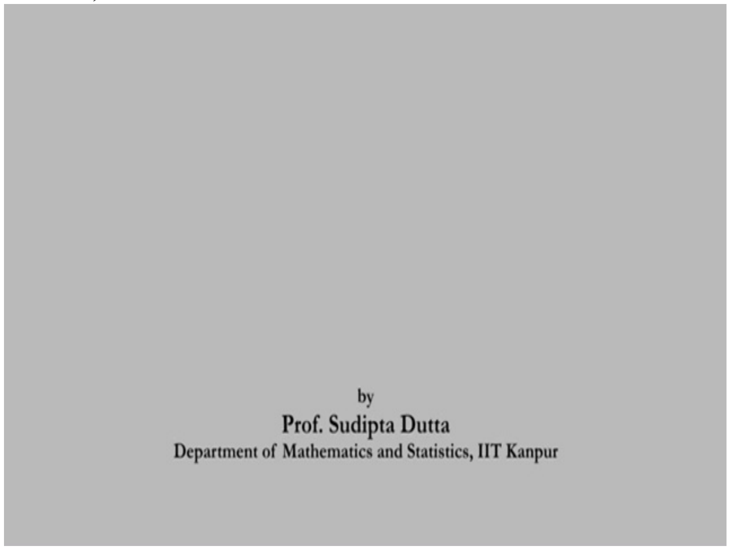

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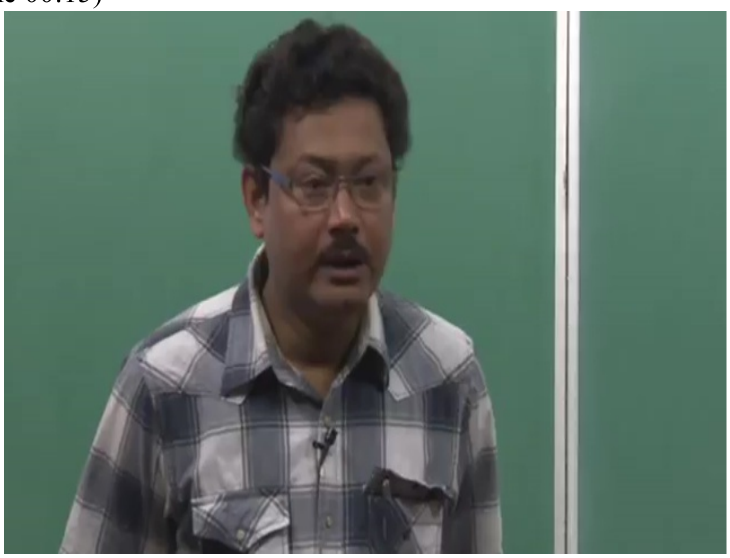

For this week in the last module, we will discuss implicit functions theorem and its application

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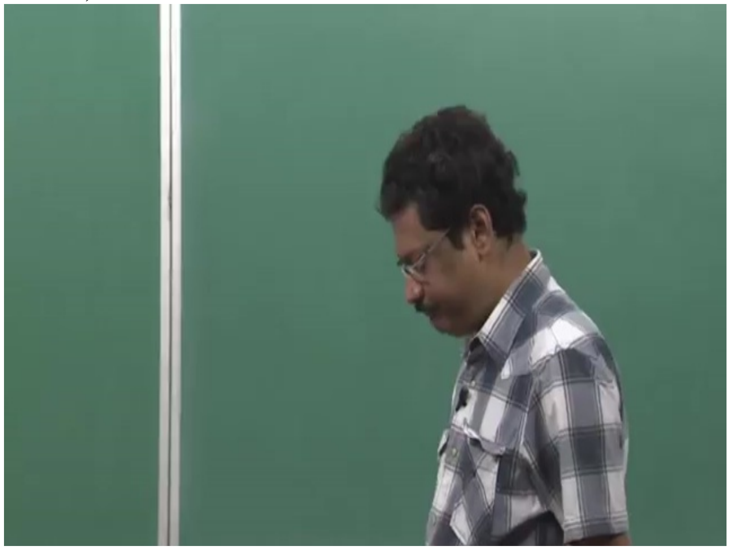

So implicit function theorem one can say, so far whatever we have done is just, we tried to lift whatever we had from function of 1 variable to function of several variables

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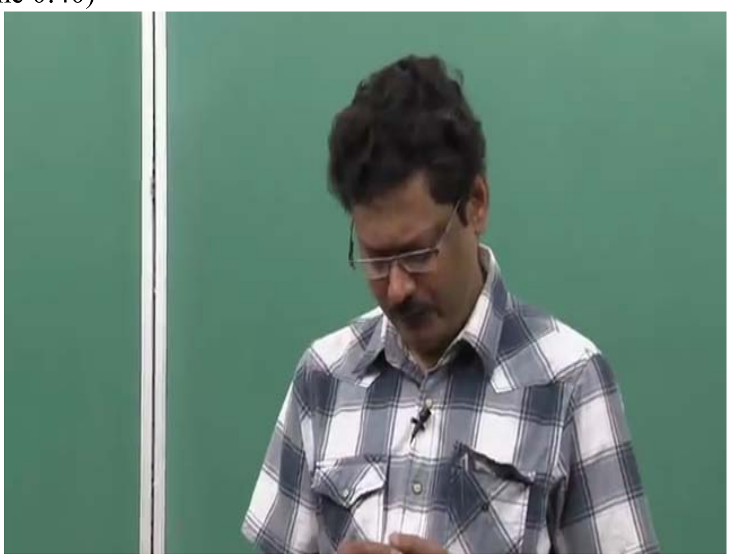

and for that we need to develop, needed to develop certain techniques, certain technical issues we had to deal with.

But as far as implicit functions theorem and its applications are concerned, this is typically a statement in several variables.

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And one can say this is the one of first non-trivial theorem what one encounters in differential calculus dealing with functions of several variables. And if is often confusing.

The statement seems to be confusing if it is written in a ...if you just see a book and just try to turn the pages

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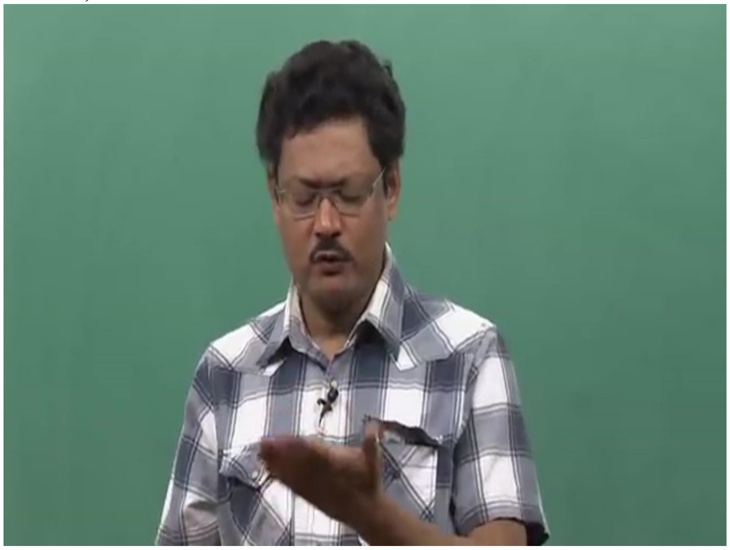

… to find the implicit function theorem from the first reading or even the second reading it is very difficult to understand what is going on. So I try to do it in a very simple fashion and

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I will do it in detail for function of 2 variables.

And you will see that, from function of 2 variables to generalize a function of any variables taking values in any "R n", this is just mere formalities.

So what is the idea here?

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The idea, the basic question begins here.

If suppose, another nice function of 1 variable; nice means let us say, smooth enough; if I want some interval...

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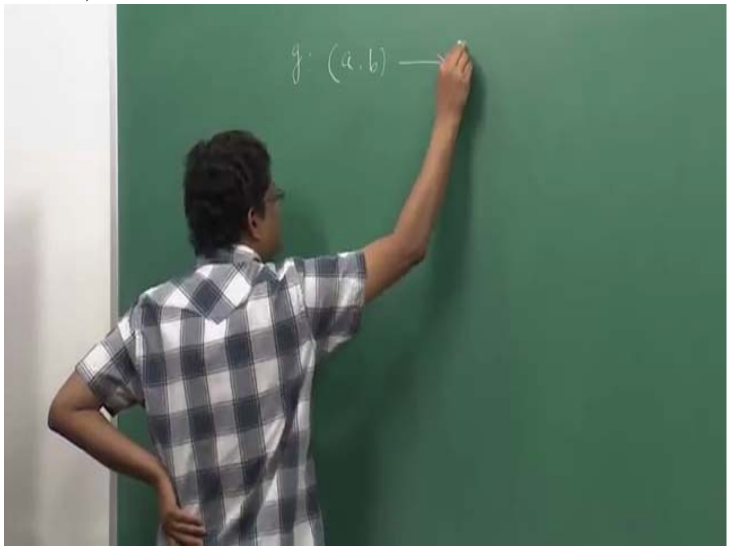

Now given a function, first one tries to plot it, right

So here is my interval “a” “b”

one tries to plot a nice function “g” out there.

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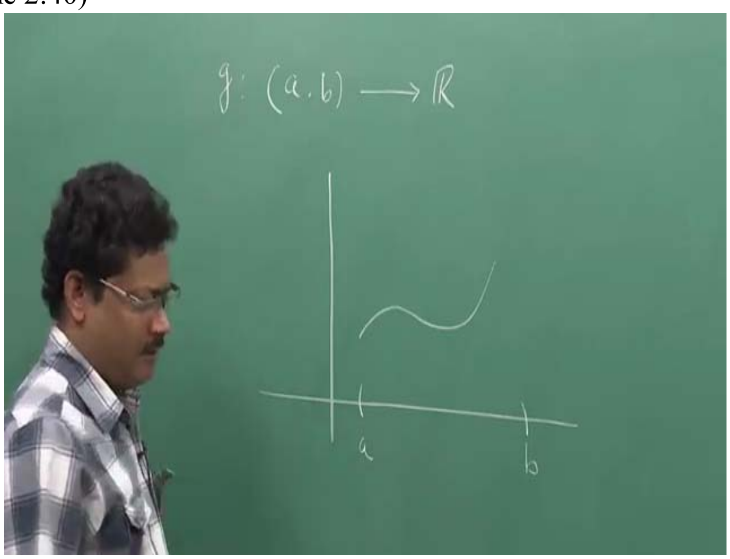

So what it means, for each value of “x”, so I have “g x” over here

Now if I look at this curve, this is a curve in R2, right, curve in R2

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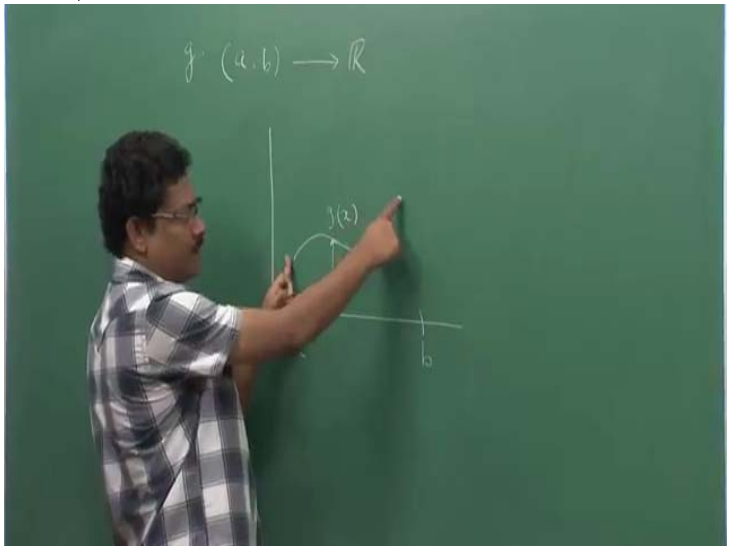

So points on the curve are basically “x”, “g x”

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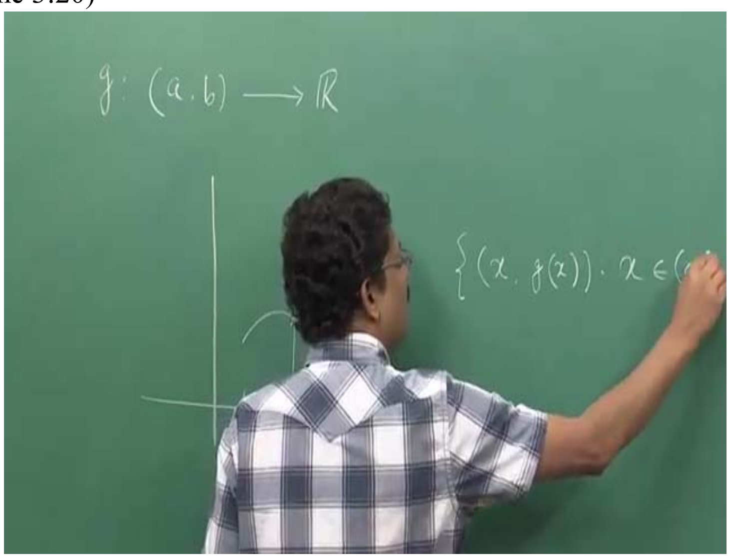

So this point is “x”, “g x”, “x” in “a” “b”

And all of you know that such a thing is called the graphs and you have been doing it for....the first moment you introduced functions, we have been drawing graphs of function

This is graph of a function “g”

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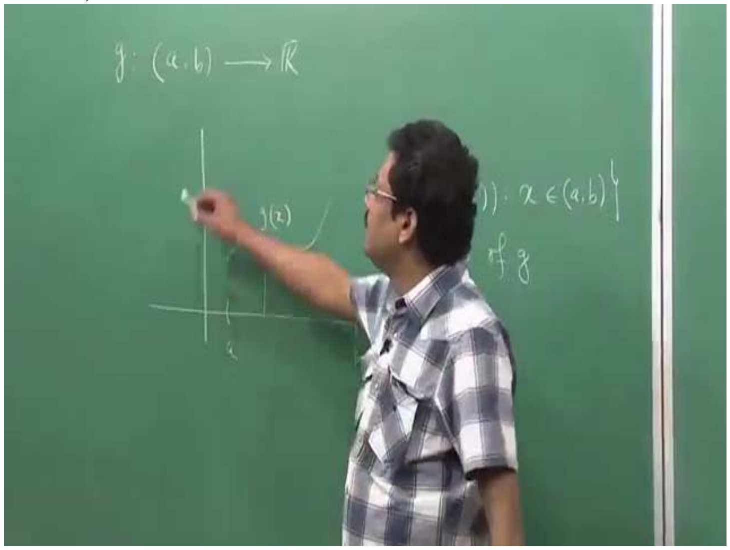

and if “g” is nice, smooth, then this graph will be nice in the sense that I can draw tangent to this upper curve, I can draw normal

So a function gives rise to a graph of a curve which is a graph of the function

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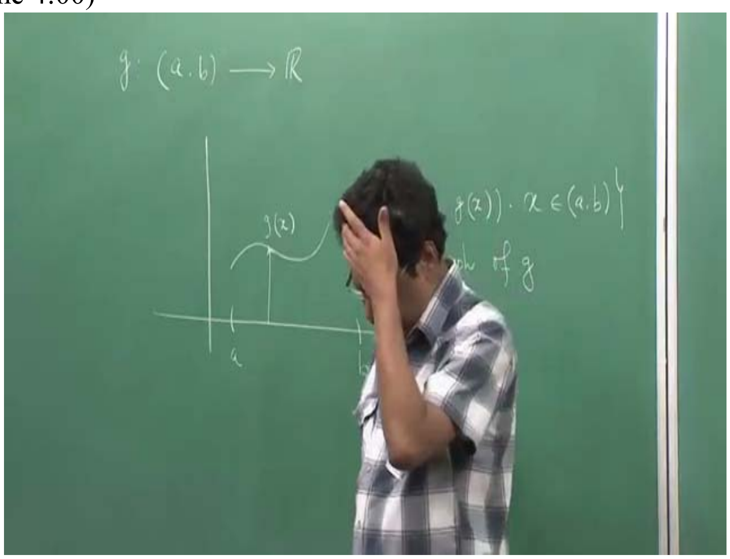

Now suppose on the other hand, I just give you a curve path in R2.

How do I say it is a curve in R2?

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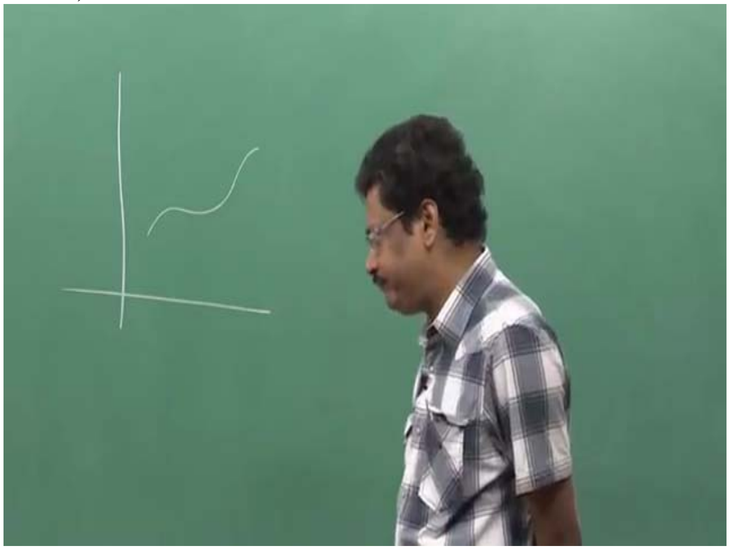

Well, this is...I say that is Ok, don't look at...I don't want function of one variable I say exist or not

But I say this is a curve which is governed by this rule

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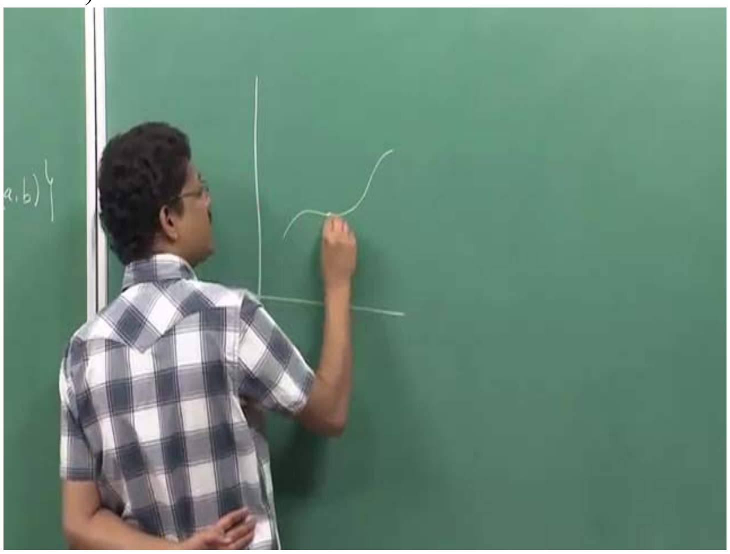

that is, any points on this curve “x, y” satisfies this equation “f” of “x, y” equals to constant for some “x” ,“f”.

For example,

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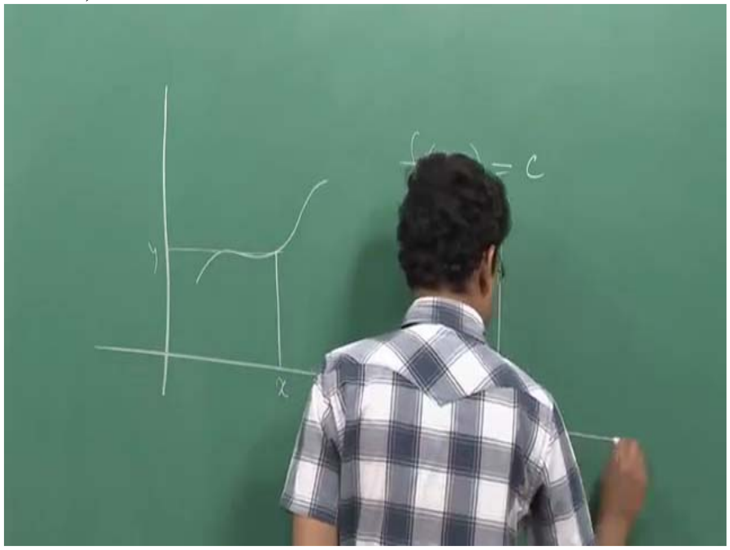

…. so I have a function “f” from some interval u to r, u in R2, sorry some open set or some set u in R2 and “f” from u to r, and look at all those points

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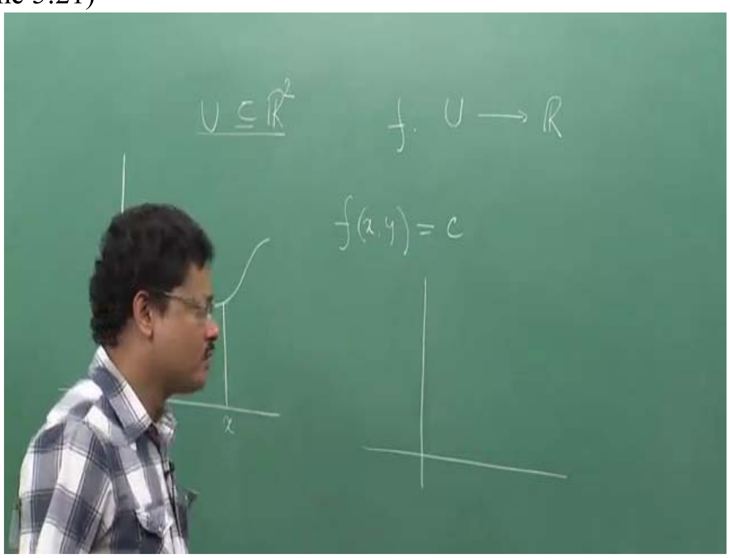

“f” “x, y” equal to “c”, so it is called a level curve

That I have a function “f” and I look at all those “x, y” such that “f” of “x, y” is constant and I call that is a curve, level curve of “f”

For example, let us take a

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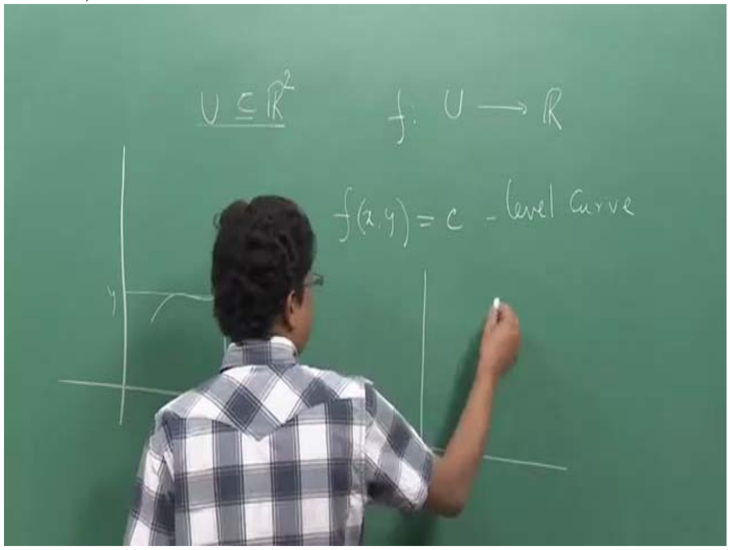

simple example

Suppose I define this function “f” from R2 to R, very simple one, “f” of “x, y”, let us say “x” plus 1

This is a function of 2 variables taking value in R

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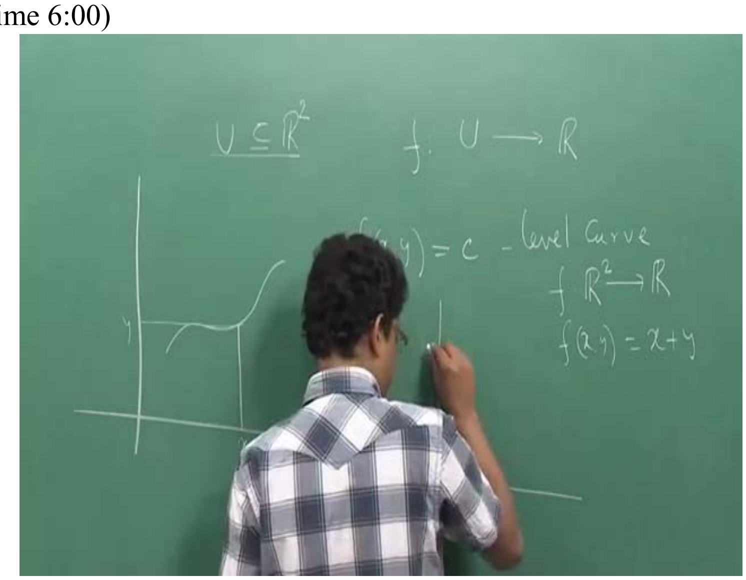

And I look at this line, the line is a curve which is....I say, this curve is nothing but...you can look at “f” “x, y” equal to let us say 1

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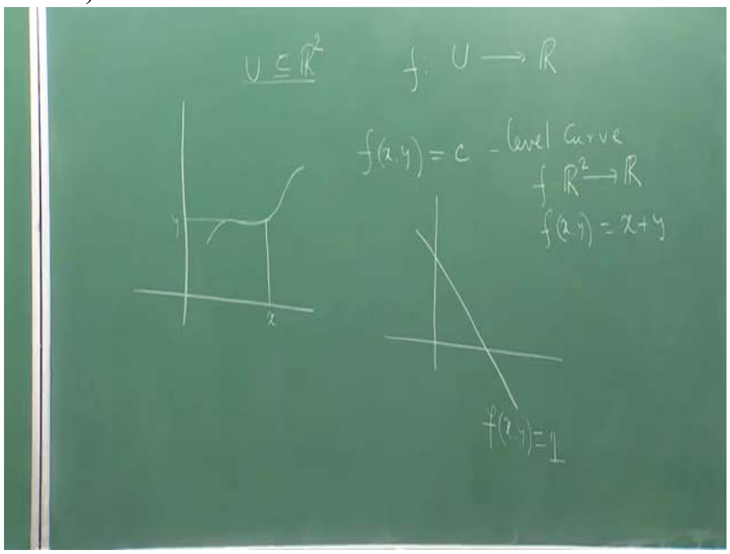

So line “x” plus, this is line “x” plus “y” equal to 1

And so, if you look at function “f” of “x, y” equal to “x” plus “y”, this is a level curve of this function.

And the question is, given the level curve in R2, you can ask

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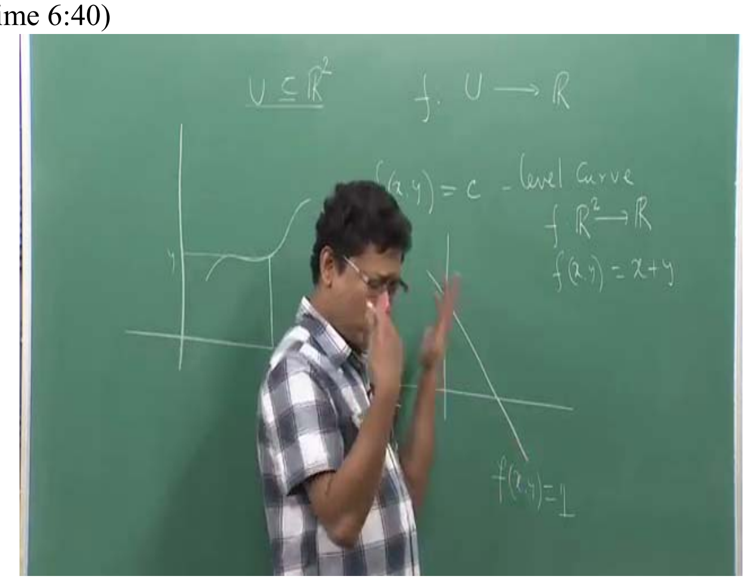

the same question in R3 when I will be giving level surface, for R3 it will be a surface so if I look at it in “f” from R3 to R,

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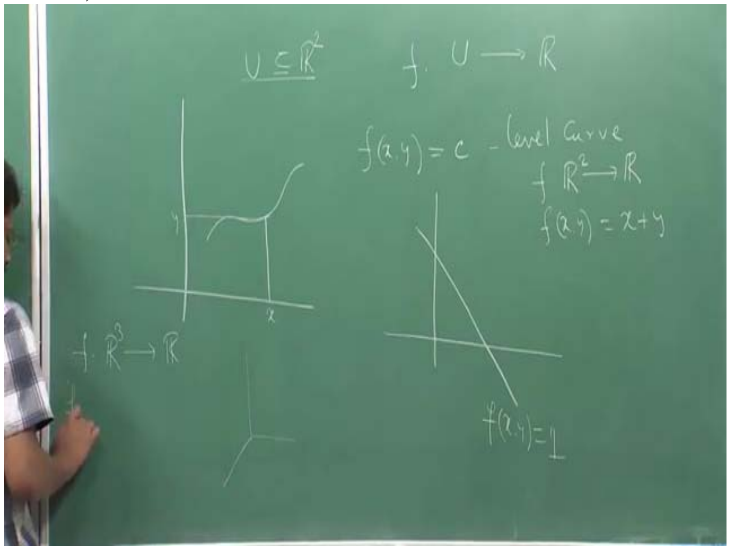

then if "x", "y", "z" equal to constant, this will be some surface in R3.

So now I can ask this question for a curve that

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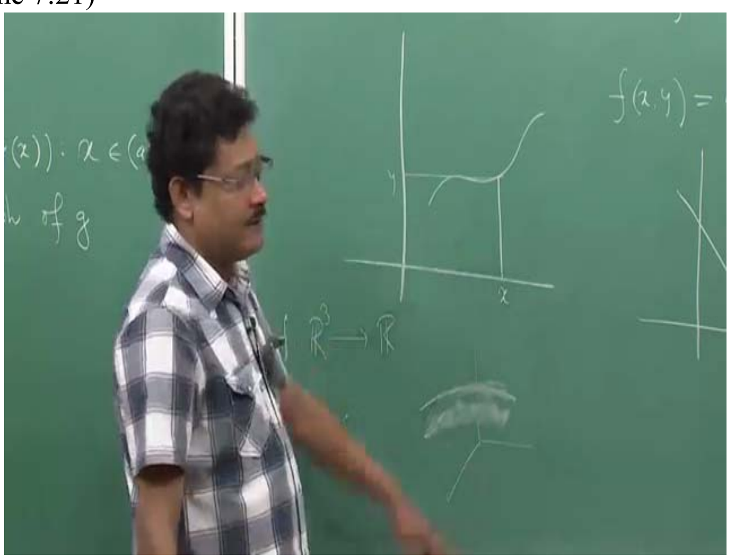

Ok, fine. I have a nice level curve; "f" is nice enough in whatever sense

Is it a graph of a function? That do I have a function "g" like that, that actually this little curve "f" of "x, y" equal to "c" comes from a graph of a function "g".

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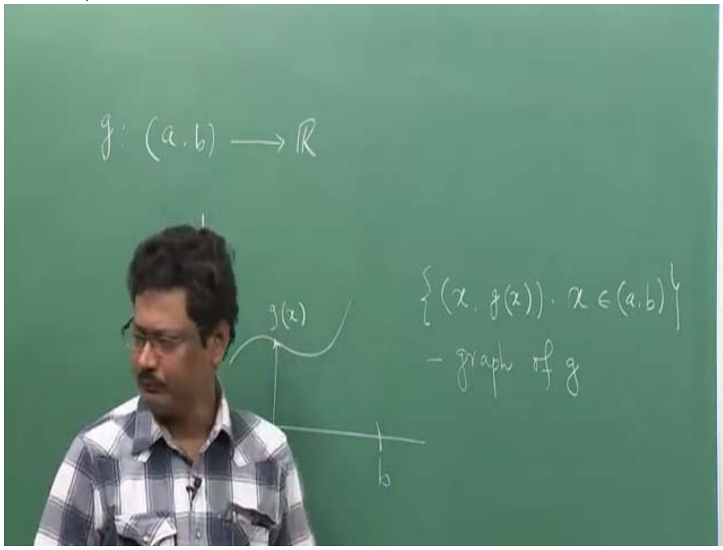

Well, for the surface, can I ask this question that does there exists a function “g” from let us say, R2 to R such that R2...so can I write

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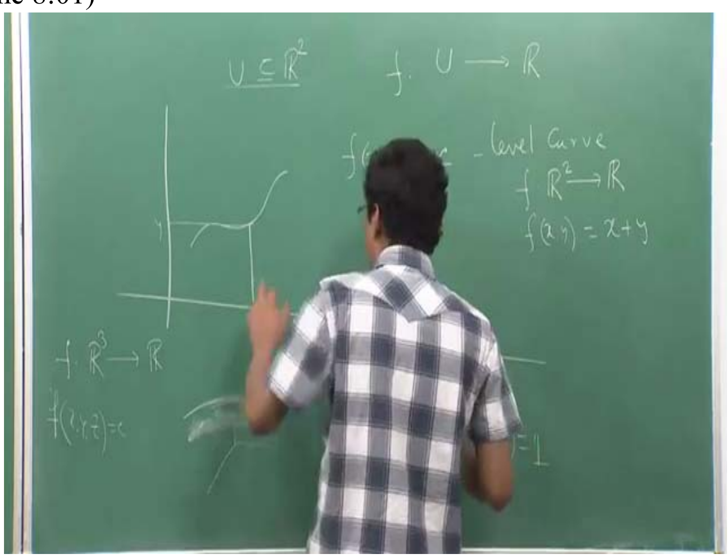

“f” “x, y” equal to “c”, this set, set of all “x, y” here, This is same as

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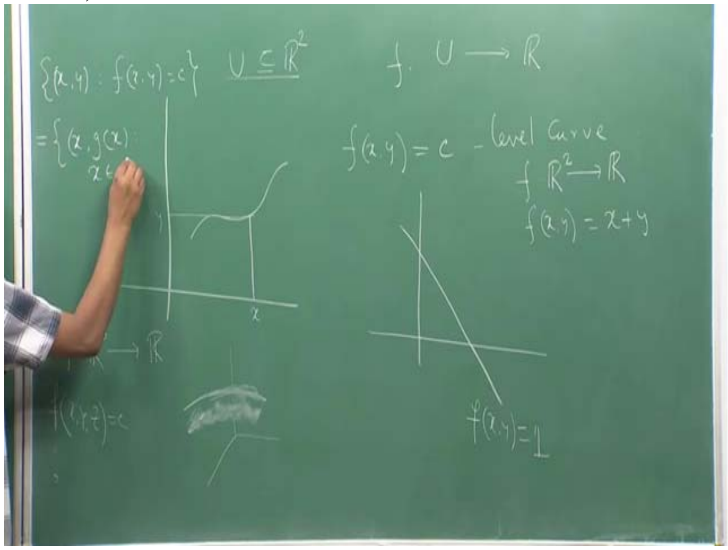

"x" belongs to some interval. Can I do this?

Of here, if "x, y" "z" equal to "c", can I write it as some set "x", "y",

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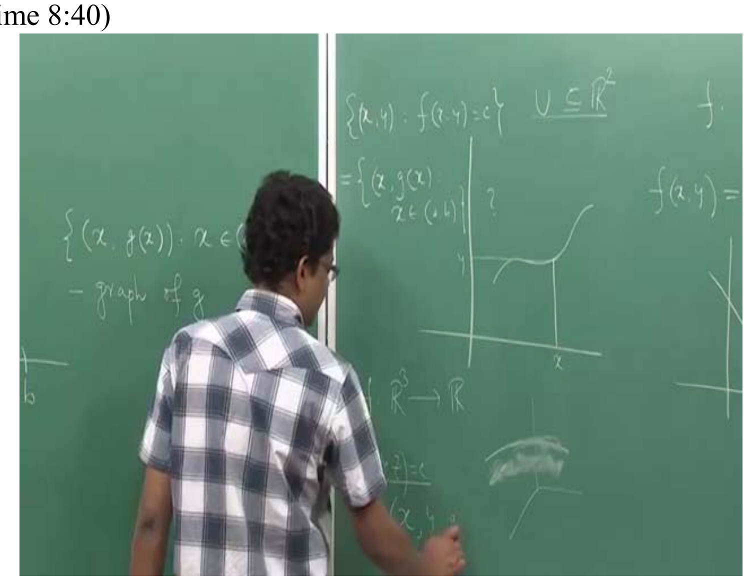

"g x, y", "g" from some set in R2 to R

So the question is, is the level curve

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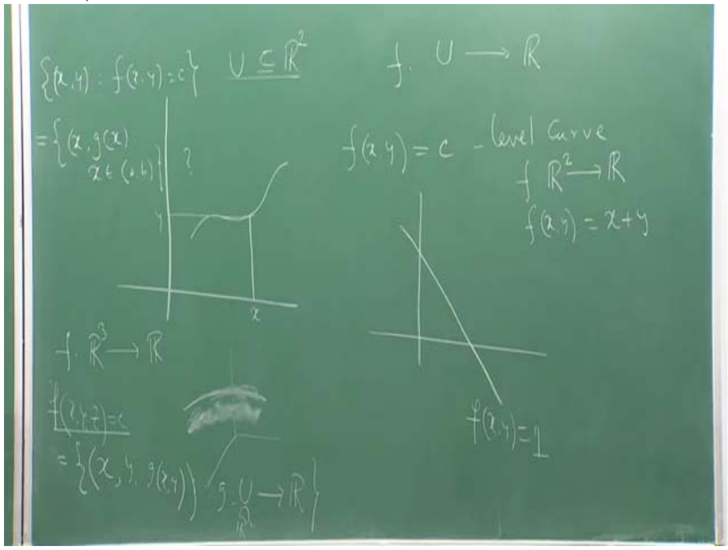

… a graph?

Well, if you look at this example, “f” “x, y” equal to 1, this you can easily see, this is actually set of all points “x” and

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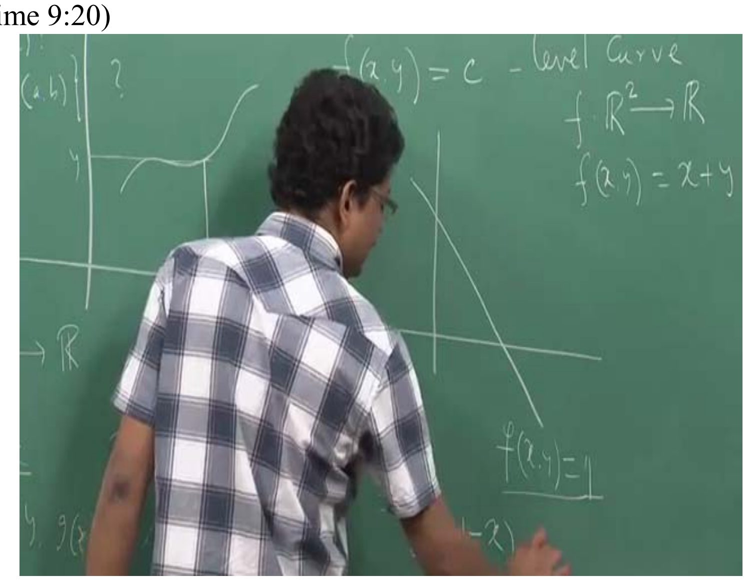

“y” equal to 1 minus “x”, “x” in R

So this is graph of the function, graph of “g” where

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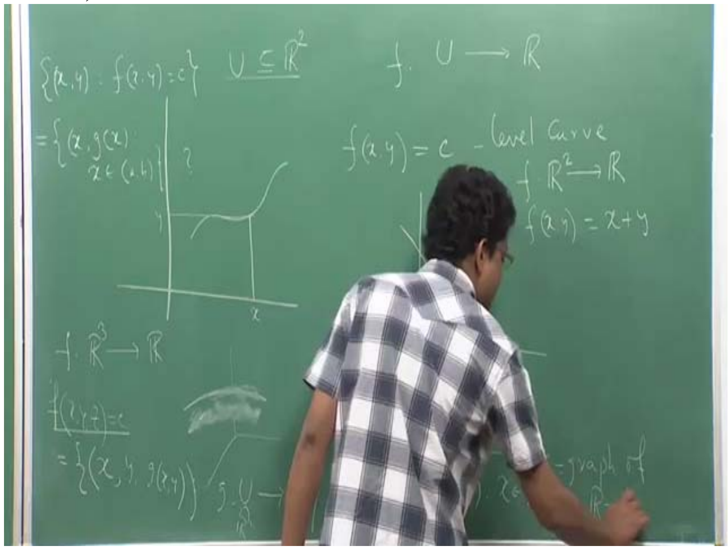

"g" is from R to R and "g" of "x" equal to 1 minus "x".

So, instead of saying "f" "x, y" is equal to 1, if I give you, say set of points, "x", 1 minus "x", or "x", "g x", on graph you will get the same curve.

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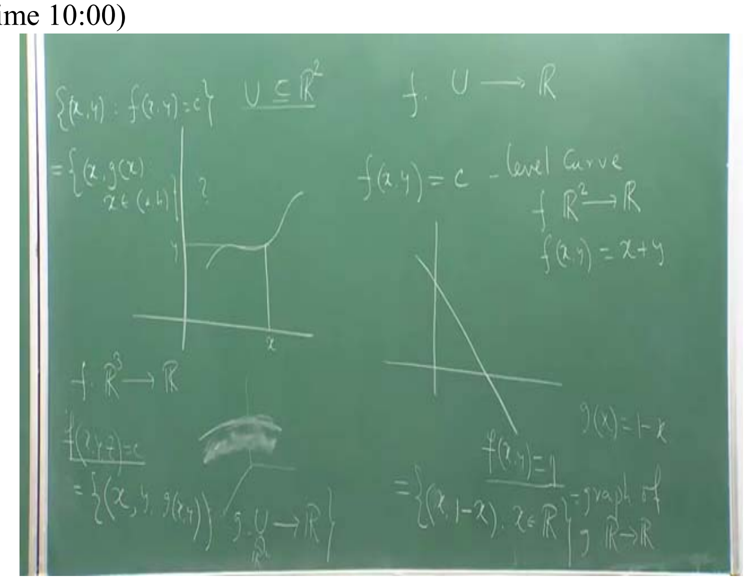

Since very nice...that this curve we can write graph of a function but a simple thought, if extended a little bit, you will see that this answer may be negative as well.

Because suppose I look at this function

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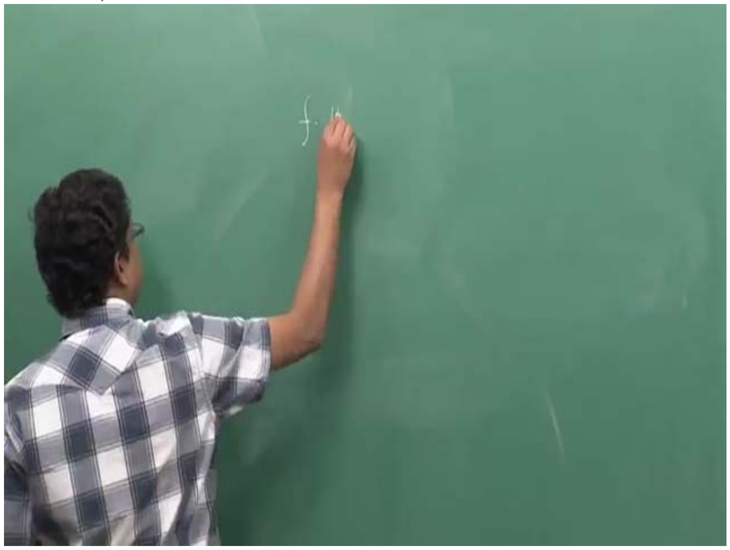

This is a very nice function “x” square plus “y” square and I look at the level curve.

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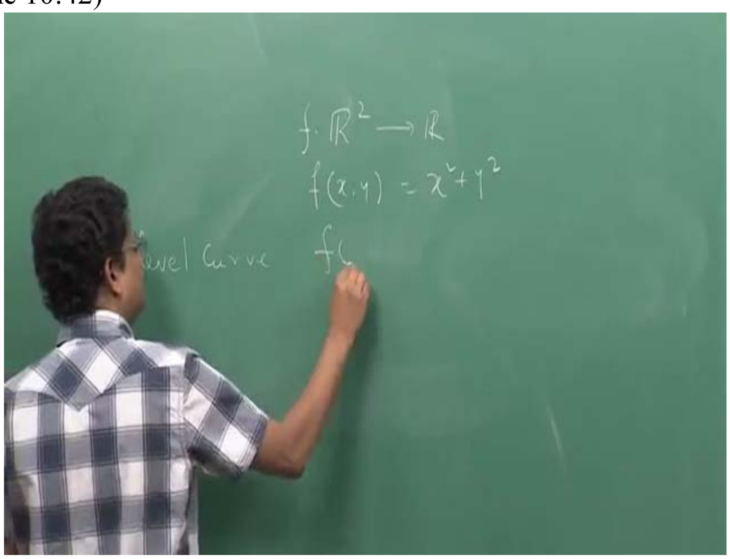

All of you know....if I plot it, what I will get, circle of radius 1 in R2

Now if I want to write it...now the question, if I ask this question now, can I write,

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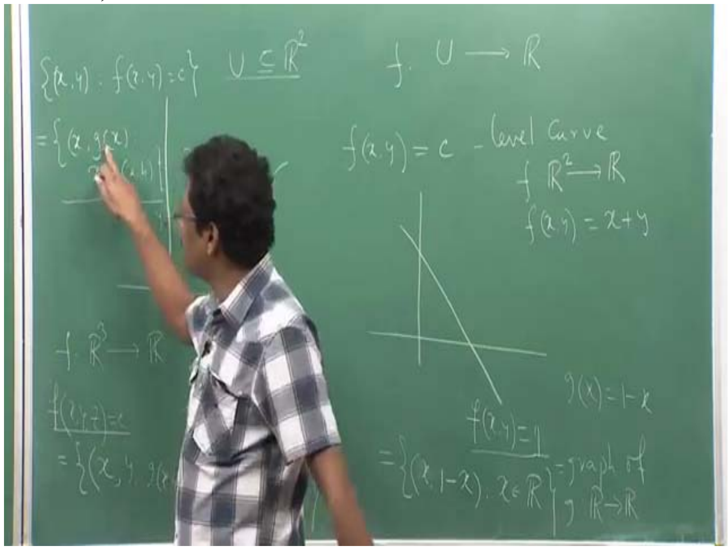

can I have a function “g” defined on some interval such that I can write this curve, any point on the curve as “x”, “g x”, no matter how you try, you may not be able to do it, right?

Because whatever “x” you choose...you choose “x” here

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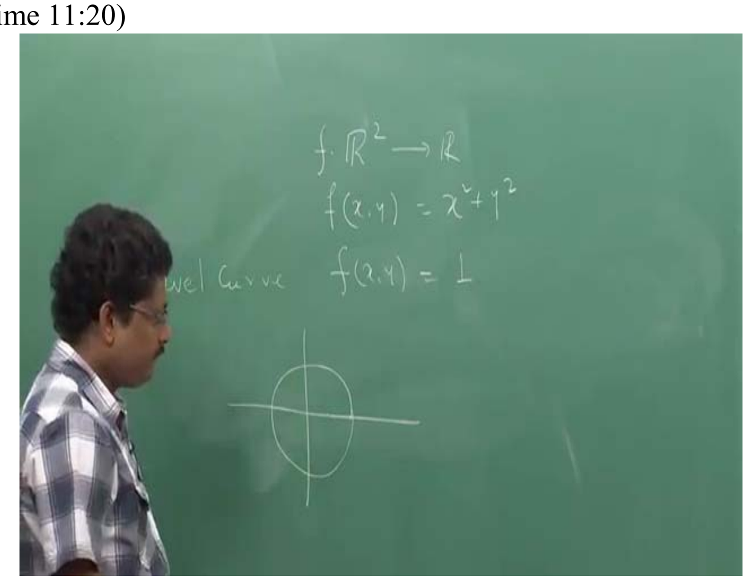

if I have a function, this value will be “g x” but corresponding to this point “x”, there are two “y”

Similarly, other way round, corresponding to any “y” here, there are 2 “x”s

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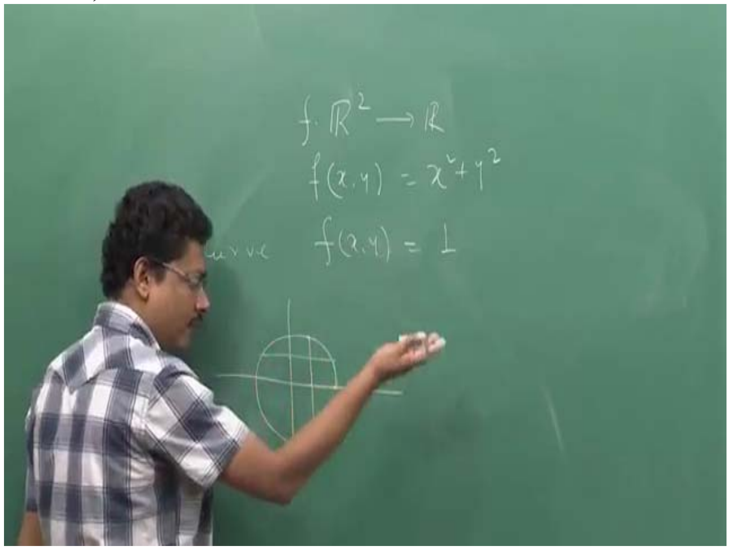

So, neither as a function of “x” nor function of “y”

I cannot write. I need at least 2 functions here. So need at least 2 functions here

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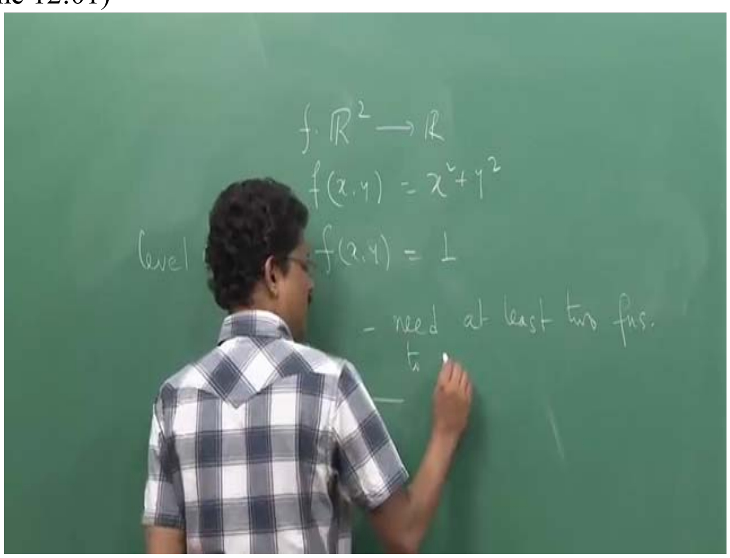

to write it as graph, namely “g x” equal to, one is positive square root, which will give you this part,

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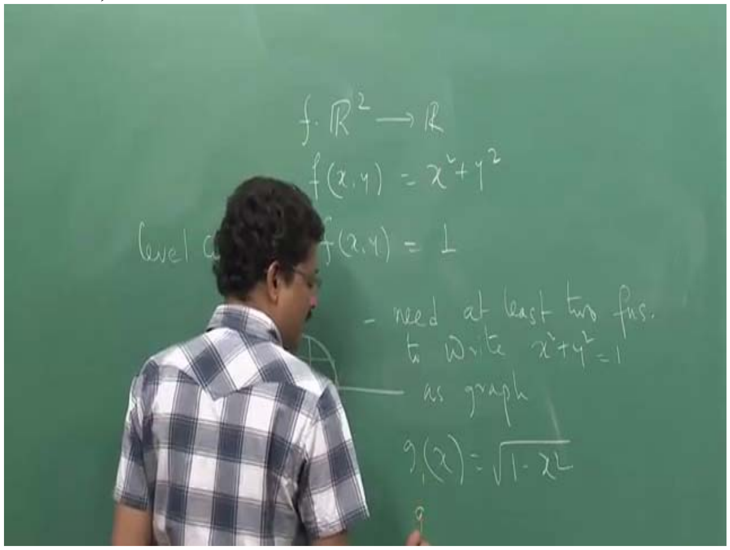

and another is negative square root which will give you this part.

So this level curve is actually union of “x”, “g x”

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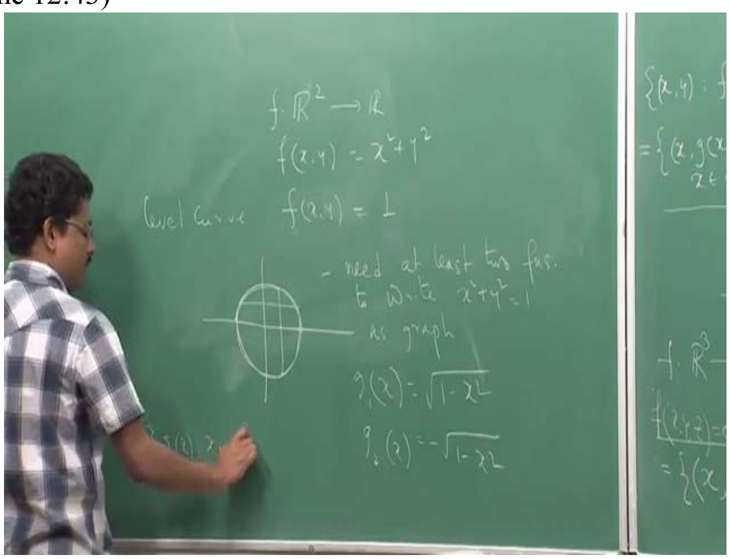

, “x” in R and “x”, “g x”, “g” to “x”,

“x” in R and these 2 here are not disjoint. Are they? Yeah, they are not disjoint.

“x” equal to 1,

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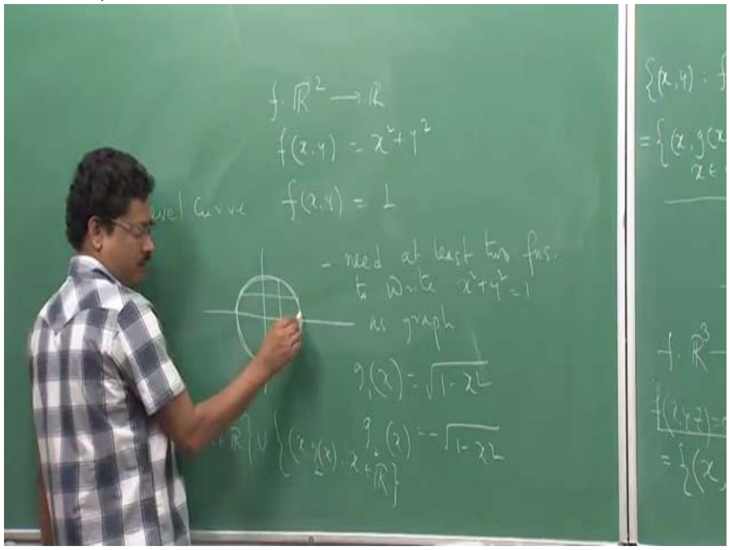

then “y” is zero, right? Yeah, “x” is 1 or minus 1, “y” is 0.

So this is not a disjoint union.

So even for this simple level curve, which is very nice, which has tangent, everything but I cannot write it as a graph of a function.

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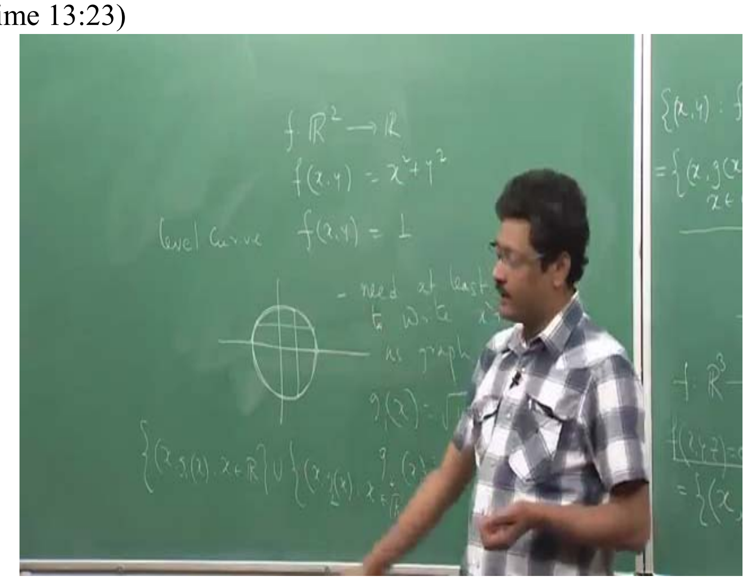

But one thing is sure.

But if I want to look at a portion of this curve, let us say this portion

This portion, of course I can write as a function. So this I can of course as write

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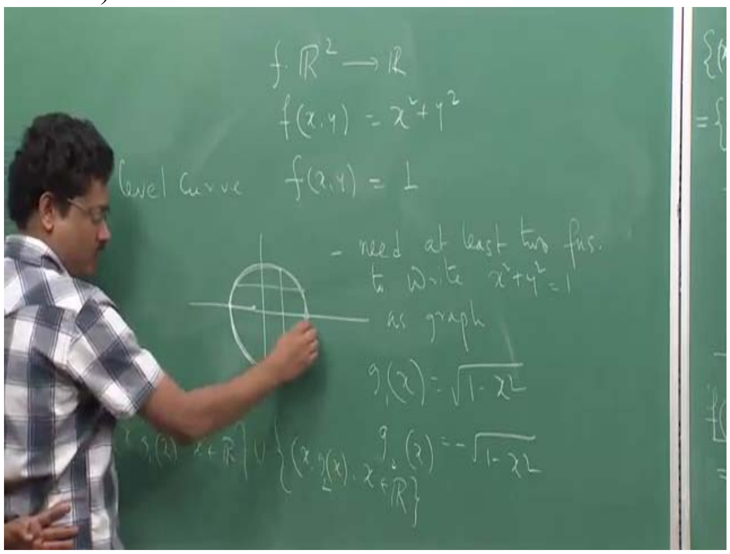

as graph of g2

This portion also, I can write graph of g2, some proper interval "x, y"

Here also

So this gives us, that may be, we should not be asking the question for the entire curve. We should be asking the question....

So let me form the question.

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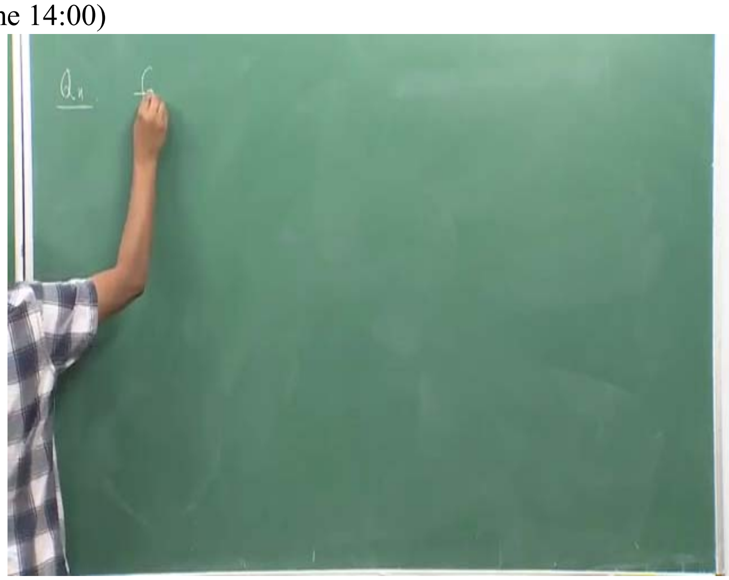

Now in full generality, that suppose I have a function "f" n, "f" from "R n" to "R m".

And I look at "g" o i level surface "f" of "x" equal to some...

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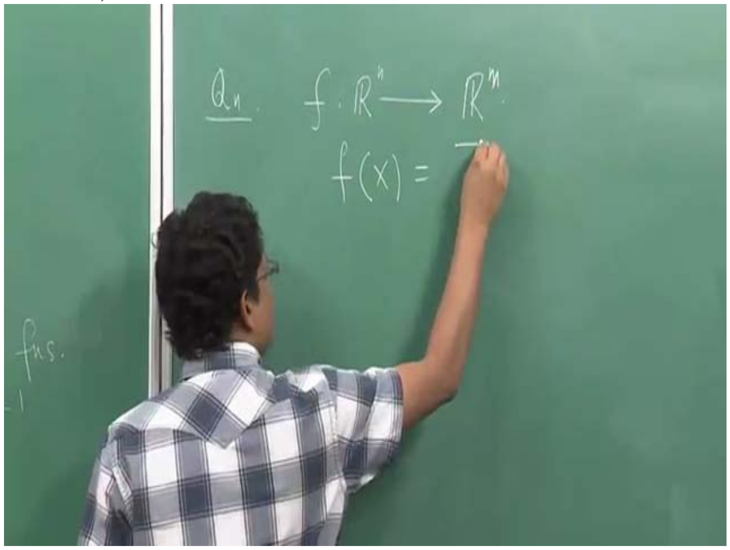

Ok let us say some “y naught”; “x” in R n, “y naught” fixed vector in “R m”

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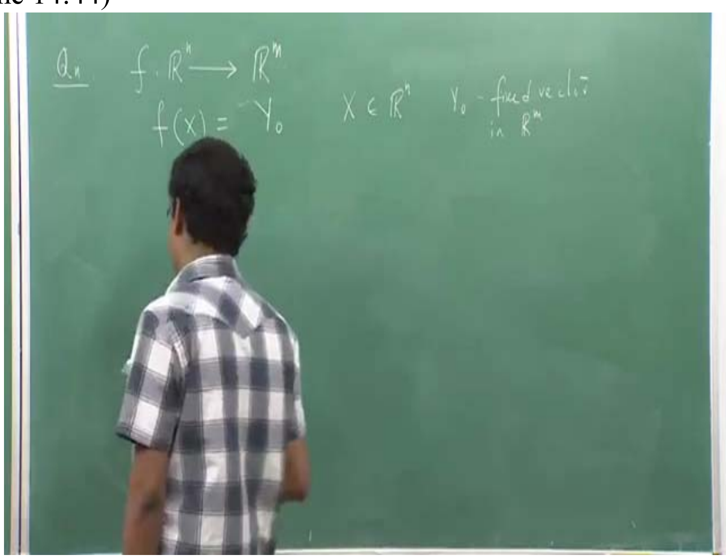

Question is, can I write it as graph of a function that is, does there exist

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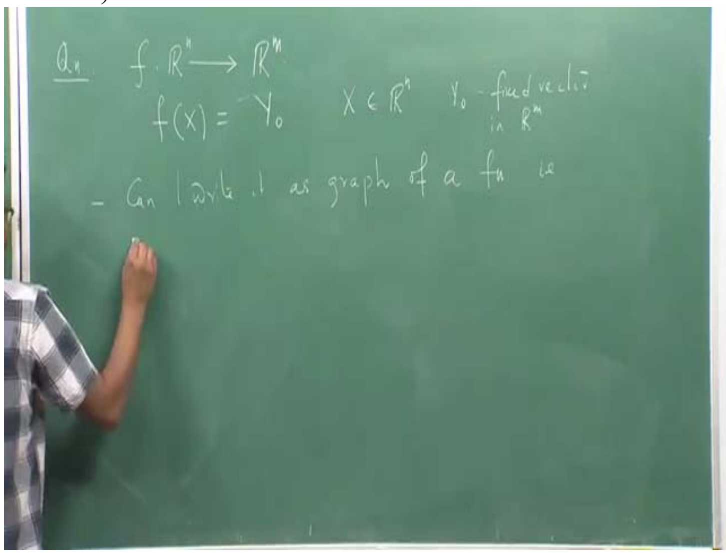

a “g” from “R n minus m”, so here “R n” is bigger than m

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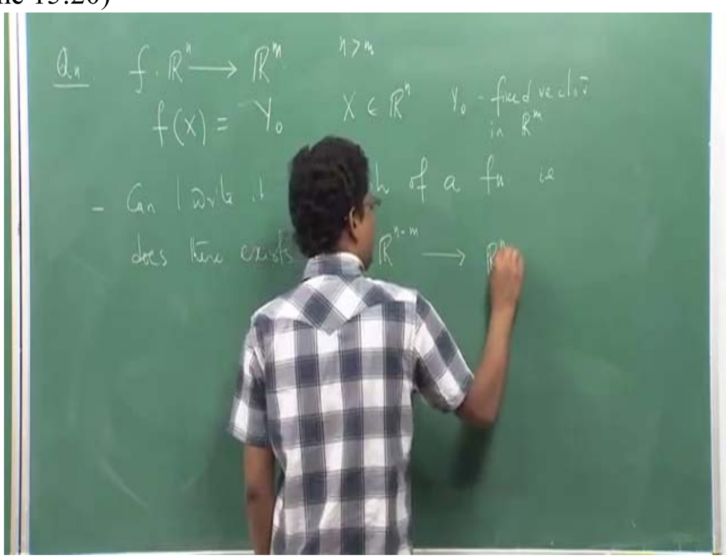

…. to “R n” such that this set of points “x, y”, “f” of “x, y” equal to “y naught”

“x” in “R n”, “R n minus m”

“y” is in “R n” equal to “x” g “x” for some “g” in some interval theta in “R n minus m”

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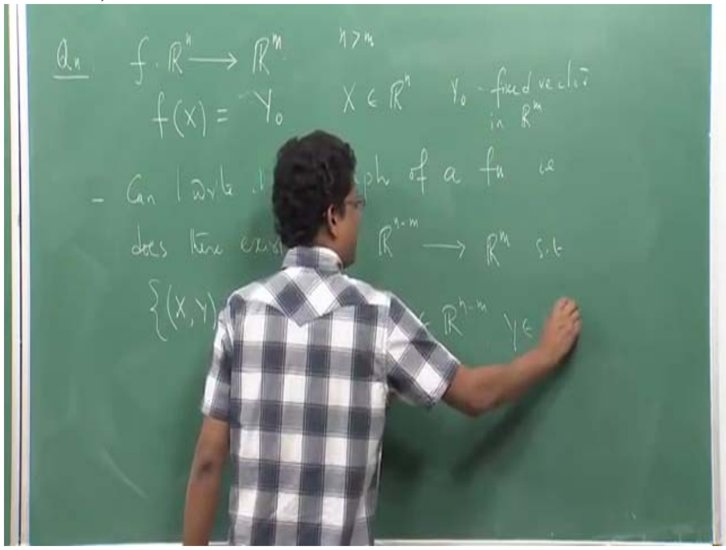

“y” is in “R n” equal to “x”, “g x” for some “g” in some interval theta in “R n minus m”

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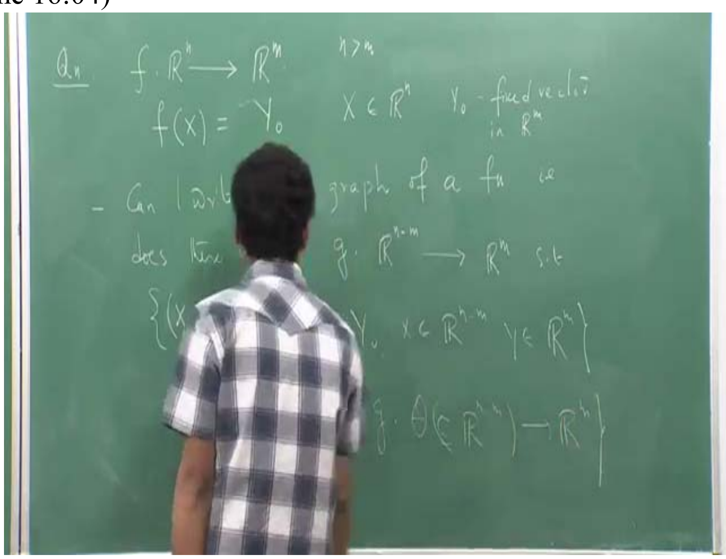

So there is a question written in full generality.

So in particular, for R2 to R, I ask the question, does there exist a function of 1 variable such that this is a graph of the function

So, question formulated in layman's term

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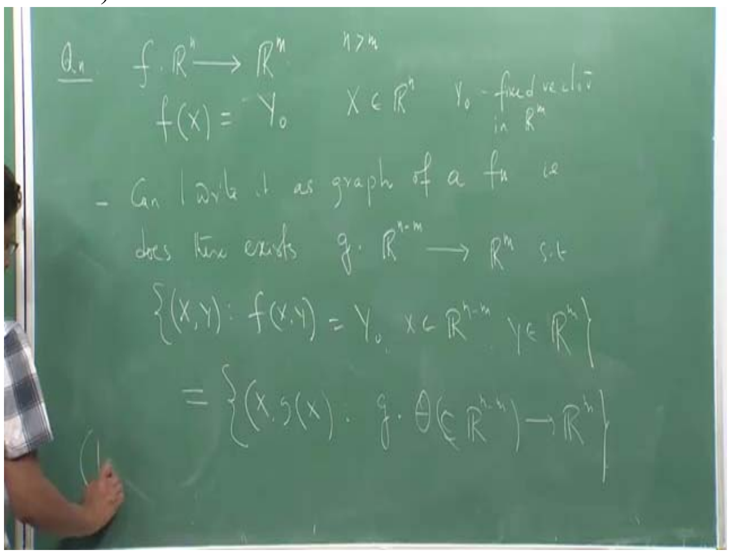

is every level curve or surface is a graph? Answer is no, as I said,

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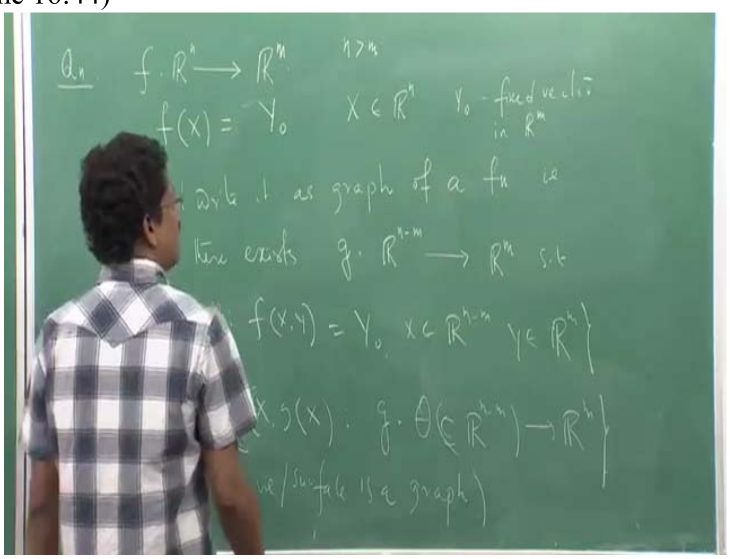

so I should...what I should do....Are you sure the answer is no, so the question should be, should not be this because if I do it, then the answer is no.

So now, we should add one term here.

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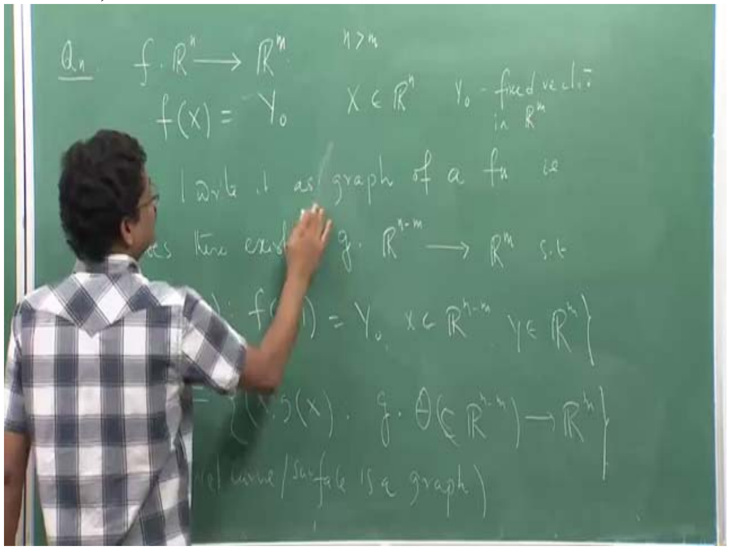

Can I write it locally as a graph of a function? Does this, does there exist this...

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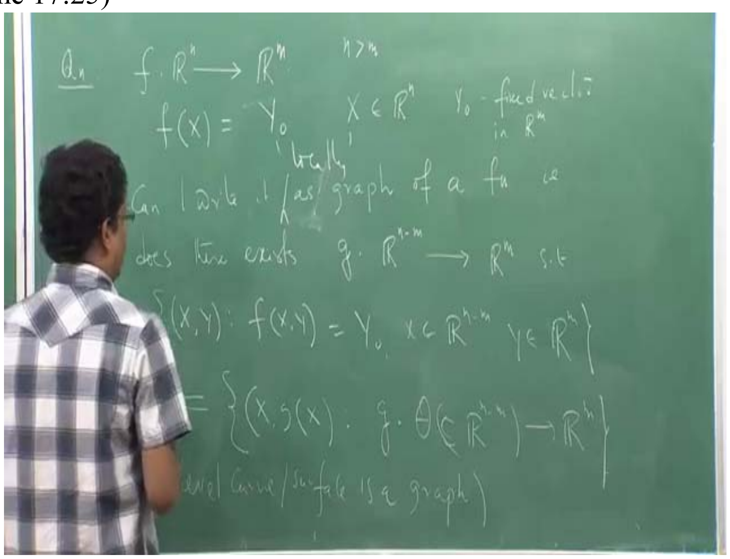

so that is, given “x naught” “y naught” in the curve such that “f” of “x naught” equal to “y naught”,

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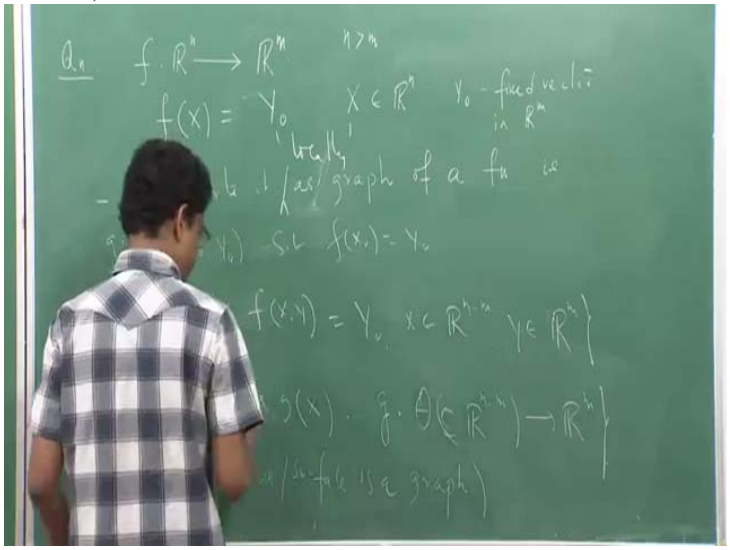

does there exist a neighborhood

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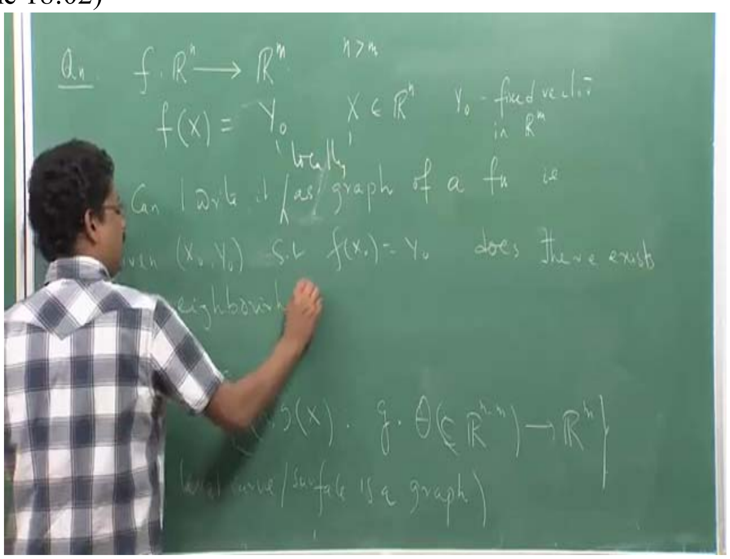

of “x naught” “y naught” and “g” from “R n minus m” to “R m”

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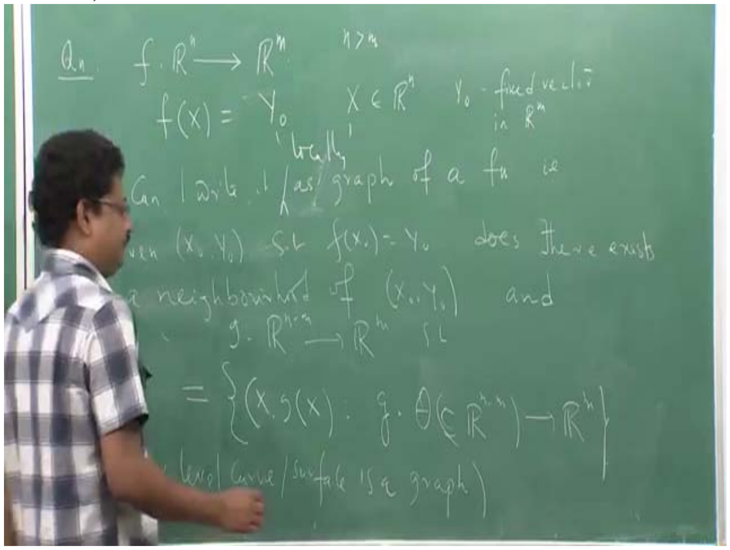

such that, in that neighborhood,

if inverse... if in that neighborhood let us say does there exists a neighborhood

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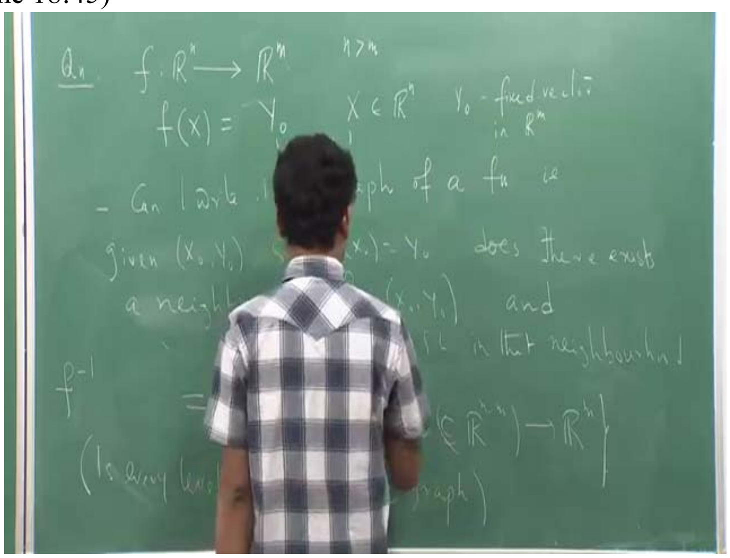

um...let us give it a name U, in that neighborhood U; if x inverse “y naught” intersection U is a graph

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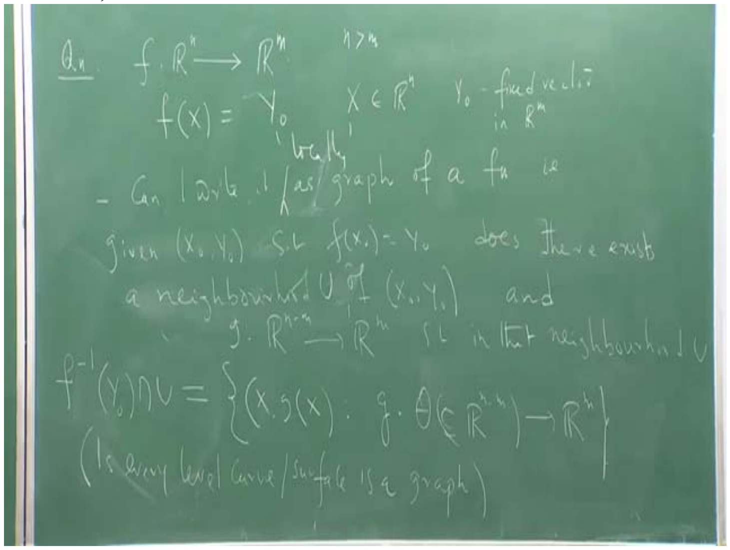

There is a confusion because this “y naught”, maybe I should change it to some constant, right, some “c”; “c” is a fixed vector in “R m”

So “f” inverse of “c” is this; So this is the function, and

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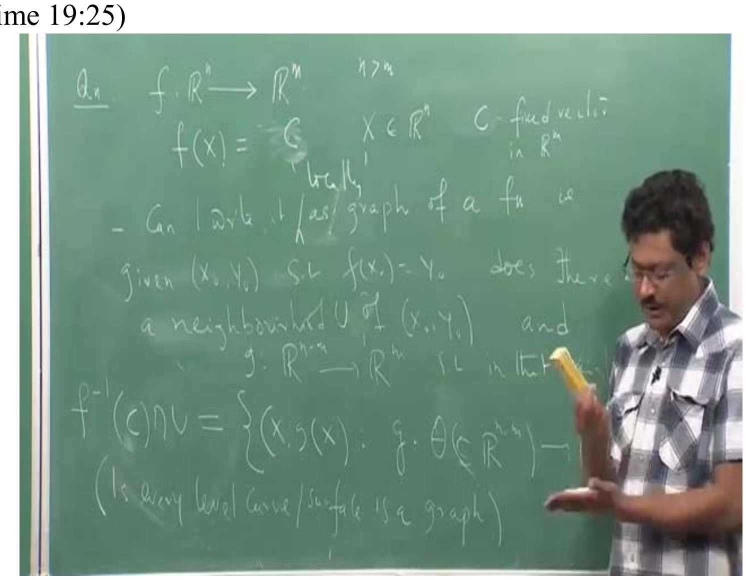

of course, implicit function theorem says the answer is yes. That’s it.

But it is a very useful theorem in calculus and, as I said...how will I state it...

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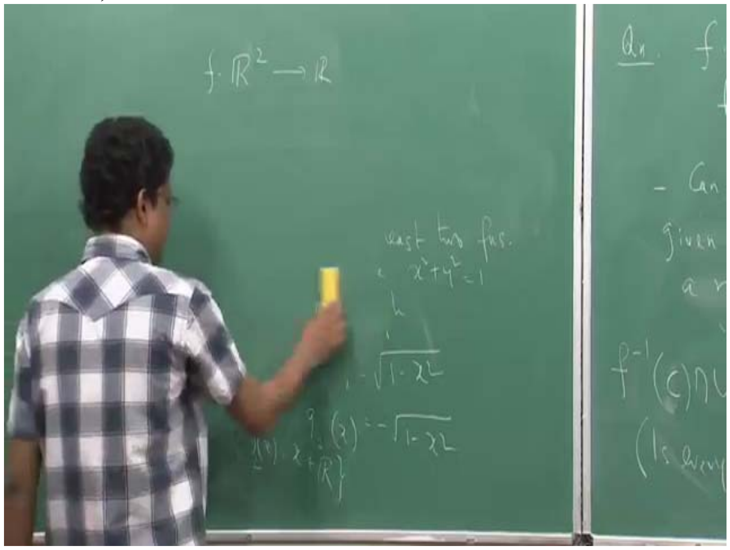

for curves in R2 to R and then tell you how to do it for any general...to answer this question in generality

So where is the question now? So I look at “f” of “x, y” equal to this...instead of “c” lets take zero.

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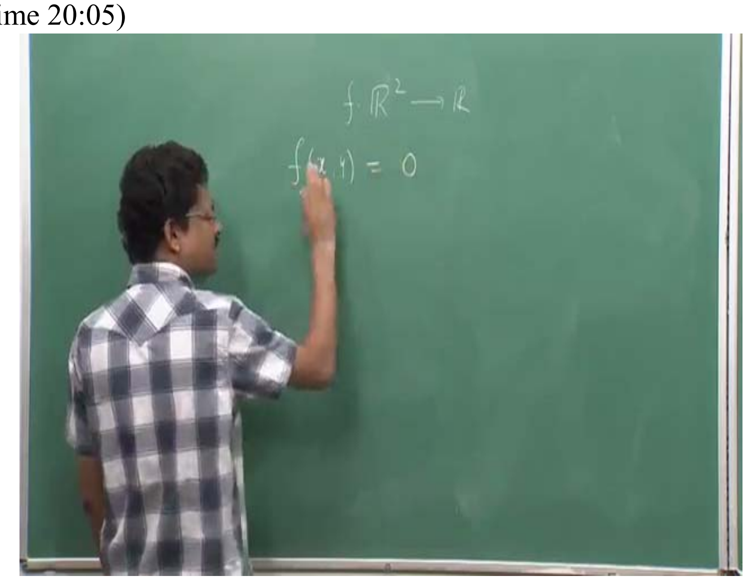

Doesn't matter, right? I can also take “c” consume in “f”.

So what I am look for? Looking for, given...so “x naught”, “y naught” is such a point

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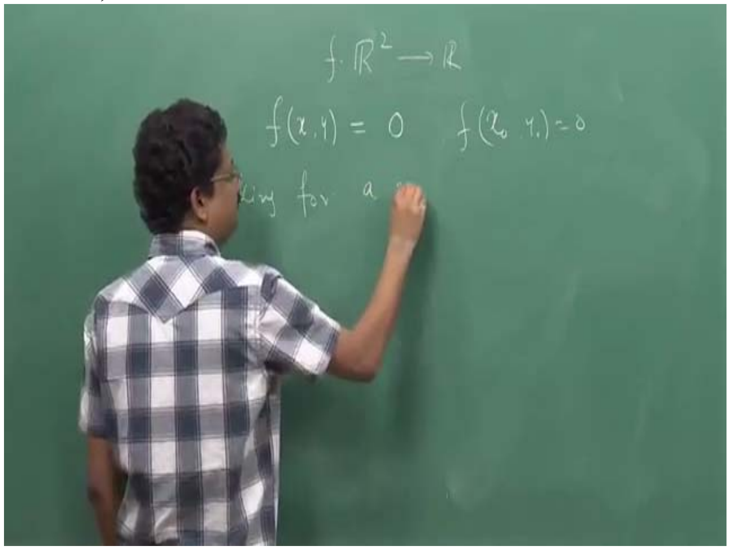

Looking for neighborhood of “x naught” “y naught”, let us say of the form some “x” minus “epsilon 1”,

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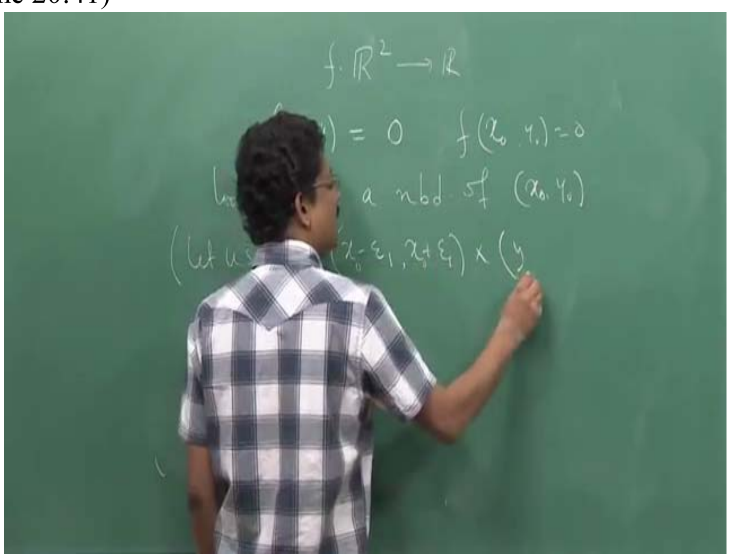

“x” plus “epsilon 1”, “x naught” “y naught”, “y naught” minus “epsilon 2”, “y naught” plus “epsilon 2”

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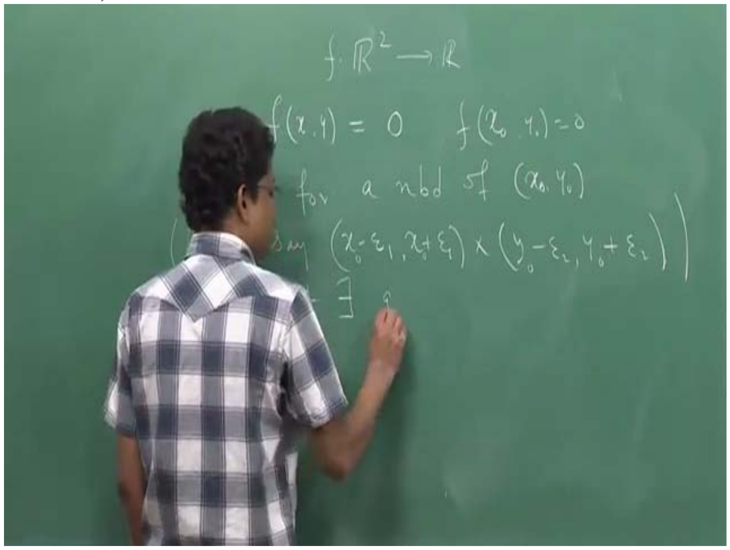

such that there exist "g" defined on "x naught" minus "epsilon 1" to "x naught" plus "epsilon 1" to R

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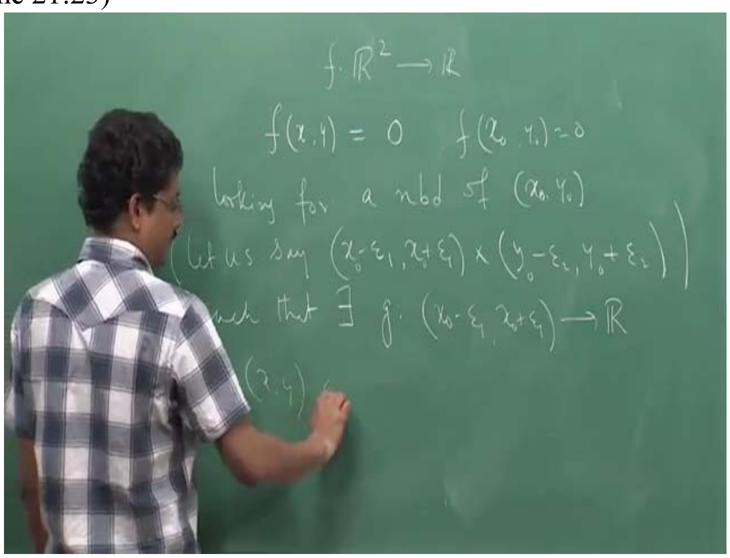

and if "x, y" belongs to this neighborhood, then "f" inverse of zero intersection this neighborhood

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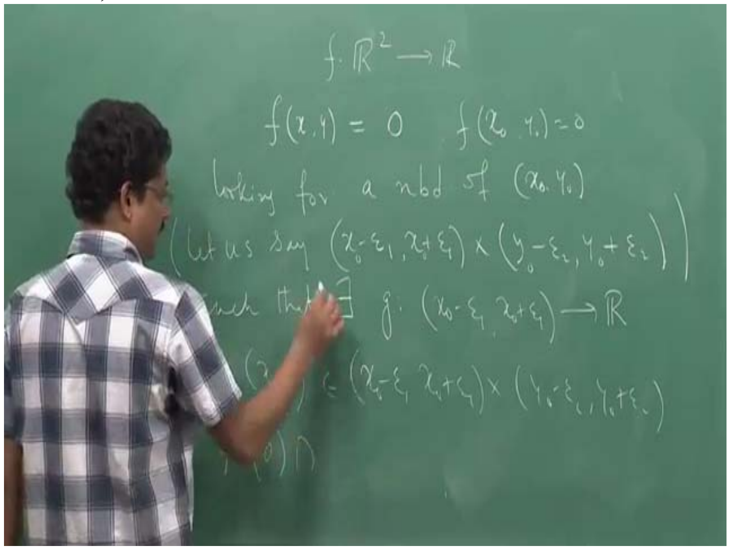

...Ok let us say, instead of writing it

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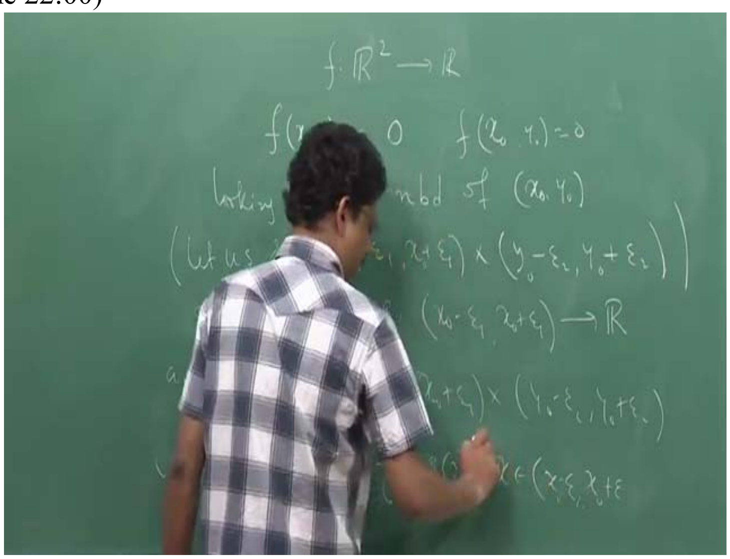

And of course, I should have "g" of "x naught" equal to "y naught" because it must be satisfied.

So does there exist such a "g"?

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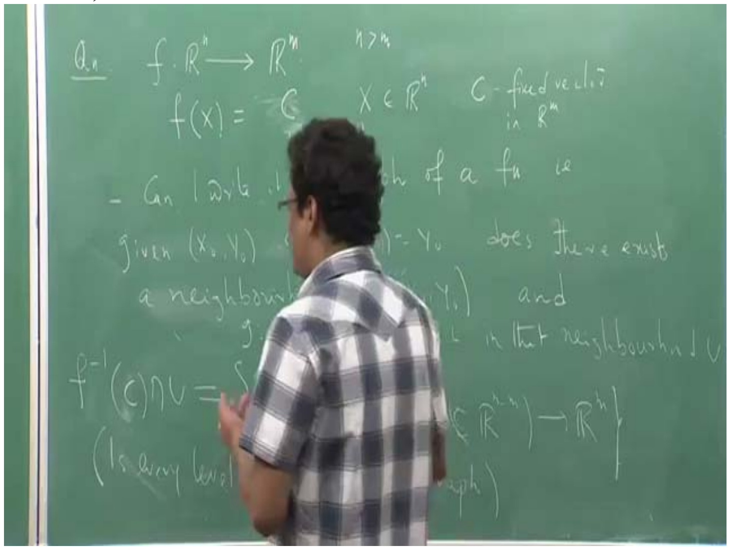

So this question simplified for a curve is this one.

So try to understand the statement first, question first.

Then we will see how we will get that, Ok? Statement clear I guess,

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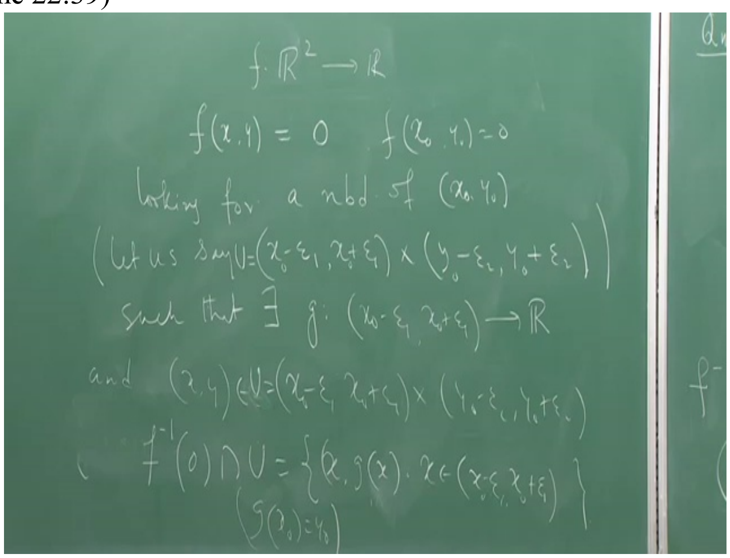

otherwise rewind your video and see it again

Well I don't want to state the implicit function theorem straightaway...

Before anything else, let us see if I want to achieve such a thing, what we need?

So from the statement, whatever information we gather...

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So what we need to define such a “g”

Let us see, one by one

So this should be a function, right? So in that neighborhood, so I need to decide

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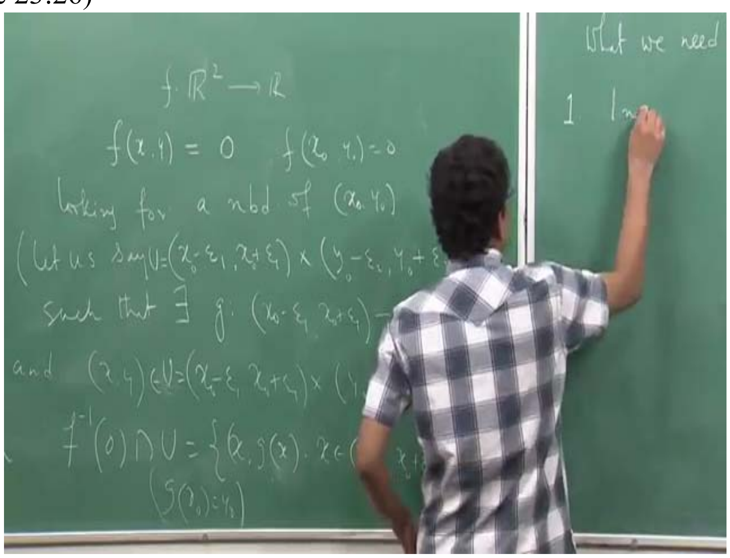

a neighborhood first, U

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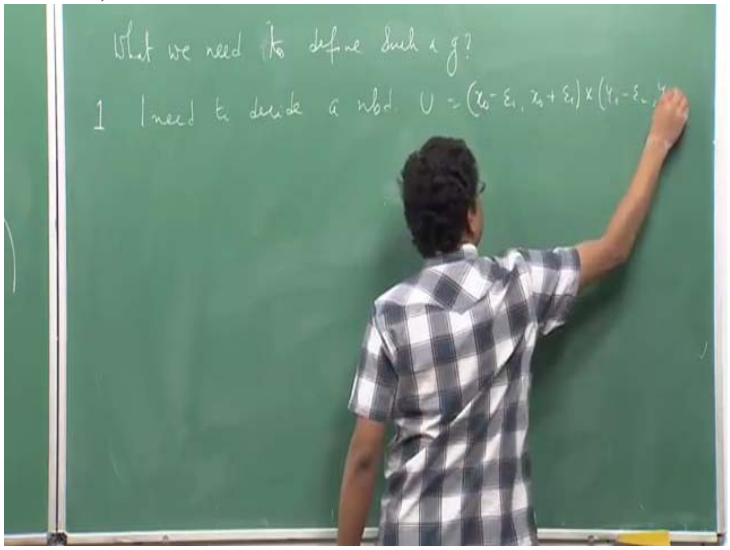

I need "g" as a function here and this level curve is a graph

So what I need there? I need to decide a neighborhood such that,

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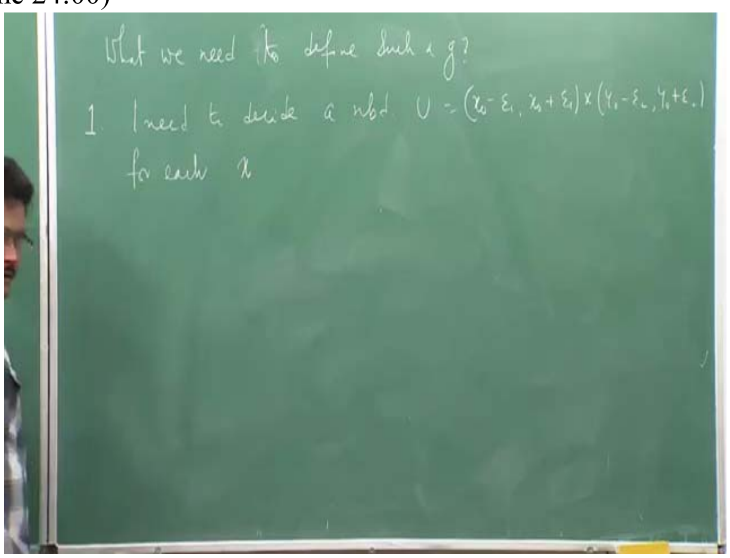

for each "x", there exists...this is the part, why "g" is a function, there exists

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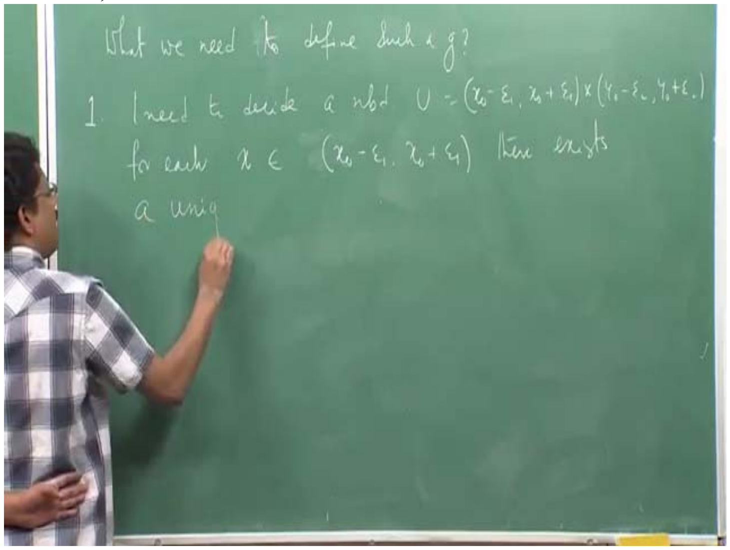

unique “y”, right, such that

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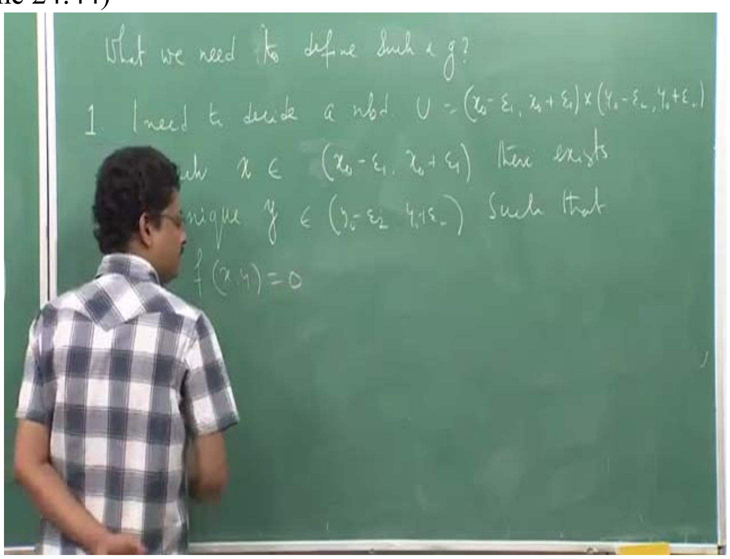

“f” of “x, y” equal to zero If there is no unique “y” I cannot define “g”.

That was a problem here, for the circle. Given this “x”, any “x” here, there are 2 “y”

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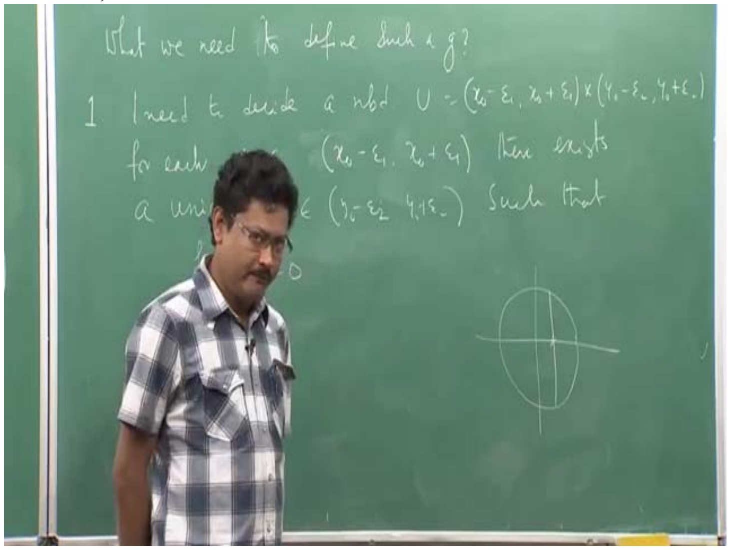

Ok So I have to decide...but now I am asking this question locally...so I will not look at the entire curve. I will look at; let us say this portion of the curve only. Ok, I will treat this...this part of the curve as

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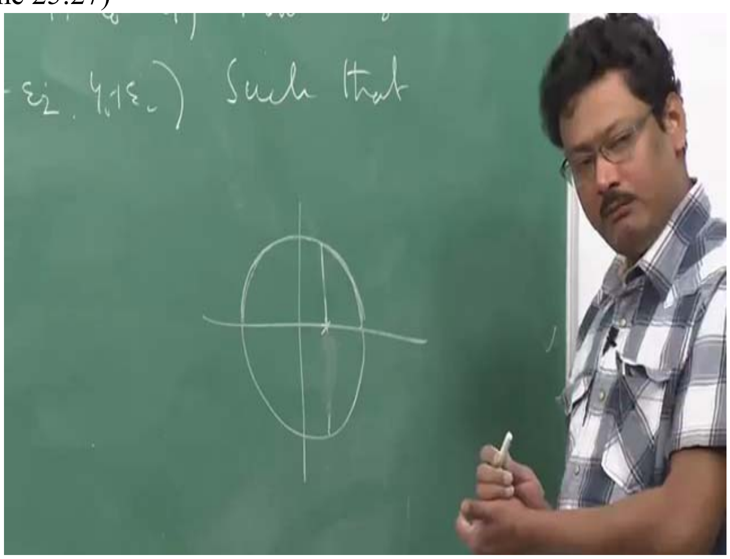

local level surface and here I have to...I can decide the neighborhood...of course I can take here, entire thing...this is a very nice picture so that for each “x” there is only “y”. So while I am asking this question, I am not looking at this part. I am looking at

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… this part only That is what...that is what is meant by looking at locally

Ok, so how do you do that? How to achieve 1? To achieve 1, one may go like this.

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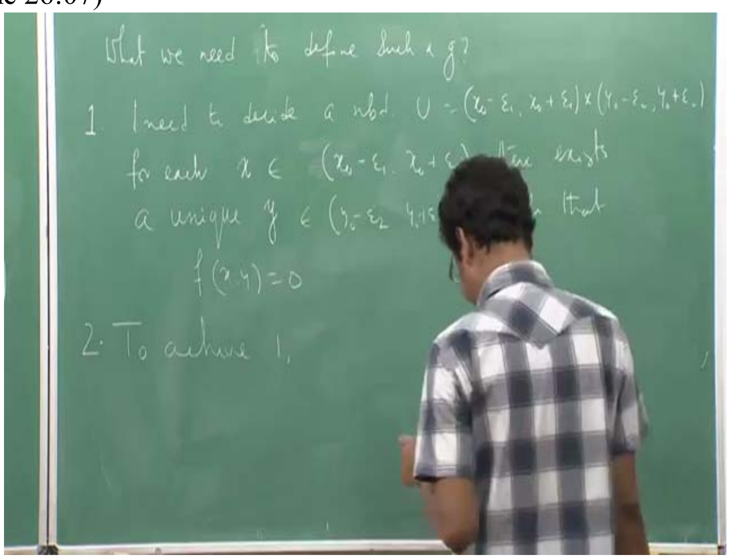

That .... so fix “x” in “x naught” minus epsilon to “x naught” plus “epsilon 1” and consider

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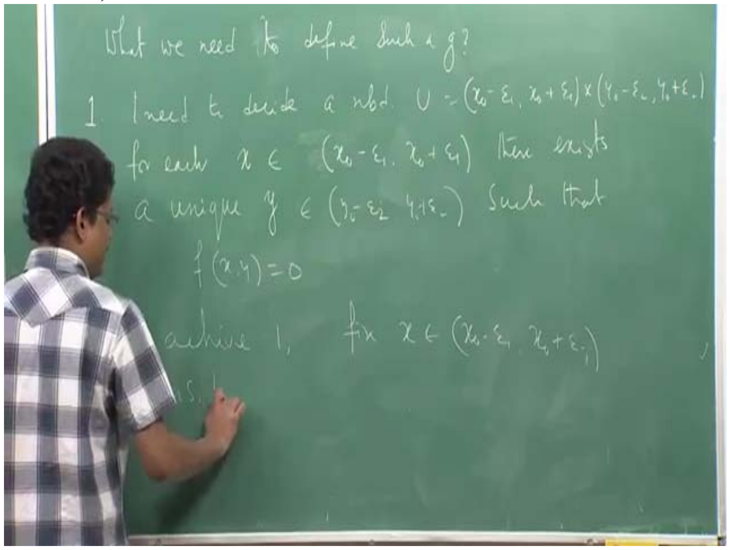

"y" going to "f" "x, y" What I need, I need an interval "y naught" minus "epsilon 2", "y naught" plus "epsilon 2"

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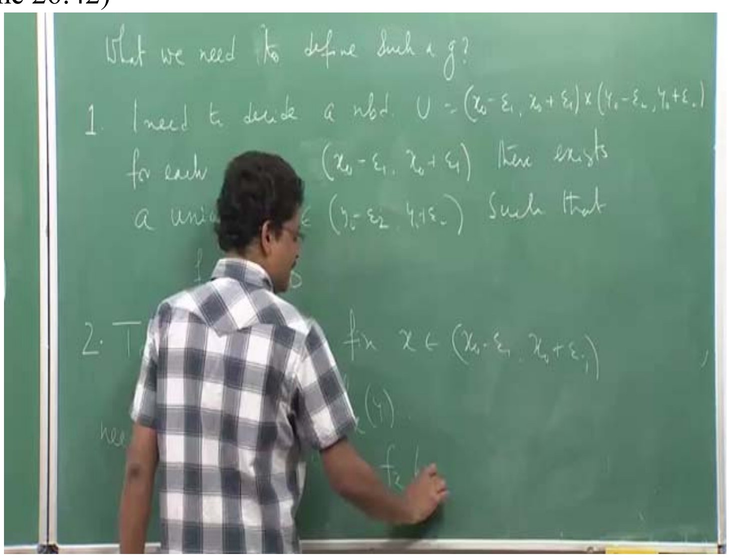

where this function has a unique zero, right?

That is what I have translated this part. Now how do you get unique zero? First of all, I need a zero. So I need a zero means...

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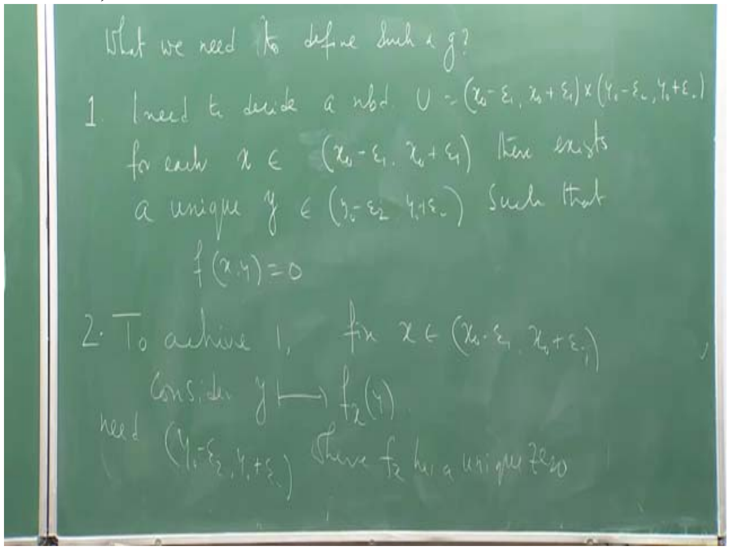

so I...some continuous function having zero, so if I do something like this...suppose “f” “x” at “y naught” minus “epsilon 2” and “f” “x” at “y naught” plus “epsilon 2”, they are defined sign...then...one is positive, one is negative,

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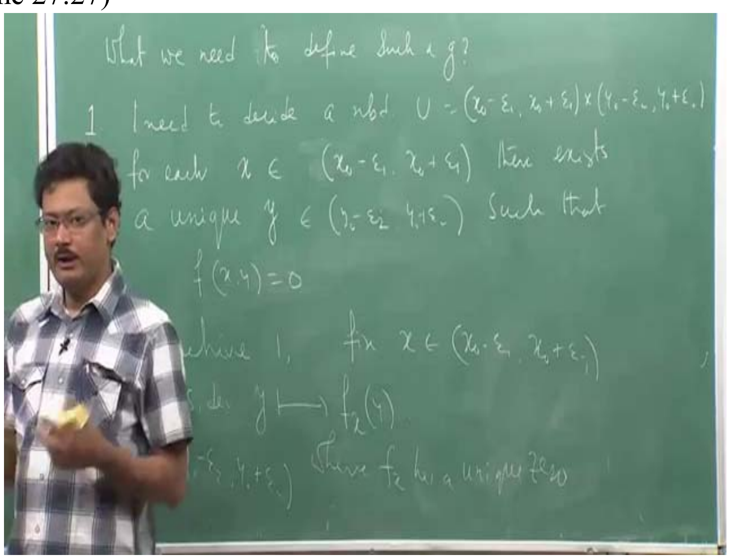

Then there will be, by continuity, there will be one 0. But I need unique 0. And how do you...so this will give a 0. How do I get a unique 0?

Well, if I ensure that

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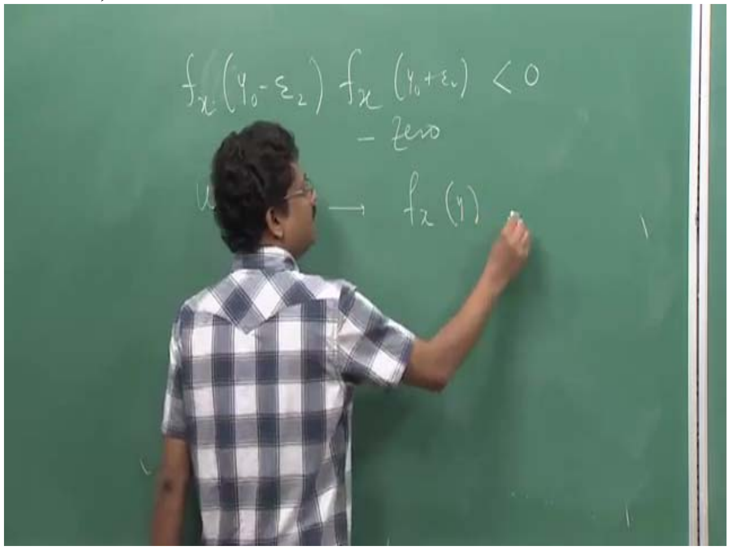

"f" of "x, y" is strictly increasing or decreasing in....

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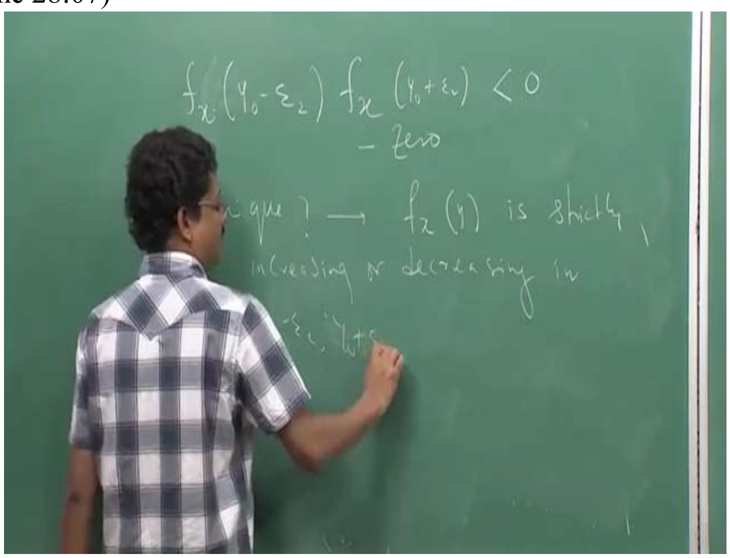

then it would be unique. There cannot be more than one. So this is the second reduction we will need. So we will continue from here in the next lecture.

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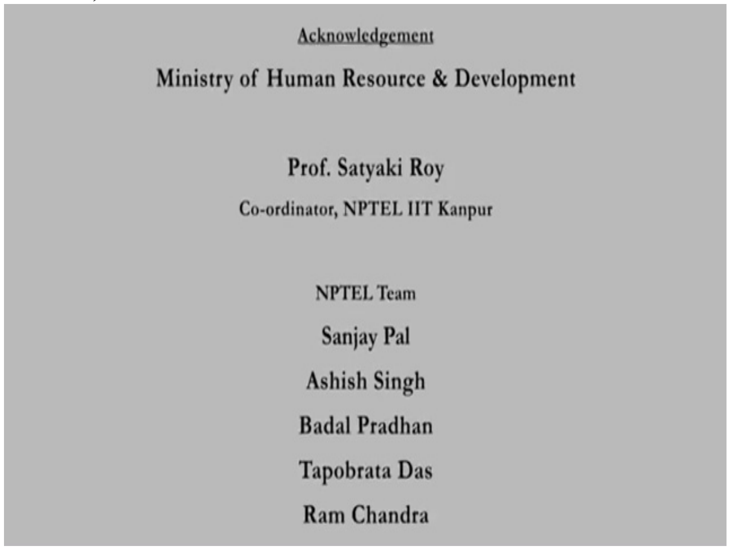

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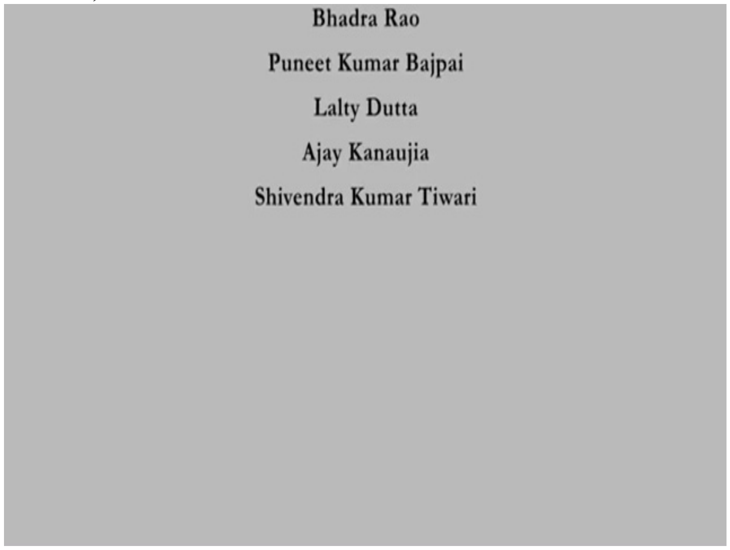

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