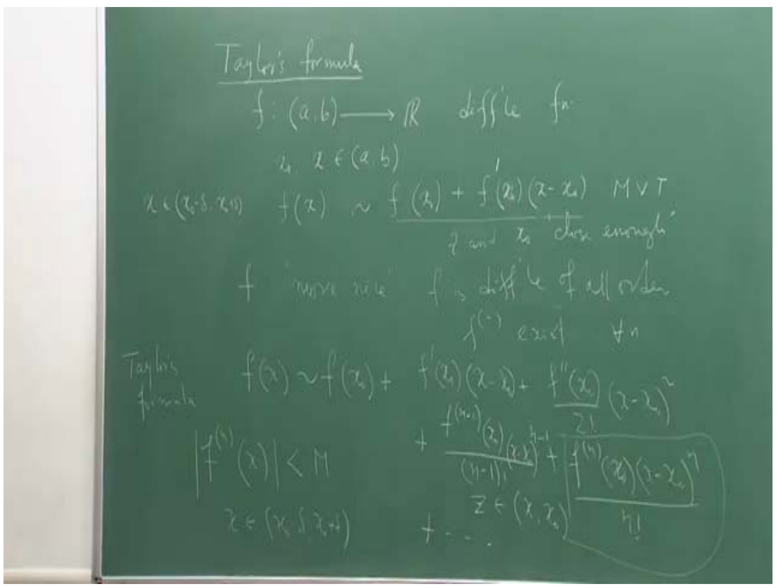
Differential Calculus of Several Variables
Professor: Sudipta Dutta
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur
Module 03
Lecture No 11
Taylor's formula.

Okay, so we start the third module today. This module will be mainly devoted to discussion on maxima minima for functions taking value in real. I mean to discuss about maxima minima we need to compare the values of F define on RN and you can compare only if they are defined, their domain is R.

[Refer Slide Time: 00:58]



So let us start again, the motivation from function of 1 value, so let it be a differentiable function, differentiable on entire A B, open interval A B. So you know that there one beautiful theorem is called Taylor's formula and I mean one can motivate this maxima and minima problem in many ways but explanation of this comes more beautifully if you understand the Taylor's formula.

So today let us concentrate on this beautiful theorem called Taylor's formula. So we will discuss a several variable part of it but let us recall what it says, I told you in the beginning, while you were discussing about the motivation for differentiability that… Suppose I have X not the point and X not and X two point in A B, maybe they are closed enough. Then this F X, so let us say X in some interval X not minus delta to X not plus delta then in this interval F X is approximated, right, by F X F prime at X not plus F at some point Z X minus X not this is…

Actually if you say MVT then you can write it as exactly this form, but Z and X not are close enough then you actually kind of approximate F X by this, right, Z and X not 'close enough'.

Close enough have to be justified with the other thing and that comes in the statement of the theorem, we will do it do not worry. What it says that, that F is differentiable then around X not in some interval X not F X linearly approximated, this is a linear function, X going to F X not plus F prime X not X minus X not, this is linear function.

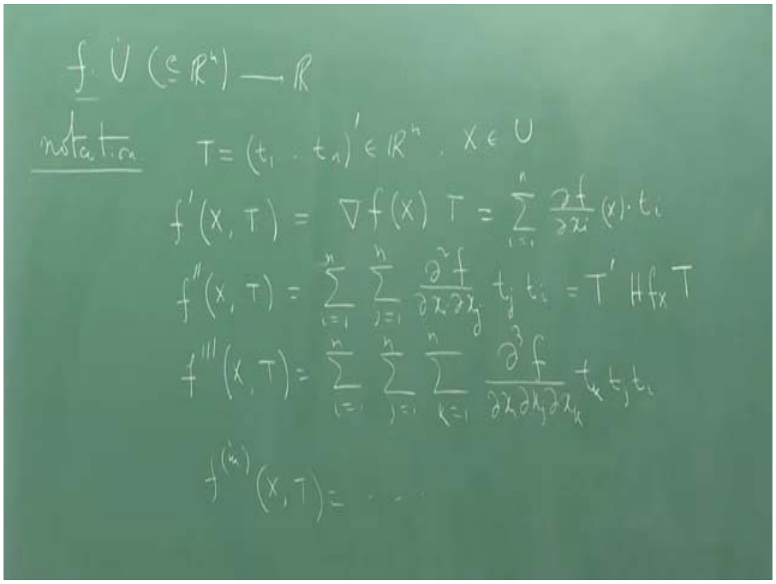
So F X is linearly approximated around F X not, now F is suppose, F is nice enough F is more nice in the sense that F is N times differentiable that is F prime X exists for 1 to N. Then this Taylor's formula tells you that F X is approximated around F X not, we have a better approximation, there is a first degree approximation but here we can have end degree approximation by uppernomial and it is much better approximation than the derivative part here goes faster than, derivative part here goes faster to 0 as X goes to X not than here.

And actually if you want to write exact expression then I have to write here then replace here by some Z, Z in X to X not, so for 'close enough' X not this is so called Taylor's formula. And this part is called as reminder term, right. So up to N minus 1 degree find this is all coefficient F prime X not and (())(6:28) X minus X not and N at coefficient is Z. So the previous term here was F N minus 1 X not, X minus X not power N minus 1 divided by N minus 1.

And this gives to so called Taylor's series also that if F is differentiable of all order that is F N exists for all N then to F X I can associate this series which goes on and if you put conditions like for all F N in for some delta, FNX is bounded for all X in X not minus delta X not plus delta then this series is actually converged to FX and in that case we will write equal to, that is the Taylor's theorem.

But in general given F X which is differentiable of all order we can associate with this series, Taylor's series it converges or not depends on certain nice conditions like this but there is a one way to express if it does, if it does there is a one way to F X as power (())(8:05) expression. So called linearity function. So far so good for one variable.

[Refer Slide Time: 08:25]



Now how do you generalize it to F from U in RN to RM, U open set. Now how do you do it once again the idea is that as you have done it before that is F is from U to RM so you have M components so do it component wise, so we will assume M to be 1, and if we can write down the generalize form of Taylor's formula, here we can write down for any F taking values in RM component wise.

Well to do this I have to talk about this higher derivative. F prime, F double prime we have already discussion this, but then I have to talk about any derivatives, and as you have seen that F N in the function for real variable so second derivative is already a matrix. So third derivative onwards you cannot write it do nicely and to put so what we try to do in mathematics that if you cannot write things nicely we try to take help of notations. We put some nice notations so that the final formula looks nice.

But putting it notation does not mean that the calculation becomes easy, but any way to put it or to look it nicely we will fix up some notations, well. So suppose I have a T, TN vector in RN, so this F prime at so, at some X in RN, X in U, X T just means Grad F X dot T so this is summation I equal to 1 to N del X I at X dot TI.

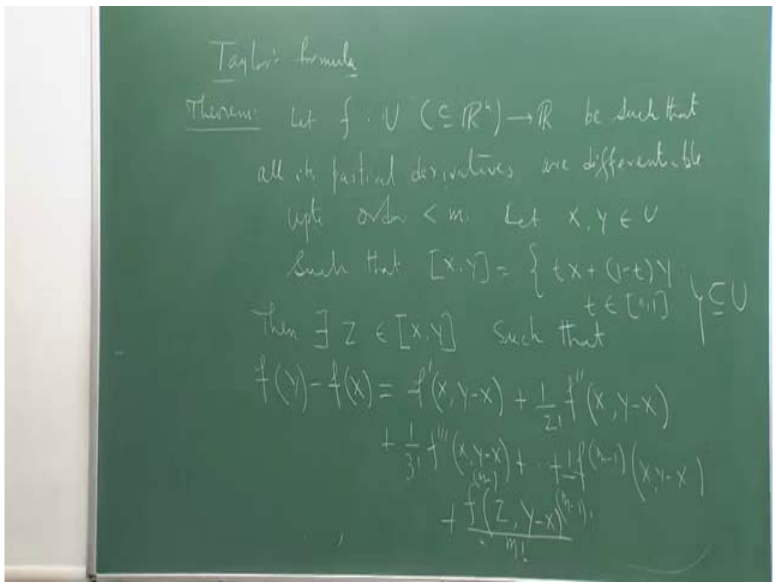
Similarly F double prime X, capital T equal to I write it in the summation for again then you try to recognize what it is, summation I equal to 1 to N, summation J equal to 1 to N, del square F, del XI del XJ (I am assuming everything exists of course) TJ TI. There is notation for this thing and if you recognize this is actually T prime Hession of F we have already introduced as FX.

Similarly F three prime X T summation I equal to 1 to N, summation J equal to 1 to N, summation K equal to 1 to N, del cube F, del XI XJ XK TK TJ TI if you want to write it in a compact, nicer looking form this will be actually T prime H FX T dot T. So this is (())(12:29) TI.

Do not worry about this part; it becomes more difficult when you write in general for any M or N. So you can write it in the summation formula.

There is a small hint that actually I have written I equal to 1 to N, XI XJ TJ TI but this is real numbers all Ts are real numbers, TIs, I could have written TI TJ, here also I could have written TI TJ TK but I have written it in this order because when you talk about complex function then certain things happen, then you have to put bars complex conjugate so therefore this is the standard notation we can easily generalize to complex.

[Refer Slide Time: 13:40]

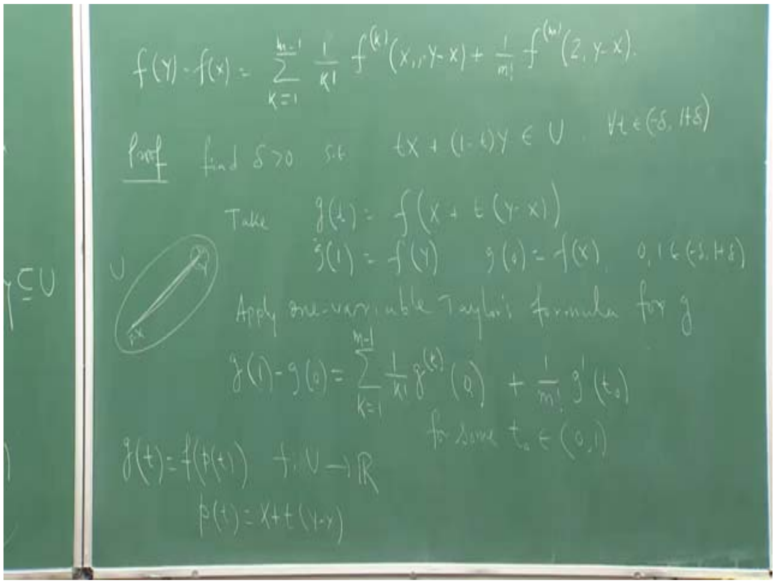


Now with this I can state the Taylor's formula for several variables which will of course match, so I will write Taylor's formula in this notation. So here is the theorem, let F from an open set U in RN to R be such that all its partial derivatives exists up to order M, up to order say less than M and differentiable, so partial derivative up to order M exists and they are also differentiable. I do not say F is differentiable up to order M or not, partial derivative up to order less than M exists and also partial derivative of M exists.

So I can put it in one, maybe not to confuse you too much, partial derivative are differentiable so it automatically assumes that they exist, differentiable up to order less than M for some M. Let X and Y be two points in U such that the line joining X Y we have used this notation before, this is set of all point T X plus 1 minus TY T in 0 1, this is also in U. So if you use a convex set there is no problem here.

Then there exists a Z in the line X Y, such that F of Y, let me write it in this form F X equal to how do I write it, okay, F prime X at Y minus X (I am following that rotation) half of Y minus, half means 1 by 2 factorial, 1 by minus 1 factorial plus F M Z at, so this is a eroterm divided by M. This is the generalization of Taylor's formula, so written more compactly.

[Refer Slide Time: 17:25]



This will look more nice you will agree, so this is the writing, this approximation part plus ero part. Well if I prove it, the same proof will follow for N equal to 1 what we started and proof is again very easy. Proof is just using first variable for Taylor's formula (18:04) variable function. And we have already done it before in case of last week what we are doing Mean Value Theorem.

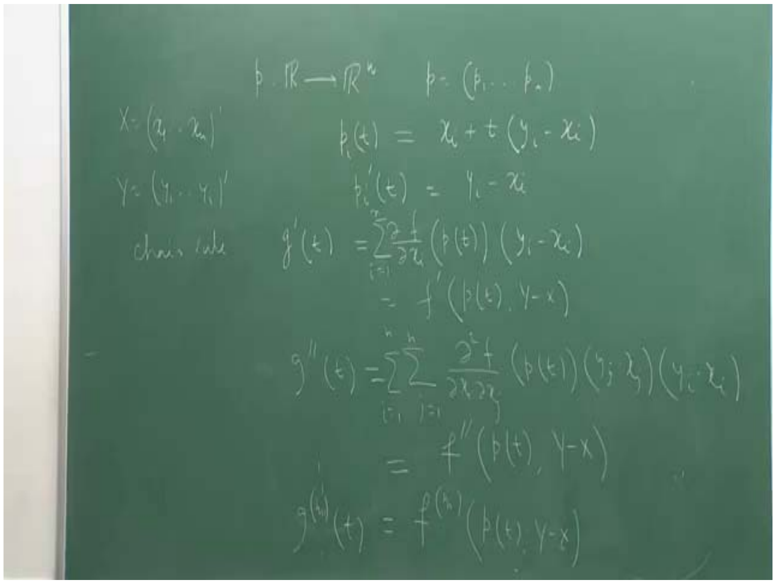
So let's say we proof here, as you can guess I will start with U is an open set so what we will do, I will find a delta greater than 0 such that TX plus 1 minus TY this belongs to U for all Tin this is again as I said to avoid one sided derivativeness at N point. So here in X and Y since this line is there and there is the ball like I can also extend little bit, I can extend here little bit, so I have extended this line, this line is there, little bit extended like will also be there because U is open.

And we look at the same kind of function here that we did for MVT. So take GT equal to F of X plus T Y minus X, okay. Then F 1 equal to G, sorry GFY, G1 equal to FY and G 0 equal to FX and as you can, and you see 0 and 1 both are in this interval so I will apply the Taylor's formula for G, so what I will apply one variable Taylor's formula for G and what we get G1 minus G0 equal to summation K equal to 1 to M minus 1. If F is differentiable, this is the composition of two functions, F and a linear function.

So just observe, GT is composition of two functions if PT where F is as according to our assumption U to RM R and P T is simply X plus T Y minus X. This is a linear function so differentiable of all order and F is assumed to be all partial derivative are differentiable up to order less than M, G will have the same property by Chain rule.

So I will write it 1 by K factorial, K derivative of G at 0, 1 minus 0, 1 minus 0 is 1, so I should have 1 minus 0, 1 minus 0 is 1, so this part plus the (())(21:49) part 1 by M factorial for some T not in 0, 1. Correct. Now what I have to do to get this formula I have to see what are these fellows are. Let us see we will apply Chain rule here.

[Refer Slide Time: 22:20]



Well what will happen, P is from R2 to RN so P can be written as P1, P2, PN components where each P I T so if my X is X1 X2 XN, Y is Y1 Y2 YN then each P I T is X I plus T, YI minus XI so PI, PI prime T is just YI minus XI so that will be me my Chain rule. G prime of T is how much, F prime PT and following that notation at derivative of PI that is YI minus XI, sorry, which is by your notation F prime PT Y minus X.

Now you apply G double prime T, this is what will be there, again apply Chain rule, now there will be summation I equal to 1 to N, J equal to 1 to N, del square F, del XI del XJ if at PT YJ minis XJ YI minus XI which is according to our notation is F double prime PT Y minus X, so on you continue you will see GMT equal to FM ET Y minus X.

Now when I put it back I will have G K at 0 and now observe P 0, P T here, P 0 is simply, where is P, here, P 0 is simply X. So put it back you will get the formula I have written for, Taylor's formula I have written in the theorem, so put them back altogether. So this is the straight forwards proof of Taylor's formula. So next time we start from the Taylor' formula and talk about maxima minima.

Thank you!