Probability and Stochastics for finance
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Lecture - 09
Brownian Motion-I

So if you go to the stock market and look at the price of say a favourite company's share. So what you would observe is the following.

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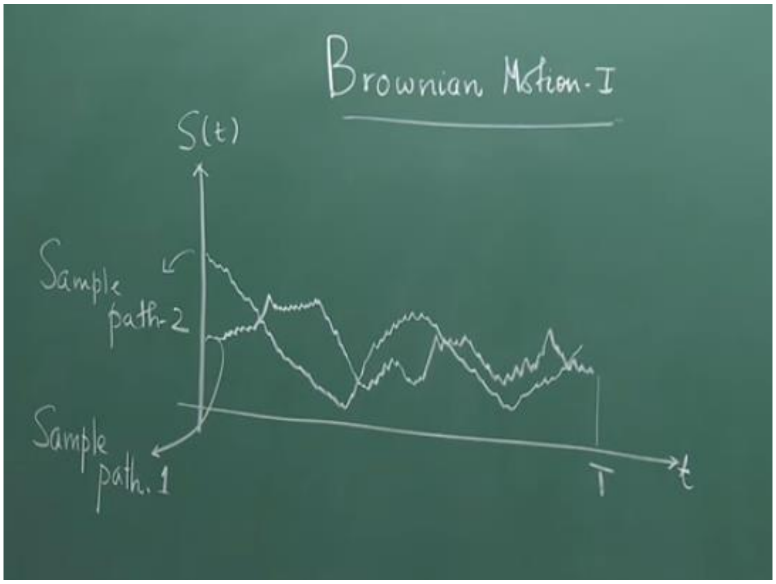

So if the horizontal axis is the time axis and if this is the price axis it tells you what is the price then you will see starting from a certain time price you will see some zigzagging motions like this. So up observe it up to time T and you observe this zigzagging motion.

This is of course random. Nobody knows what is the next price is. So if a particular scenario evolves you have a particular path. This is called a sample path. So if another scenario evolves, if another scenario evolves there would be another path, for example it could be like this. The stock price is going down down down down and you are in a bad shape and then it again climbs up and again it falls down down down and again then again climbs up.

So under a different scenario it has a different path. So it is sample path 1 it is sample path 2. So it is what type of scenario one evolves.Now of course you can ask me what is this term scenario that you are talking about, what is the meaning of this God damn scenario. We will come to this very soon. But how do I model such zigzagging paths, what way to model it. Is there any mathematical way to say that a, or can I construct the stochastic process whose sample paths are represented in this form?

Let us do, to do that we need to study what is called Brownian motion. Brownian motion is a type of stochastic process which will help us to model stock prices at the end. So the whole term Brownian motion comes from the name of Robert Brown who first studied the movement of pollen grains in water and he found that they were having a zigzag haphazard movements. But it is not so immediately apparent that you can just start writing about this particular stochastic process.

We need to have some more idea and built upon some simpler stochastic process. So we will begin by introducing what is called symmetric random works.

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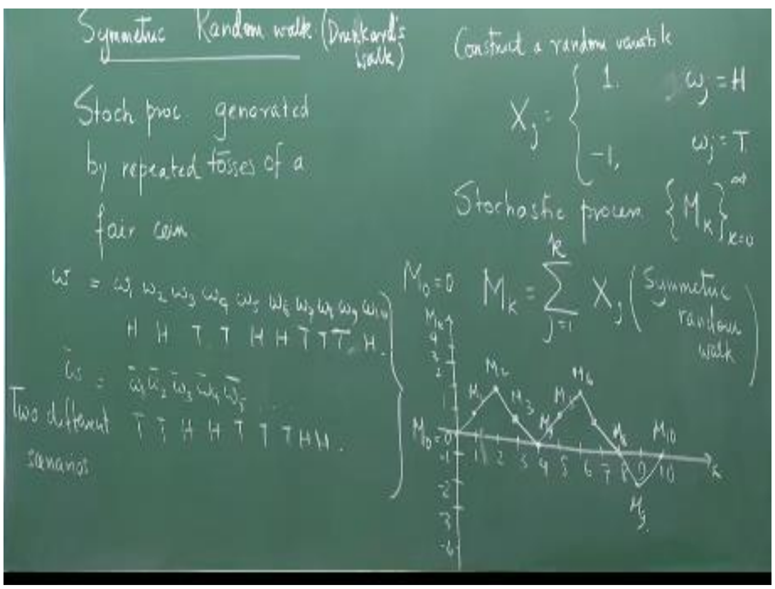

Where there are only 2 possibilities you can either go up or down that is like a coin toss head or tail and that here I can have infinite such possibilities, infinite such sample paths and there are infinite possibilities also. Here also we will have infinite possibilities but generated out of only 2 possibilities. So symmetric random walk. So when you take a fair coin and then you keep on repeatedly tossing it.

So it is a so this is a stochastic process which I will write in short form now as stochastic process generated by repeated tosses of a fair coin okay and if you look at it very carefully what I mean by this? So you start tossing the coin so repeatedly you are tossing omega 1 omega 2; so omega 1 is either head or tail; omega 2, omega 3, omega 4, 5 and so and so forth. Suppose you have here head, head, tail, tail, head, head, head, tail, tail, tail and it goes on.

So this is one particular scenario that has evolved. You could have another scenario say omega bar which is consisting of say omega 1 bar, omega 2 bar, omega 3 bar, omega 4 bar, omega 5 bar, it could be something like this; tail, tail, head, head, head, tail, tail, tail, head, head, head and so on. So these 2 are different scenarios and these 2 each would generate 2 different sample paths. So how do we generate this symmetric random walk?

So these are 2 different scenarios, 2 different scenarios. Now construct a random variable Xj which takes the value 1 if j is equal sorry if omega j is equal to head and takes the value -1 if omega j is equal to tail if tail appears and 1 if head appears. So now you define a stochastic process, define a new stochastic process Mk, k=0 to infinity or let us we can need not bother we can also fix it after some time.

It could be some time capital say K is say 25 something here 25 or 30 whatever. But in general it is alright to take plus infinity just a sequence where M k is given as follows. Each of these M ks are calculated by starting from M 0 equal to 0. M k is equal to j is the sum from j from 1 to k to X j. So let us see what would happen if one particular scenario like this evolves. Let us see then what is the sample path of this. This symmetric random walk is also called a drunkards walk.

So somebody has had a good drink and he has become drunk and if you look at his walk so a drunkard would walk like this if I am here so I start from here then I can just go like this and he goes like this just it is just or like this you know I am coming here and then I am going there something like this. So this sort of thing you will immediately observe as I start say checking out with this scenario.

So here is my k and here is my M k value. Now the first one here has turned out to be head. So M0 is 0. Let me write -1 -2 -3 and so on -4 here 1 2 3 4 and so on and of course here also you have to have k values which is 1 or maybe 1 2 3 4 5 6 7 8 9 10 and so and so forth. So M0 is 0 this is 0 is 0. Now you toss a coin and you have head.

Omega 1 is head so you go up by +1 because X j will take +1 because m1 is just X 1 so here is the value of M1 this is your M1 so you join the M0 and 1 by line so 0 is M0 and then you again had head so M2 is again 1. I am looking at this scenario M2 is you go by 1 so it is 1+1 now 2. So M2 is 2. So you join again by this line. But M 3 is tail so you will drop by 1 so it will be -1 so it will again drop back to the point 1.

So this is your M2 and this is M3 and then omega 4 is again tail so it drops back to 0 again you -1 subtract. So this is your M4 0. Again then you have head for omega 5 say so here you have omega 6, omega 7, omega 8, omega 9, so for M4 you have tail you have come to 0 again then it goes up again for m it goes to plus 1 again.

So this is your M5. Again, it goes up to 2 M6 but then you have tail again so M7 comes down to +1. Again, you have tail so M8 comes down to 0 because you are adding up everything. At every step you are going up or down. So you add up in this fashion and you move like this. So M8 I have here omega it is again omega is tail so again I have to go down so I go down by so from 0 I will have to go down by 1 so I will come to -1.

So this will be your M9 and if suppose M10 omega 10 is head then it will again go up to 0 at the 10th place because you will again add 1 so it will become your M10. So what you see that the symmetric random walk is providing me some zigzag looking curve which might tempt you to think that possibly these 2 have some relationships. They are they do have some relationships and we will talk about that slightly down the talk.

But let me tell you some more properties of this symmetric random walk M k. So here is my stochastic process and this stochastic process is called the symmetric random walk. Of course we are not mentioning but underlying we are always taking some probability space and all of those things.

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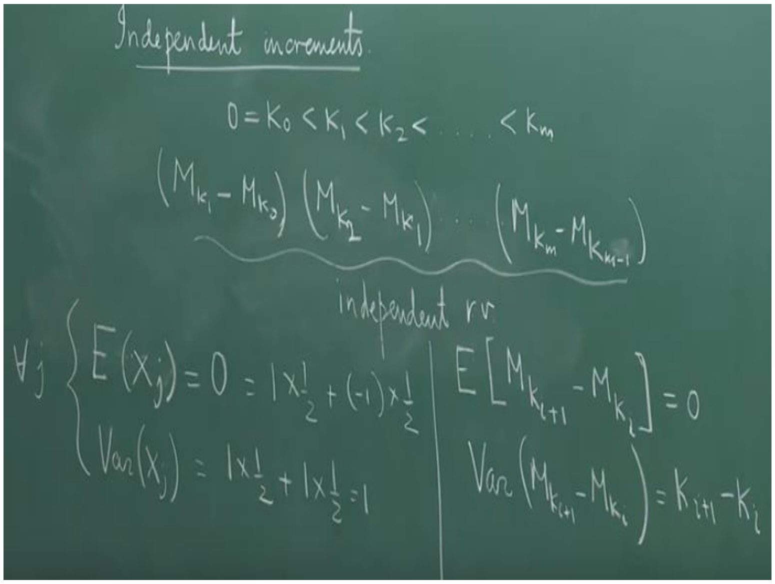

So now we will these random this particular random process of stochastic process has independent increments. What do I mean by the fact that they have independent increments? What I mean is the following. So if you have you take certain numbers sometime some K m then you have the following.

You have M k1 that is once you consider non-overlapping intervals then this difference is independent because they depend on independent coin tosses because coin tosses are independent when the coin tossed at the second level really does not the second outcome does not really depend on the first outcome right when you do a repeated coin toss sorry M k2 - M k1… M k m - M k m-1.

So these random variables are independent. All of these random variables these form a set of independent random variables. So that is once this happens this is when we say that it has independent increments and this actually has independent increment. These are independent because their difference which really does not depend on the coin tosses here does not depend on the coin tosses here and here and so you have independent increments.

So this change that you see here does not depend on the change that you see in this interval or the change that you see in this interval right. So you can still observe that it is like a drunkard walk so drunkard walks like this goes down goes up. In George Gamow's famous book One, Two, Three Infinity this has been described in a very very nice way.

How do you, see here, my success probability this occurs with probability half and this also occurs with probability half so if you observe that exponential of X j sorry expectation of X j is 0 because this is one into half plus minus one into half. Variance of X j so what is variance of X j exponential X minus X j whole square which is 0 so it is 1 into half plus minus 1 minus 0 whole square plus 1 into half which is 1.

Once you have this information this is true for all j. It is immediate that exponential M of k i plus 1 minus M of k i is 0 and the variance of M of k i plus 1 minus M of k i is equal to I leave it to you to calculate these stuffs. Our second property about this random walk is to show that this is also a Martingale. You see Martingale thing comes up. So symmetric random walk is a discrete Martingale.

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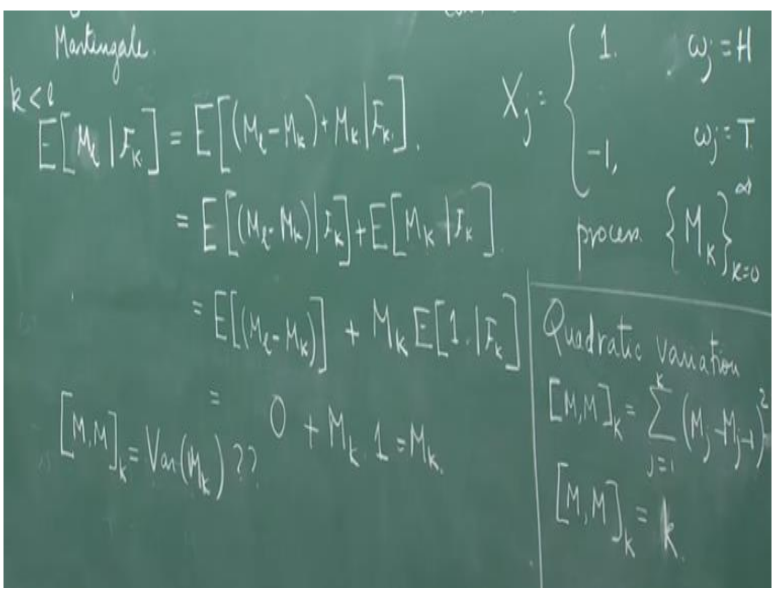

So you can easily prove that it is a Martingale. So you take any k strictly less than l and look at the expectation of M l conditioned on the filtration, the Sigma-algebra F k which is the part of the filtration okay.

So you can write this as M l minus M k plus M k. So these can be summed up just like expectation can be some conditional expectation this random variable can be decomposed into 2 parts which you can actually prove which will be a part of your exercise but we are just using this fact here so, anyway I should rub the board a bit.

Now let us look at the first part. Since l is strictly bigger than k this increment M l minus M k is independent of F k. F k does not have the information of anything which is beyond the time k. So here by one of our rules for conditional expectation this is nothing but M l minus M k and here at time k everything about M k is known. So F k contains all information about M k. So the first law was taking out what is known, I can write this M k as M k dot 1 where 1 is the constant random variable 1. So whatever be the scenario it will just give you the value 1.

So I can write this as M k so I will write this as M k dot 1 so I can take out what is known 1 dot F k. Of course, 1 is a constant random variable. It does not really depend on is independent of F k so it will be E of 1 which is a constant which will be just 1. So everything will be 1 so the sum of the probabilities will sum up to 1. So the expectation will be just the number.

So this again is 0 which we already know plus M k into 1 which is M k and so this shows that M l is a this symmetric random walk this thing forms a discrete Martingale. Of course, F k is M k has to be adapted to this filtration that is the basic definition of Martingale. There is another notion which crops up in the study of these sort of processes is called the quadratic variation. So you essentially look at path by path.

You look at how much the random variable values are varying between one end of the path to other end of the path that is between k1 and k2 say how much it is varying but do not take just the sum of those variations they might just be 0 so you have would not get any information but take the square of the variation. It is like a mean square error type thing so we again take here and introduce the notion of a quadratic variation.

So the quadratic variation is expressed in the following way. M, M k is defined as summation j equal to 1 to k M j minus M j minus 1 whole square is equal to and this if you look M j minus M j minus 1 whole square this value is always 1. If you sum them up what will be left here, X j would be left here, the X j. If you take the difference between M j and M j minus 1 you will have the value X j left. X j is either plus 1 or minus 1 so the squaring will always give you 1.

So this expression of the square errors basically or the square changes around every path of a given sample I can take the changes but whatever be the path independent of the path it turns out to be k. If you do up to the kth level it turns out to be the k independent of the path that you have taken which is very very interesting.

It does not happen for suppose you want to compute the variance so M, M k is actually variance of M of k think about it how it is possible. But you see to compute this I really do not need to bother about the path but to take variance of M k we are essentially averaging over all the paths. So this is a difference.

Now how do I can I do something with this process. Can I increase the jiggling of this process a bit this symmetric random walk a bit and generate some sort of an approximation of a Brownian motion. Generate this sort of zigzagging that we had just seen in the beginning when I had drawn the picture of the stock price that this sort of zigzagging can we generate this sort of zigzagging this sort of zigzagging can be generated by using the symmetric random walk and that leads to what is called a scaled symmetric random walk.

We will not go too much of details into it because that might you know take you off track and you might feel a little bit of discomfort for those who are not so very comfortable with very complicated analysis. So what we are going to now show by this scaled random walk is that what we are going to show by scaled random walk is that we can construct an nth level approximation for the Brownian motion.

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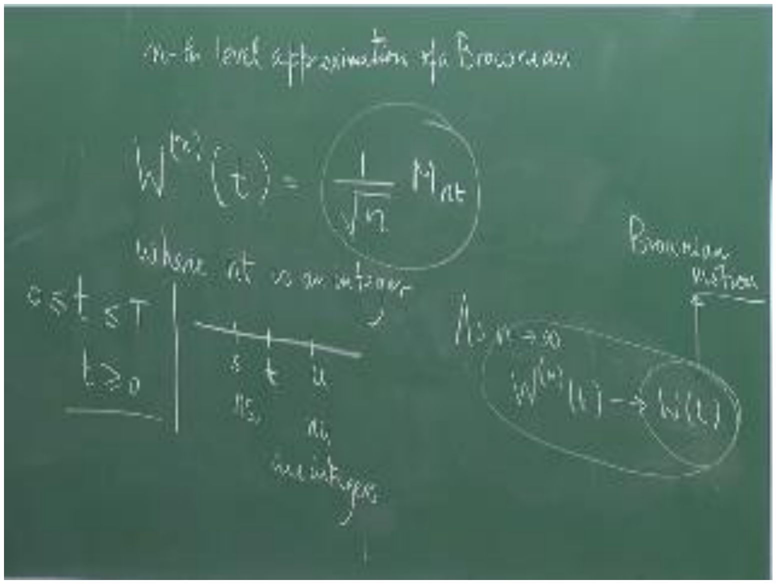

Let us construct an nth level approximation. So if you are zigzagging by say +1 and -1 I might zigzag by 1 by 10th and minus 1 by 10th. So I will decrease my zigzagging steps but increases my time size right so we construct the scaled symmetric random walk which is the nth level approximation of a Brownian motion. So all these are stochastic processes so this is a discrete stochastic process from which I am trying to go to a continuous stochastic process. I define it like this.

You see if I do not have nt to be an integer I cannot define this. So here my t is a t say t between t starts from 0 and say it is up to t or even t goes to infinity so basically for me here this t is just greater than equal to 0. So using the discrete thing I am trying to construct a continuous stochastic process but I have to be aware that if I really want to use it so at the nth level approximation this m and t this nt has to be integer if I want to actually compute this.

Otherwise m is a discrete thing it is computed only at integer points you cannot compute it at non integer points. So what happens if it is not computed, if nt does not turn out to be an integer? So basically what you are considering for n very large at various time points nt would be an integer and you are actually computing out of nt. So if nt is not an integer take the t for which is not an integer then take some u and take some s which is nearest to t such that ns and nu.

These are integers and then compute the value of Wnu and Wns and then make an interpolation linear interpolation to approximate the value of Wnt and that is how you can actually do you can generate it in a machine by taking a sample so you can take a sample of say so you can do the coin tossing 400 times with one by tenth you can toss the coin 400 times with probability of half of going you go one by tenth if it is h and you go minus one by tenth if it is tail.

So you decrease your movement so you actually increase the zigzag by decreasing the movement and at every time you have to observe that your Mnt nt has to be an integer. Once you do that you will find all the properties that you had for here is in here provided that this M n into t is an integer. So you first do it only for n into t is a integer. Whatever is left you do the interpolation and you will see you will start getting a zigzagging curve much zigzagged than the symmetric random walk itself.

Actually it can be shown that as n tends to infinity as n tends to infinity this W nt converges this random variable converges almost surely sorry not almost surely I made a mistake converges in distribution rather converges in distribution.

Okay these are terms which I have not mentioned. Just forget them for a while, converges. In some sense W nt as n becomes infinity. This this stochastic processes gets changed into what is called a Brownian motion. So this is what we are going to talk about in the next class. So tomorrow we are going to study the properties of Brownian motion for the next 2 classes.

So tomorrow’s class would be the last for the second week of the course. In the third week we continue our discussion on Brownian motion and then go to understand stochastic integrals or Ito integrals and doing Ito calculus which is the foundation of any financial mathematics that you do. Thank you very much.