Probability and Stochastics for finance

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Lecture - 08
Martingales

Okay so we are going to talk about Martingales today. So what are Martingales? We cannot immediately approach that Martingales are particular type of stochastic processes because stochastic process behaves in a certain way, we will call it Martingale.

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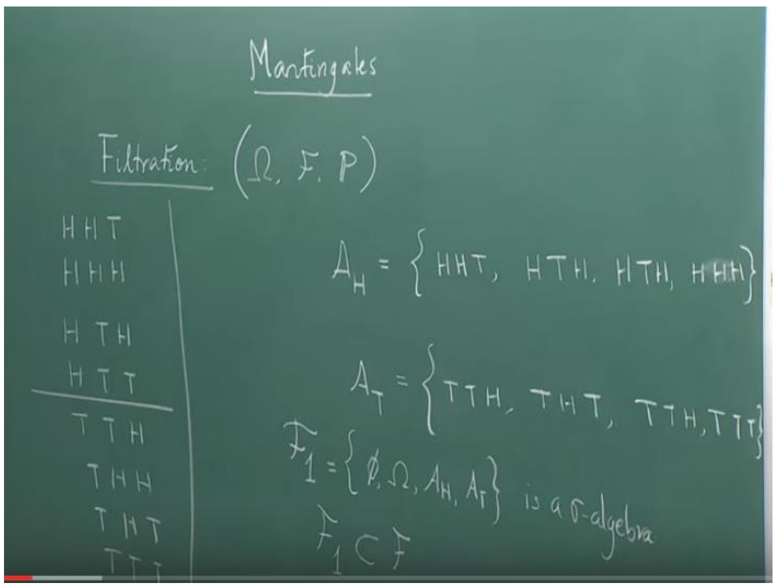

In order to understand martingales, we need to first talk about the notion of a filtration. Of course in any discussion that we begin with always keep in mind that even if I do not mention it underlying everything is the following probability space okay. Now what essentially is filtration? Filtration is the part what happens is that as a random experiment progresses and new information becomes available you know which part of the Sigma-algebra you already know.

So some part of the Sigma-algebra would be completely revealed. So some part of F or subset of F would be revealed once we start getting more information. For example I throw a fair coin. So if I throw a fair coin, there are 8 possibilities okay. So let me write down what are the 8 possibilities.

This is 1 or we can have all heads; head, tail, and then head; head, tail and tail; tail, tail, and head; tail, head, and head; tail, head, tail; and tail, tail, tail. These are the 8 possibilities if you throw 3 coins or 1 coin thrice in succession. So you are doing a repeated experiment. Now suppose I know that the first coin I tell you okay suppose I do not allow you to see the experiment, I am conducting the experiment and I tell you okay the first coin has turned out to be head. Then what are revealed to you?

What events are revealed to you? If the first coin is head is the situation is this one, means you know if the first coin is head the only possibilities now are the following sorry. So once I know that the first coin is head there are only these possibilities that can occur or maybe if the first coin is tail so if the first coin is head then these are the possibilities that can occur. If the first coin comes out to be tail then these are the possibilities.

So if the first coin, the outcome of the first toss is known to me, then given any omega, if you give me any omega in any sequence of 3 tosses, I can tell you whether this will be revealed or this would not be revealed, this will be an outcome or this will be not an outcome. So what I tell you okay the first coin has come out to be head then you say okay can tail tail tail come, no. the tail tail tail come will come in the compliment of this H this is the compliment.

So what we have done? We have essentially segregated this thing, separated this out. Made a finer division of these 2. So given any omega now, any omega does not belong to the empty set, every omega is belonging to the whole sample space which means the sample space and the empty set is always revealed. You know that either there will be nothing or there will be everything basically.

But the interesting part is that now given an omega I can tell you whether that event will now occur or will not occur. I have the information. If you say first toss is actually head, I can tell you what will actually happen, what are the next consequences any of those 4, right. If you say okay what about T H T can this consequence will be there no, it cannot be.

So if you look at it the following sets of the Sigma-algebra F is now revealed which I called F1 which is, see if I know I have a knowledge about the first toss. These are the sets of the Sigma-algebra which is revealed and this itself F1 itself is a Sigma-algebra because it follows all the rules of the Sigma-algebra because if you take the union of these 2 it will become omega.

Now if you take the intersection it will become phi. If you take the compliment of H it is A T compliment of A T is H. So this is a Sigma-algebra. So this part of the Sigma-algebra, so this is of course you immediately see that F1 is a subset of the Sigma-algebra F. So part of the Sigma-algebra gets revealed when some information is revealed and the next stage I said okay good I tell you what has happened in the second toss.

Then I will have more finer revealing. I can again partition this into more finer parts when basically I am breaking up the space F the Sigma-algebra F.

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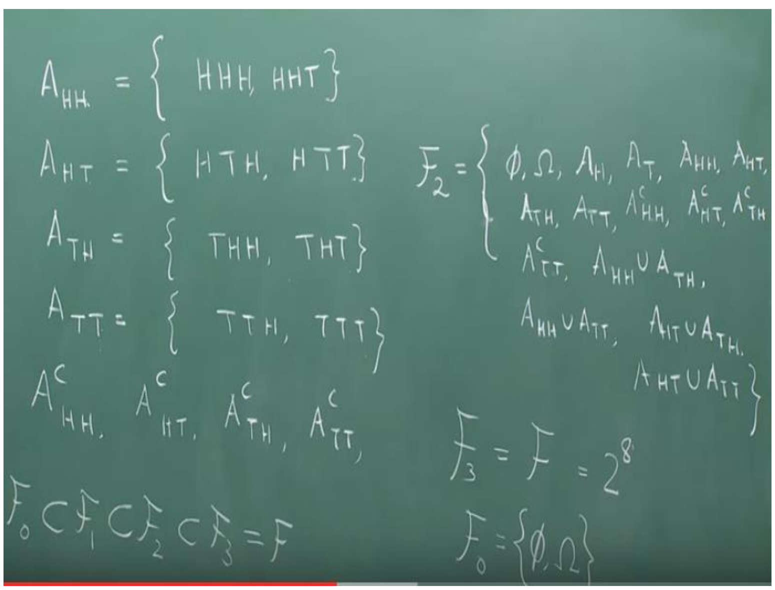

So you tell me that so what can be the second toss either both can be head first head and then can be tail or the first can be tail the second can be head the first can be tail and the second can be tail. So if I tell you what are the 2 consecutive things you know what can occur now. If I have the knowledge of what is also the second outcome and also I know both the outcomes I know what are the occurrences.

So here it will be this, nothing but just augmenting with head or tail that is all. So you have made more finer divisions of this basically. So I know if you said that the first coin is head and the second coin is head if you said the second coin is head so these are the 2 things that can happen, anyone of the things can come. Second coin is head because the first coin is could be head could be tail. So any one of these 2 things can come.

Now you see can I now make some Sigma-algebra out of this information. So this sigma that sigma here for example this Sigma-algebra totally encodes the information that the first toss is known. If I know the second toss, can a Sigma-algebra be constructed which can encode all the information. So of course you should have once if you want to put construct a Sigma-algebra if they are all inside that Sigma-algebra then all this has to be also part of the Sigma-algebra because A C H H is not any one of them but this whole the union basically.

So you basically do not have to when you construct the set you do not have to write the union of this union of this union of this 3 or union of any of the 3 because that is the compliment of the remaining. So you can take the compliment. So that is it. So you take the compliment and that is so you take the compliment. Of course you have to take some unions also. For example if I take this union such a union does not such a union for example does not appear in any one of these sets already known.

For example if you take this set and this set and take their union there is no where I can find anything right. But if you take this set and this set and take their union and this set is A T. So I do not need to bother about the union of these set but I can I have to bother about the union of these 2 sets, union of these 2 sets which were not there to create the Sigma-algebra.

You see H and A T is anyway revealed even if I know the second choices. The first choice must be either head or tail so these are already there. So whatever is known at the first stage will always be carried on to the second stage because they are anyway revealed. For example if I take the union of these 2 and take the compliment of that that would anyway give me A H. So A H and A T anyway will continue to be revealed right.

So essentially if I want to construct F2 you see how far the size of the cardinality of this set will grow. So A H and A T will anyway be there okay. You will also have these sets A HH, A HT, A TH, A TT. Of course then you have to line up their compliments okay. Now you can combine this with this. Do not combine this with this because this with this will give you A H so this is already there so you do not have to show that combination.

So A HH union A TH must be there must be considered. A HH union A TT must be considered because they would not form anything which is already given. A HT union A TT must be A TH must be considered these 2 and A HT union A TT must be considered means those things which do not appear at all. You see your from just 4 elements here we have increased to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 sorry 1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18 just blowed up actually.

Now it comes F3, all is known. All is known, means F3 means I know all possibilities so I basically know the whole Sigma-algebra F. So F3 and the third if I know all the coin tosses are known and F3 is nothing but F and that will have 256 possibilities, 2 to the power 8 combinations because there are 8 possible elements in the Sigma-algebra.

So what I have, now let us talk about what is F 0 then. F 0 means when nothing is revealed to me. Then I have only 2 choices. Either nothing will happen or every possible choice might happen right. So F 0 consist of just so this is under no information. This is under 1 information that first toss is something. This is under more information the second toss is revealed and this is under the third we have revealed everything.

So everything is revealed so which means if you look at them you have F of 0 belonging to F of 1 belonging to F 2 belonging to F of 3 belonging to F where F of 3 is equal to F you do not write have to write F of 3 subset of F or equal to F. So what I have done is a chain of Sigma-algebra each containing the other. So as time evolves more information is revealed, you know more about the structure of the Sigma-algebra.

So such a sequence of Sigma-algebras which each of them are subset of the original Sigma-algebra and each of them are bigger than the previous one, such a thing is called a filtration which we can obviously formally define.

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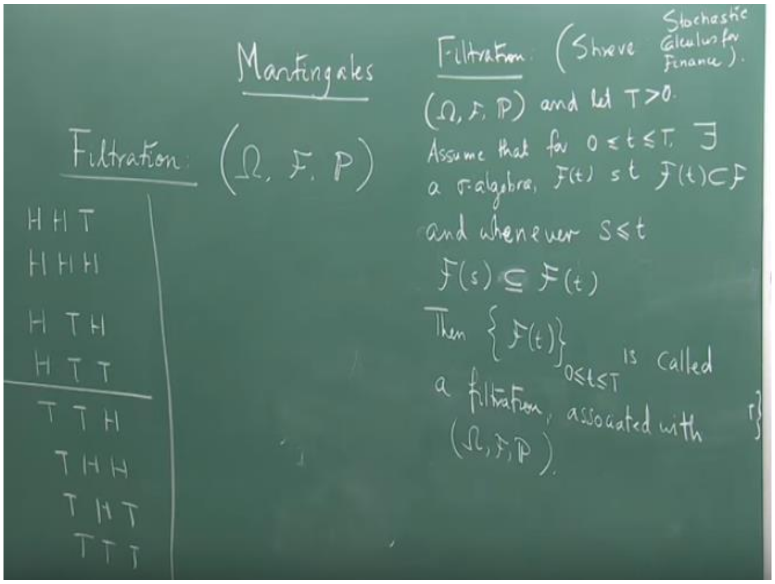

So this is in a very discrete setting. You can define a filtration in the following way. So this definition I have given from the book of Shreve, Steven Shreve's book called Stochastic Calculus for Finance okay. Now observe what I want to say. So if you have sigma F P let T be a number greater than 0. Assume that for T greater than equal to 0 less than equal to T there exists, this is the symbol of there exists, a Sigma-algebra F T.

So when we are going to write continuous things we will put it F T instead of this T just to differentiate between the discrete thing and the continuous event. When we write the discrete thing we will write it as a lowered index and when we write the continuous thing it is as if it is a function.

So there exists a Sigma-algebra F T such that whenever I have S less than equal to T F of S must be contained F T such that F T is contained in F and whenever S is less than equal to T will F of S would be contained in F of T say F of T will contain in F of S. So this condition has to satisfy it. Then if these are all satisfied then the family of Sigma-algebras F T where T is from 0 to T is called a filtration associated with the probability space.

So it might, the definition might look tricky, but I think you can go and read it up in books or just think about it a little bit just think about the definition you can if you want rerun the whole lecture. So you can see it how many times you want so that you get your concepts cleared once for all. So then we are going to talk about something called a stochastic process adapted to the filtration.

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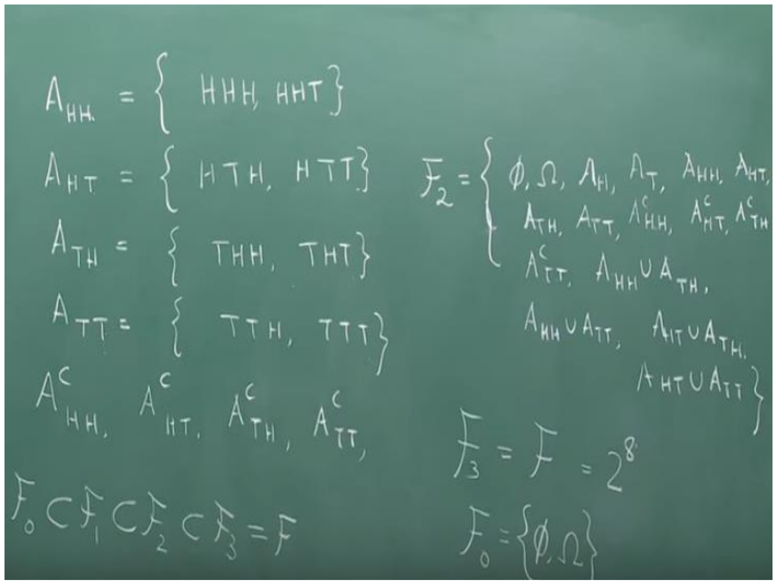

So given a stochastic process X t say is of this form, so this is the stochastic process given to you. We say that this X t is adapted to the filtration F t if X t is F t measurable so all these are these are all random variables. For every t between 0 to t X t is some random variable; adapted to the filtration if X t is F t measurable and X t measurable that is X t inverse of B belongs to F t for all Borel set B in R okay. This is the meaning of an adapted stochastic process.

Martingales are special type of adapted stochastic process. So first we will talk about discrete Martingales and then we will talk about continuous Martingales. Do not get too much bothered about their properties right now except the one which we will require.

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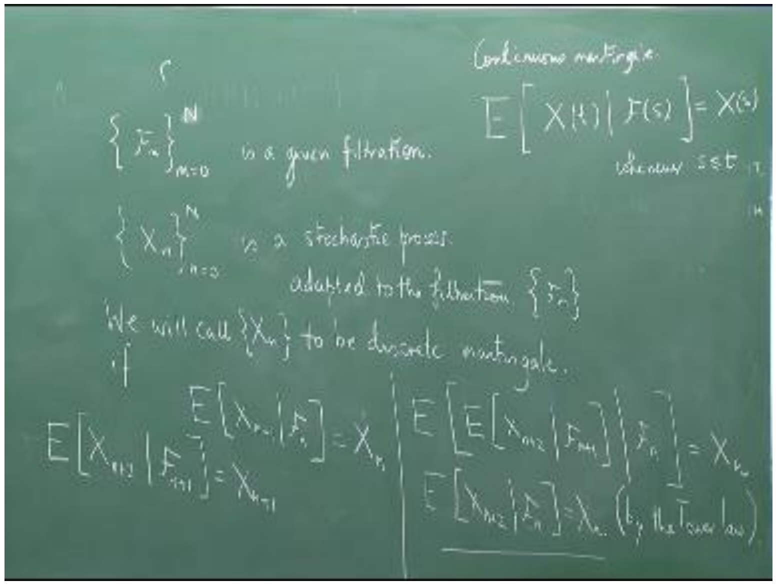

So suppose you have a discrete filtration say F n so 0 to say m equal to 0 to capital N why I am using a finite class because I am essentially talking about finance. In finance you have a trading horizon, trading time, so trading starts at 0 and ends at T so ends at time N. So that is why I am taking the simplest definition. You can possibly take N equal to 0 to infinity it does not matter and you can have a stochastic process which is infinite.

Stochastic process are nothing but sequences of random variables where the indexing is actually overtime. F n is a given filtration. And then you have X n n equal to 0 to capital N, a stochastic process adapted to this filtration. So you will hear this term adapted to filtration repeatedly after this. Stochastic process adapted to the filtration F n. So now we will call X n to be a distinct Martingale if a conditional expectation of the random variable at n+1 that is X n+1 having the information till time n is X n.

So how it is related to gambling possibly? It says that a stochastic process is a Martingale if your expected payoff at the n+1 th move is same as whatever payoff you have received at the nth time provided you have only information up to nth time which is the fact. You will not know what will happen at n+1 it is uncertain. Up to nth time you have information, you know that this is X n is what I have got is a random variable or the random process which was revealed.

When I had done the gambling at nth instant but when I am going to do at the n+1th instant conditioning upon the fact that this is known then this is my answer, when I cannot expect to improve. Once I improve then I call it something like a super Martingale if I do bad then I call it like a sub Martingale.

So this is what one should expect that I can only expect I will do as much as I did last time. My payoff would be as much as I had last time nothing else. Of course you can ask that, this will go on basically, n-2 F n+1 this will again become X n+1. So this is what is known to me. Now I will put this fact here. Let me see what I get out of this. Some little game with conditional expectation.

So this itself is again this. So if a scenario has revealed at the n+1 th time then what I expect to get for a given scenario is same as what I would get at the previous time. That is the meaning of, I cannot expect something more. I have to remain status quo. That is what I can expect. See if I am putting this fact here then by using the tower law of conditional expectation because F n+1 is the largest Sigma-algebra than F n. I can write this is nothing but a conditional expectation of X n+2 conditioned on F n.

So it does not matter whether it is now n+2 or n+5 or n+ 6 you can so this is an application of the tower law. So it does not matter what is your X n, after X n however large n you put here put any n does not matter n is n+5 or n+100 or n+10. If you just have information up to n level move whatever X n you put n is any number bigger than n conditional expectation of X m conditioned on the fact that I know only things up to n is nothing but X n.

That is the meaning of the Martingale and this idea this fact that does not matter whatever you put here. If you just know up to F n you know you can only expect what you have got at the nth level and this fact this beautiful fact has some beautiful properties and so this idea actually allows you to also look into the continuous case. If you have a continuous Martingale you just have to write this.

So a continuous stochastic process X t given the filtration F s would remain to be X s whenever s is less than equal to t. So this is the meaning of a, this is just the definition of a continuous Martingale okay.

Now I will just tell you one property is that it does not matter whatever be your position. If you take the expected value just calculate this expectation of any one of the random variables that is the expectation for each of the random variables. So you have the stochastic process X 1 X 2 X n like this so X 1 X 2 dot dot dot to capital X N X of capital N then if you take each of them as separate random variables and take their expectation provided that this sequence is a Martingale then every random variable will have the same expectation.

So this calculation can be done very simply, I will just do it on the top and end our talk today and tomorrow we will start some lovely thing called Brownian motion and that is the hardcore stuff because Brownian motion is to be known because that will allow us to model the behavior of stock prices in a financial market in a stock exchange possibly.

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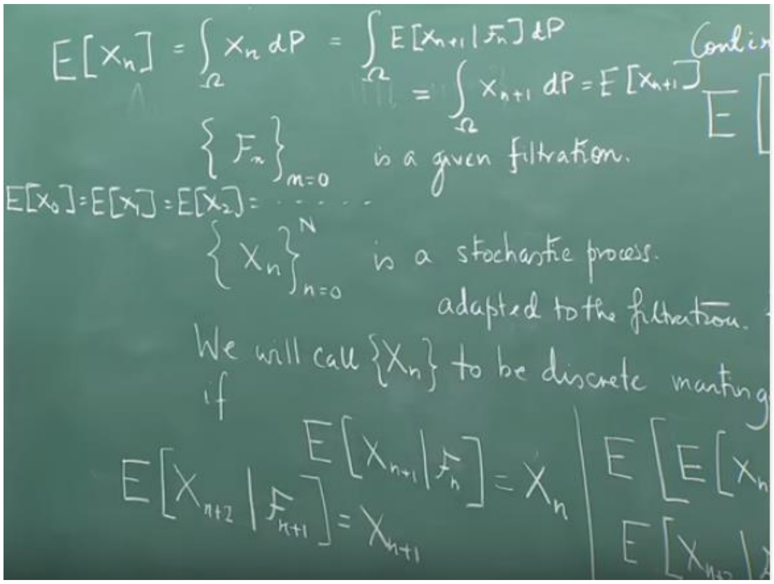

So here we just do it in a very simple way. If you take expectation of X n so this okay I am just writing maybe I should write in a much more different way but okay I am just writing in a standard symbolic way. So this is nothing but integral but by the very definition of conditional expectation when you do it for the general case general Sigma-algebra F n not the one generated by a partition you know you have to use this partial averaging idea that is essentially you define it in that way. This is nothing this one is nothing but X n+1 d P. This is the definition of conditional expectation for the general Sigma-algebra.

This is called the partial averaging and this is nothing but expectation of X n+1. So does not matter. So which means if X 0 is equal to X 1 is equal to E of X 2 so what you finally get is E of X 0 putting N equal to 0 that is equal to E of X 1 that is equal to E of X 2. So with this lovely property of the Martingale we will stop our discussion here and tomorrow we are going into this wonderful stuff called Brownian motion and we will take our discussion off from there. Thank you.