Probability and Stochastics for finance
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Lecture - 07
Conditional Expectation-II

So, now we are going to talk about how to get conditional expectation, when I do not have the Sigma-algebra G given by a countable partition of omega, so how you generalize. So, in the case where we had this issue of countable partition, you know that we proved that there would exist a G measureable random variable E X by G such that integral E X by G dP over any A and G is same as integral of X dP over A, A belonging to G.

Now this idea, cannot be so easily proved, if you consider that G is just a Sigma-algebra. So, in many books, especially books of finance, they define the conditional expectation X by G as follows.

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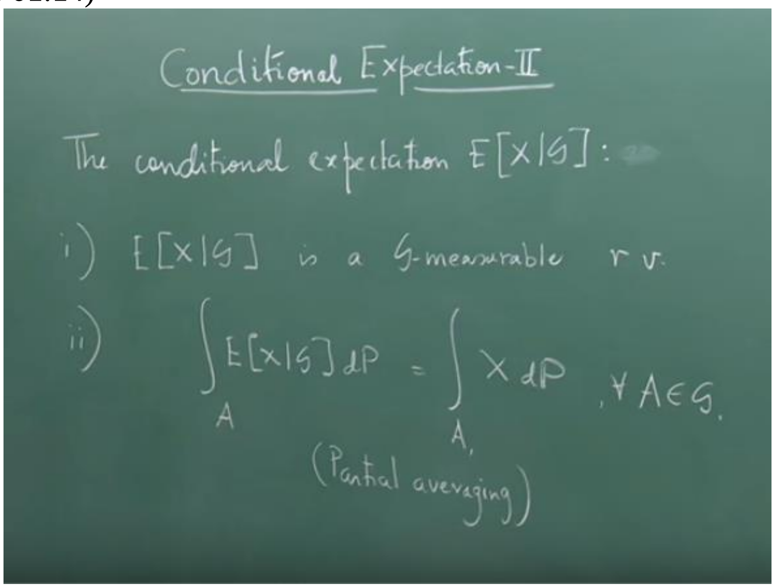

So, it is a random variable, is a G measurable random variable, because that is what we proved earlier, so we are assuming it okay, random variable r.v. And number 2, whatever we have proved in the last class, in the last result, we want to assume that such a thing actually happens, as integrating over this and integrating X are the same thing.

So, this variable and X does not have much of a difference, that is what the proved. So, this property is sometimes called partial averaging. So, this is the definition. So, conditional expectation is a G measurable random variable which satisfies this, which shows that, this may not be unique, there cannot be any more than 1 G measurable random variable which can actually get you this. But the key question is, whether such a G measurable random variable would at all exist.

Okay, I am making a definition, but try to generalize my last result, of the last class, but how do I know, that this such a thing such a random variable would exist. If it does not exist, then such a thing is nonsense. To know that it exists, we need to go through some steps and get into some slightly deeper probability theory. We will not push you for proofs etc.

We will try to give you an information about how things are done what are the thought processes behind the steps that is required or tools that is required to prove that such a random variable would exist which will exhibit this property, okay.

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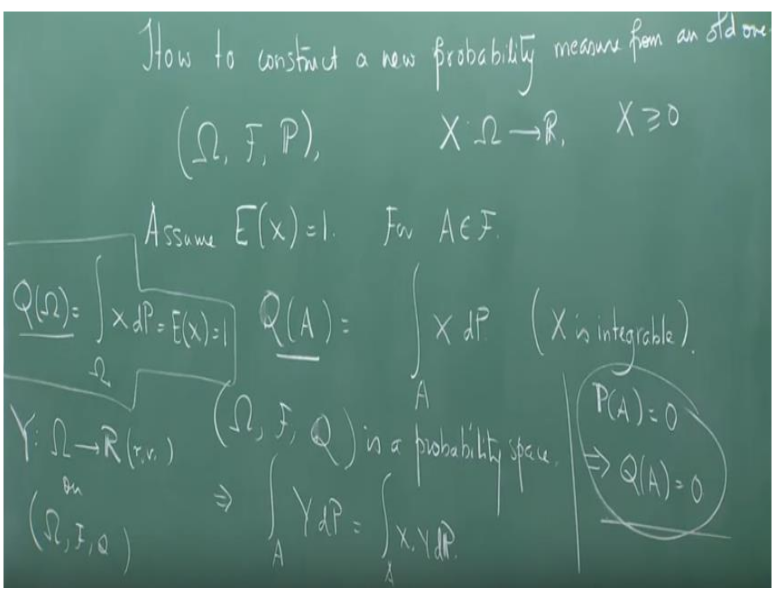

One of the, first steps is to know, in finance and in probability theory is, how to construct, a new probability measure from an old one, that is the key question. So, in that sense suppose you have a given probability space, so whatever handwritten notes I have, I will see whether, I mean, I finish delivering the lectures, I can hand it over to the book, support and staff, who can actually very good technical staff here, who can actually scan it and put it up on the portal, so that you can see all these things. So, I cannot write down each and everything in detail which is written in the notes because I am just trying to explaining the real issues.

So, here we take a probability space and we take a random variable and assume that this random variable is non-negative. You can assume it, in the sense of almost everywhere but okay let us for the moment assume that, X omega is greater than 0 for every instance omega. Now, also assume, you have X is equal to 1.

You see, once I know this, then I construct for A in the Sigma-algebra, if A is a event then, construct the following function of A. This Q, which is actually taking an element of a Sigma-algebra and mapping to R given this is integrable. So, or suppose that X is integrable actually. Then this Q generates a new probability measure. So, then Q okay maybe I will write it like this no maybe I will not do this because that will make it look like rational numbers. So, this Q is a probability space, that is absolutely fundamental. So, given a probability measure, I can construct a new probability measure in the following way.

So, another interesting property that we have is that, if Y is a random variable on and suppose I can integrate, then for any A that you have in F, so this is nothing but, so this is an additional result. So, this is the key step, where you have constructed the new probability measure.

Finance is very important, how to move from one probability measure to the other. An important result that can be shown is that, suppose P of A is 0 okay. Then, in this particular case, it would imply that Q of A is equal to 0. This is called Q have been absolutely continuous with respect to P. So, if you have an A for which P of A is 0, then Q of A is also equal to 0. This is fundamental which comes out from here, this definition.

So, here you see Q, now how do I know that Q is a probability measure; because you see, Q of omega is integral omega X dP and this is nothing but expectation of X, which I have taken to be 1, so probability of the sure event omega that whole sample space must be 1. This is something very important that, it is truly a probability measure and you can show that this will happen. So, the interesting question that we now can ask comes from here.

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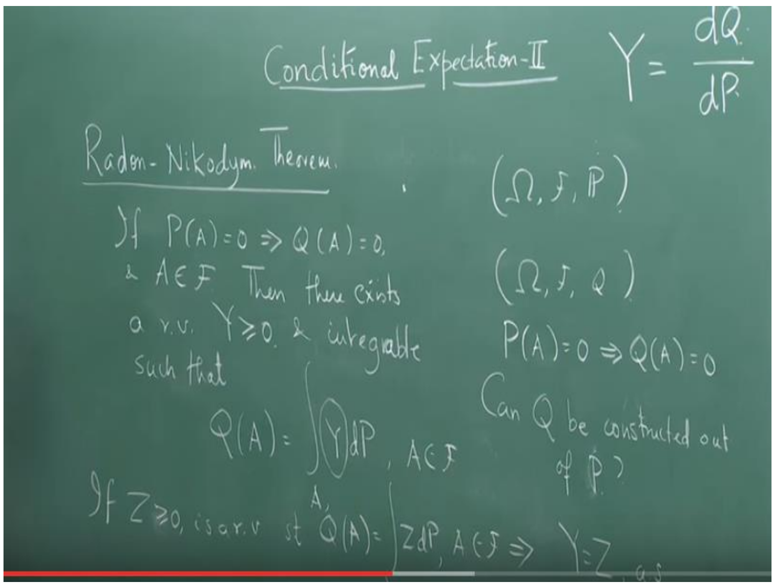

So, if I have a situation where P is the probability measure and Q is the probability measure on both on so P Q are both probability measures on omega. So, we have this probability space and we have this probability space also right. So, this probability space and this probability space, but we also have this connection that P A equal to 0 would imply Q A equal to 0. Whenever P A is 0 Q A is 0. Then is Q, a probability measure constructed out of P. So, which means the question is can Q be constructed out of P?

Answering this question leads to a deep result called the Radon-Nikodym theorem and the random variable which will require to construct Q in this manner, the one which we have already shown in this result, that random variable would be called the Radon-Nikodym derivative and this is a very-very famous result in measured theory and in probability, also it is called the

Radon-Nikodym theorem. Just take this for information do not get too worked up because you are not going to really use Radon-Nikodym theorem in such a way in your actual practice of finance, but it is good to have some additional information.

So, what does Radon-Nikodym theorem tell me? Radon-Nikodym theorem tell me the following, tells me that okay, very good, so you have this information. If P A equal to 0 implies Q A equal to 0, where A is of course element of F and A is element of F, A must be an event of course it would not compute the probability, then there exists a random variable Y which is nonnegative, an integrable.

This is a very deep result because we are proving the existence of something. Every existence here results in mathematics as deep results. It is not easy to prove that something actually exists whether you are able to really hand calculate it that is a different matter but the question of its existence is a very-very important issue in the way modern mathematical thinking goes.

So, you are telling such a thing would actually be there and so then there exists random variable this which is integrable such that, Q of A can be constructed in the following manner. See, why you need to talk about a positive random variable not just any random variable taking any value.

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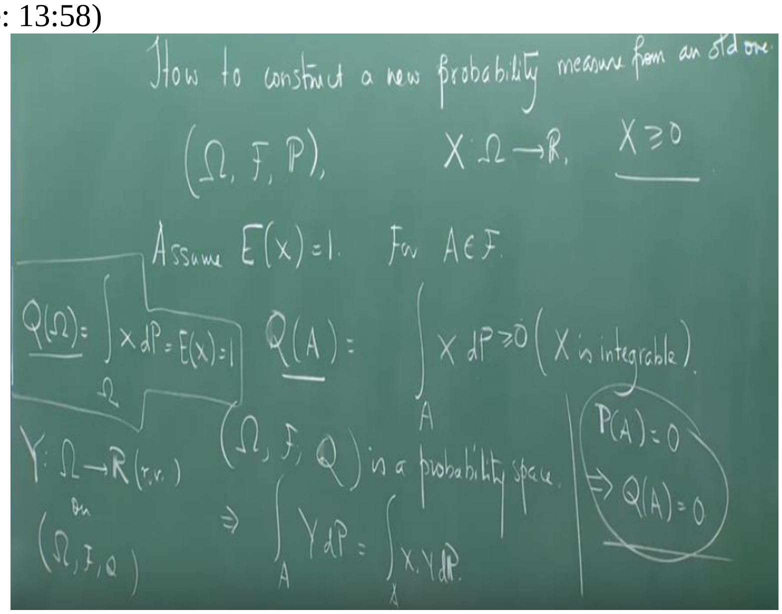

You need to talk about a positive random variable whether it is here or here because then you will always have Q A greater than equal to 0. You put phi here it would become 0. It will put omega here it is 1 so that is a true probability measure. Q A values are lying between 0 and 1 because X is nonnegative this will always be greater than equal to 0 and that is why you need to talk about our nonnegative random variable, okay.

So, you will always get this. There would always exist a nonnegative random variable such that this will happen. Of course, E of Y would be 1 because, you already now know that, Q is the probability measure so Q of omega is 1 so expectation of Y is 1. So, let now interesting fact is, that this is unique up to a measure, means it says that if Z greater than equal to 0 is a random variable, such that Q of A can also be written like this, for every A element of F. Then it implies that Y and equal to Z almost surely that Y and Z agree at every omega except on a set of omega of measure 0.

So, this is essentially unique so this Y is usually written as dQ dP the Radon-Nikodym derivative of Q with respect to Q given P and Q are absolutely continuous measure means Q is absolutely continuous with respect to P.

So, this is the Radon-Nikodym theorem which tells you that okay if you can construct a probability measure if you have random variable like this you can always construct a probability measure, for which this property will hold. And the reverse says that if this property hold this will always get a nonnegative random variable, for which would be able to through which you are able to represent the measure Q, in terms of the probability measure Q, in terms of the probability measure P.

But these two things sum up to give you, there is no time to really write down proofs, but I am trying to tell you that this two results these two ideas sum up to prove the existence of a random variable, which will satisfy exactly what we have seen in the definition of conditional expectation in the general setup.

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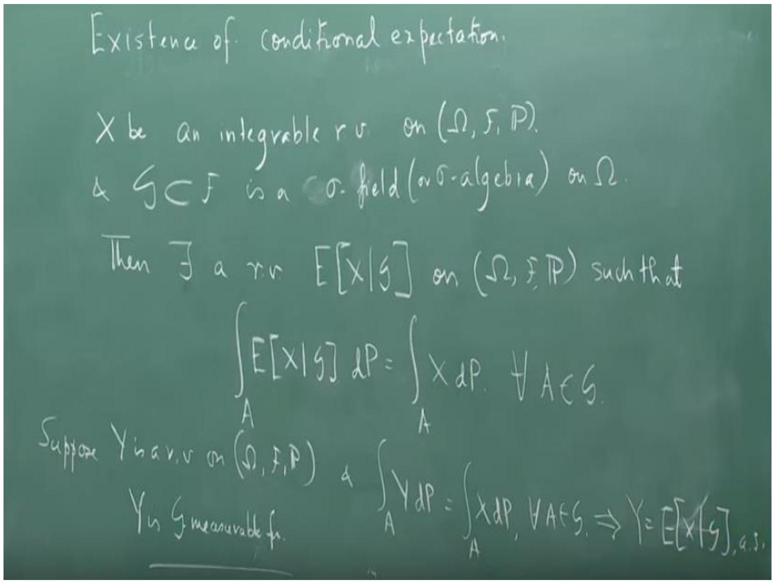

So, existence of conditional expectation. So, what does this result say. So, let us start with this fact that, X be an integrable random variable on the probability space, okay and G subset of F is a sub sigma field okay all (()) 17:45, just this sigma field or Sigma-algebra whatever term you want to use you can use, Sigma-algebra on, then there exists a random variable E X by G on.

So, this is an important result, which is integrable. Son this sign is very common to mathematician, this means there exists. So, then there exists a random variable on this, on the probability space such that, the partial averaging property, which we had written down actually holds. So, it shows, that though this random variable may not be unique, but such a random variable do exists and exhibits this partial averaging property. So, this is true for all A you take in Sigma-algebra G with respect to which you are doing the condition.

Now the interesting part is the following. So, suppose Y is a random variable on this probability space, omega F P and integral A Y dP is equal to integral X dP over A, for all A in G then it implies that, Y is equal to the conditional expectation of X Event G almost surely or almost everywhere. So, any random variable Y which satisfies this property must be same as the conditional expectation that is the fundamental idea.

So, that also shows that this is really not unique you can just find some Y which satisfies this, that is, which is G measurable. So of course, Y has to be G measurable. So, if you can find any G measurable function this will just work.

Now, I will state the properties of conditional expectation. I will not give the proof of the properties. The proof of the properties would be coming through exercises. You will be asked to give a proof of the properties and then the proof would be given in your homework. Maybe if there is time, time permits I will try to do for 1 and not each and everything. So, let me write down the properties of conditional expectation. So, these are important properties used very often in finance so it is important to know this.

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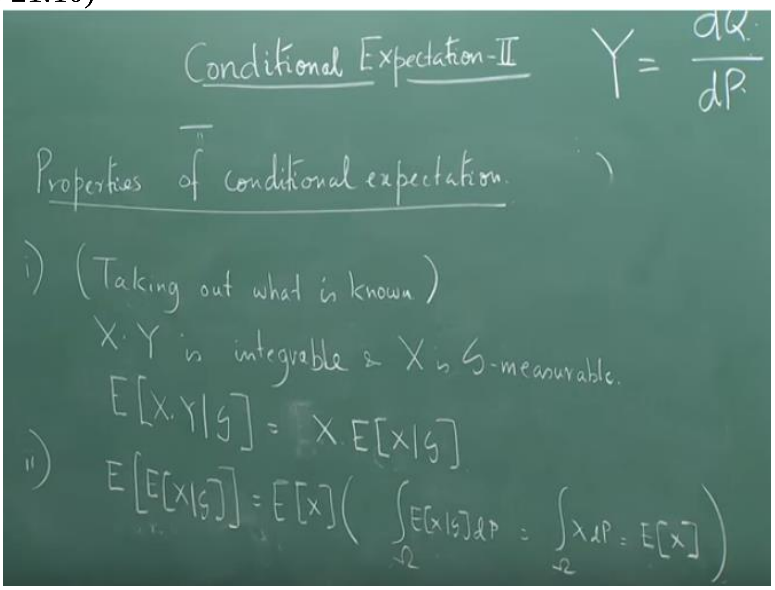

Properties of conditional expectation. So, what is the first property. First property is taking out what is known. Number 1 property, is called the taking out what is known property. So, what do I mean by this.

So, suppose I have 2 random variables X and Y and I say, that X and X into Y that is X omega into Y omega for the omegas, when you compute them is, integral, is integrable and X is G measurable, means, G contains all information about X that is the meaning. So, that is the meaning of something is known. So, X is essentially known by knowing G. So, the nature of X is completely revealed to G. So, this is nothing but X into expectation of, this is the thing.

Number 2, expectation of X by G is expectation of X. This is a very simple thing, this property because, if you use the partial averaging property, what does it mean. What does expectation of a think mean? Integral omega expectation of X by G, this means this, and this means omega X dP and that is expectation of X. So, this is the partial averaging property and this nothing, it gives me this because these 2 for any A in G and of course omega the whole space A is in G because G is a Sigma-algebra.

So, as I have told you in the very beginning that the whole omega has to be a null event and sure event has to be the whole space has to be in any sigma fields. So, that is it and this is nothing but the expectation of X by G. So, you have taken the expectation of this random variable which is this and the partial averaging it is this which gives you this.

This is property two then we come to property three and property four.

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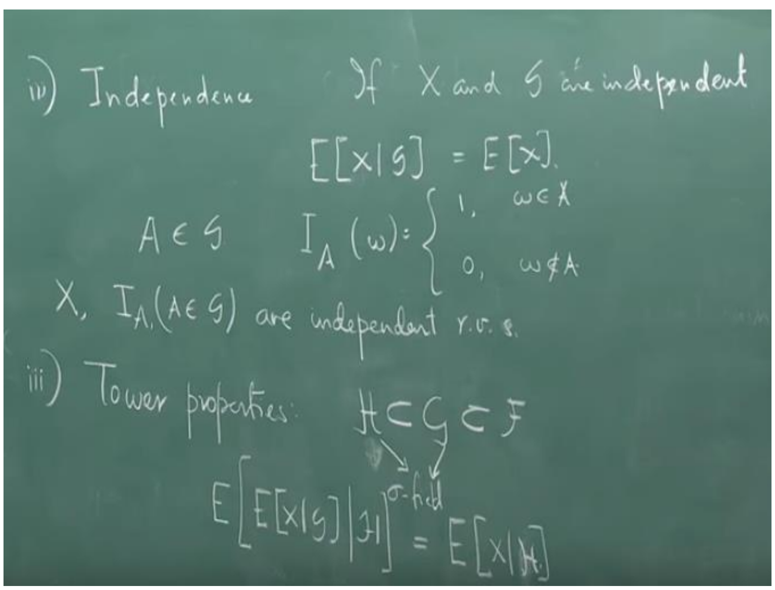

Property three is called independence. So, if X and G are independent, I will explain to you, what I mean, if X and G are independent, you will be very surprised that I am talking about independence of a random variable with respect to a Sigma-algebra. I will tell you what is that. So, mathematicians hardly mix-up lot of language.

So, if X and G are independent then basically X does not depend on G so the conditional expectation of G is basically a constant random variable; for any omega that you take this gives you the values E X. What I mean by X and G being independent. By the term X and G being independent, I mean that if you take any A element of G, how do I identify the element A, how do I say that the event A has occurred. Event A, if I take, I look at the outcome of random experiments some omega has occurred in the sample space. If that omega remains is in A then I say that the event A has occurred.

For example, I throw a die and the event A is that an odd number occurs. If I say that the number 3 has occurred, then I say that the event A has occurred. So, which means, this event A, can be identified in terms of a random variable IA the indicator function. So, where IA is a function defined as follows where IA omega is equal to 1, if omega is in A and is equal to 0, if omega is not in A.

So, which means that X and IA where A is element of G this are independent. So, this are for any A and G are independent random variables. So, this is a random variable, independent random variables. That is the meaning of the fact that X and G are independent. So, if these are independent, so X really does not depend upon the occurrence of any event in G then of course this is the case.

Third and the last property which is used pretty often is a tower property. So, it is very important for you to learn these properties. So, you will have exercises to work on. So, what do you do. You have a sigma field H, which is contained in the sigma field G, which is contained in the sigma field F, so this both sorry, so both G and H are sigma fields, which is contained in the sigma field H.

Then the property that we have is the following that, expectation of X by G given H, is same as expectation. So, conditional expectation of the random variable E X by G given that H has occurred is same as sorry, E X by H.

So, this is something I would like you to remember. So, these 4 properties that you learn about conditional expectation are just very crucial. So, lot of things you have to use repeatedly when you study finance and when you start studying Martingales.

Martingales are something used for when it comes term actually comes out with gambling, so you have gambling and then as time passes more information is revealed and if you have some information now, what is your expected gain or profit or expected income or expected, whatever you want to say, expected payoff at the next time. So, that is what Martingales talks about and that is what we are going to learn in the next class.

It is very interesting and hope because at the end stock market is gambling. At the end the stock market keeping money in the stock market is gambling, but people do keep money in the stock market because, the rate at which money grows in the stock market if everything goes on, is much faster than if you keep it in the money market which is fixed deposit like or invest in government bonds.

So, it is very important that you have to have an understanding of this. If you really want to know in the second part of the course how derivatives are priced, of course some derivative pricing will, of course come in here. We will talk about at the end when we learn Ito calculus. We will learn to apply Ito calculus. We will talk about how to compute derivative rates. We will talk about, what is the actual behavior of the stock prices, a pretty good model called the geometric Brownian motion.

All these things will come gradually, but these properties will become pivotal there and as a result, I would like you to remember and use these properties. Do these exercises which is associated with this conditional expectation in a pretty serious way. At least learn this part very well. Of course, the internet is there. You can keep on looking at it but here the course is basically, to give you some understanding. Maybe you will do slightly better than the internet, but anyway, you can use whatever resources you want but these things have to be known very well.

Thank you very much.