Probability and Stochastics for finance
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Lecture - 03
Random variables, distribution functions and independence

Welcome to the third lecture on series on probability and stochastics for finance. Let me tell you that from this lecture we are getting more and more immersed into the deeper theory associated with probability. The last 2 classes were much more of developing an insight. Of course, I expect that the course that you are now taking would be some sort of a second course for you rather than just a first course where the first course needs more detailing, which is not possible in the set of 10 hour lectures and it is essentially a pre-runner to the course on derivative pricing in financial markets.

So, it is better for me today to tell a book that I will be using from this class onwards till the end of this course, is a book called An Introduction to Stochastic Differential Equation by Lawrence C. Evans and this book is published by the American Mathematical Society.

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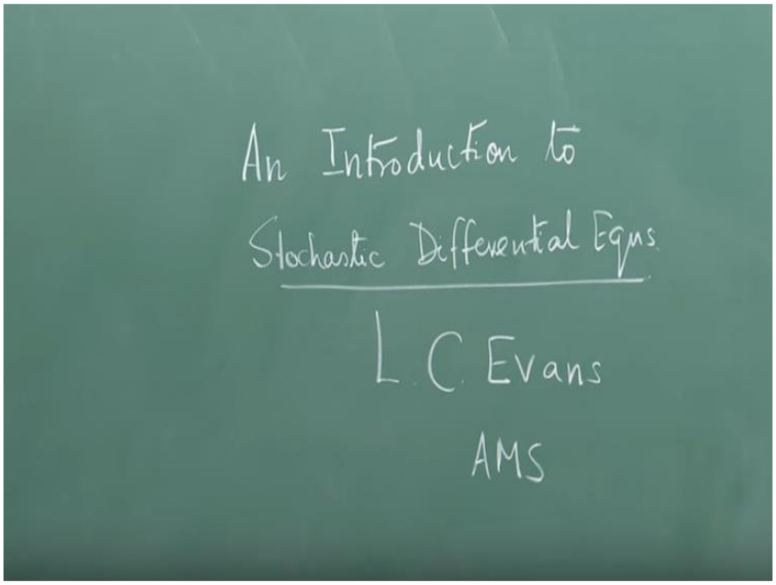

I do not expect you to have this book with you but I will in general follow this book except maybe a little bit of material taken from some other sources. Anyway, I should just show you the book in case you want to see the cover that it is better to show you the cover in detail.

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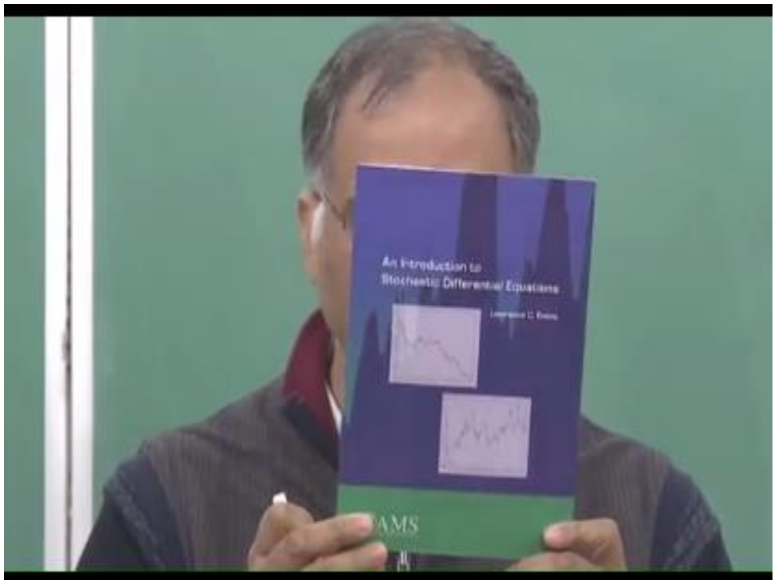

This is the book, which I will be largely using in this course and as I have told from today we are really going to get into the more theoretical issues. We will immerse more into the technicalities of probability theory.

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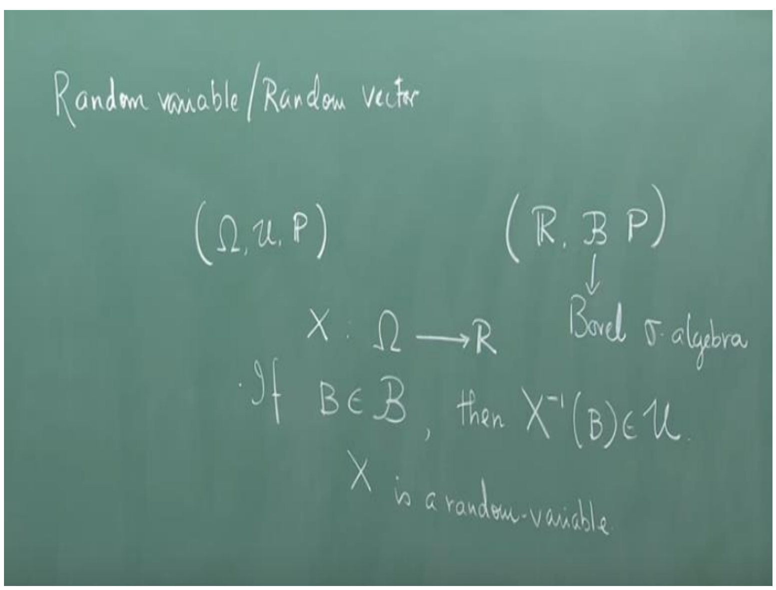

So one of the first set of technicalities that should come is the idea of a random variable or a random vector random variable/random vector. We will explain both these concepts. Of course, those of you have got a basic course in probability would know that random variable itself is slight misnomer because they are actually functions, but in a standard probability course you have been told that a random variable is nothing but a mapping from a sample space to the real line.

But there are certain other properties that needs to be attached to such a function in order to call it a random variable.So let me consider a probability space and a probability measure P and also consider a real line R and a Sigma-algebra B. On the real line R, the Sigma-algebra that we have associated with R is called the Borel Sigma-algebra, which is naturally related to its topology.

This Borel Sigma-algebra consists of all open sets; hence all closed sets, all half-open intervals, half-closed intervals, open intervals, closed intervals and so and so forth. So it is a collection of all, so it is the smallest Sigma-algebra which contains all open sets in R. Instead, is the smallest Sigma-algebra, which contains all open intervals in R, you can just think of it like that.

So, this is the information that we really need. Essentially, random variable is a way in which a outcome of a random experiment is translated into a number and a outcome of an event is translated into an interval, that is the essential idea, so the interval or union of intervals. So x is called a random variable if it is a mapping from omega to R.

But with the following property that if you take any B which belongs to the script B, the Borel Sigma-algebra, this is not the standard B, this is the script B. If B belongs to the Borel Sigma-algebra, then the inverse image of the set B must belong to the Sigma-algebra U. So, in this case x is called U measurable or just measurable. So this property has to hold. If can define a function for which this property hold, then we called x is a random variable.

So, this property is called the measurability property that inverse image of an event in the domains in the range space is also an event in the so if you take an event in the range space and pull it back may be inverse image then it will map to some event in the map, to some event in the actual space, the decision space or the domain space.

So why, so that is why this, what does this idea tells you. This idea tells you that that every image, I am sorry, every event is actually related to some interval here or some union of interval, some set here. So that is why when you pull back, take a set from here and put it back, it will always map it back to an event.

So an event is taken to another event, but in the space R. So, this is something very important that, so every event is taken to some sort of an interval or an union of interval. That is the idea. That is the key idea behind introducing random variables. So that is the key idea behind the notion of random variable that an event here has to be always taken into an interval in the real line. So, we map an event to an interval.

So, we can now handle real numbers, which are much more easy to handle instead of handling events which may consist of arbitrary things. They may consist of set of all dice that you have thrown, outcomes of a die or outcomes of a coin throw or the arbitrary random, which was using points from a rectangle for example.

So, that arbitrariness of the events, because they seem, because these are very different different random experiments are being taken to a common thing or being mapped to something which is common. So all the studies of probability can be now done in the settings of real numbers.This idea can be extended and it is useful in some cases in finance to look at it this extended idea of a random vector.

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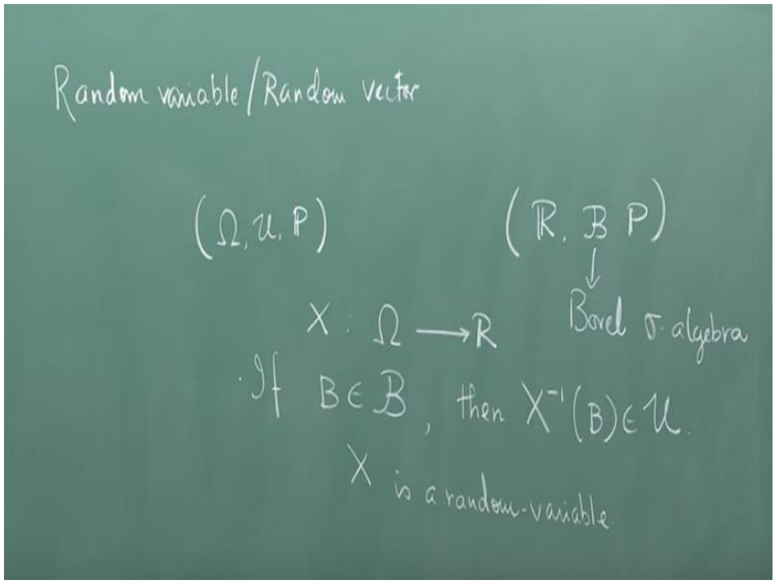

Where you have the same setting, okay, and in this case you even need not bother about any probability measure, you can just forget it, you can just look into this measured space and you can now talk about, may I just write it like this. So this sort of thing, so this B associated with R n, so this this B is the smallest Sigma-algebra, which consists of all open sets in R n.

What is the meaning of smallest Sigma-algebra? Take any Sigma-algebra which consists of all open sets. Take intersection of all possible Sigma-algebra which consists all open sets in R n. Their intersection is a Sigma-algebra, which gives you this set. Now, can I visualize it, can I write them down clearly, no.

These are basically tools for you to understand what actually is happening. So, you take a mapping x. Again you take x. Now, so this is called a random vector, so random vector. Most of the studies would largely for the domain of random variables, but this is just to give you a generalization. So, in this case the same thing would occur that if you take any B, now it is set in R n which goes to this sort of B, the Borel Sigma-algebra in R n then it implies that x inverse B must belong to U.

So the random variable concept is fundamental because it takes you arbitrariness from the notion of a random experiment. They are always in the setting of a real line, that is the key idea. So, what is the key? For example, I can give you 1 construct as 1 example, you can see it rather from standard books and many of you possibly know it.

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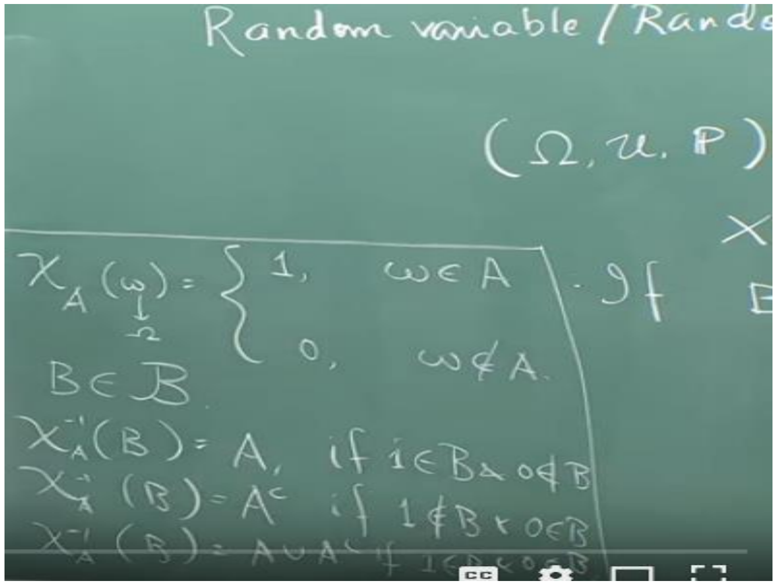

So you define this integrated function, so this is your set integrated function which says that if omega is in A that you mark it 1, you mark it 0 if omega is not in A, okay. Now, once that is done, how do I guarantee that this is a random variable? I say K A is a random variable, so I am mapping taking each omega from this and making this definition.

So if I take, these in all so if take a B from this B, how do I calculate, that is the key question. See, if 1 is in B and 0 is not in B then this would be in A. If 1 is in B and 0 not in B. Of course, sorry, it will go into the compliment of A if 1 is not in B and sorry 1 is not in B and 0 is in B. So, what would happen if, so A is of course is an event.

This is in U. Since C is a Sigma-algebra, A compliment is in U. Similarly you can talk about what happens if 0, 1 and 1 is in B and 0 is also in that set B, then what would happen? Then it will become A union A compliment. So, and that is also again in A. So whatever way you say that okay this is not anywhere. 1 is not in B, 0 is not in B and A union A compliment's compliment so then that is the way so that will again be a Sigma-algebra. So you can try out. This is 1 example.

Other simple examples are head and tail; when you toss a coin you, let us denote by random variable the number of heads that come when I toss a coin. It can be either 0 or 1. That is the sort of thing. So that is the way you define random variable.

Another sort of important way of defining random variable is to construct some random variable out of these indicator function. This indicator functions play a major role in probability theory so this is something one needs to understand and try to get a hang of the ideas.

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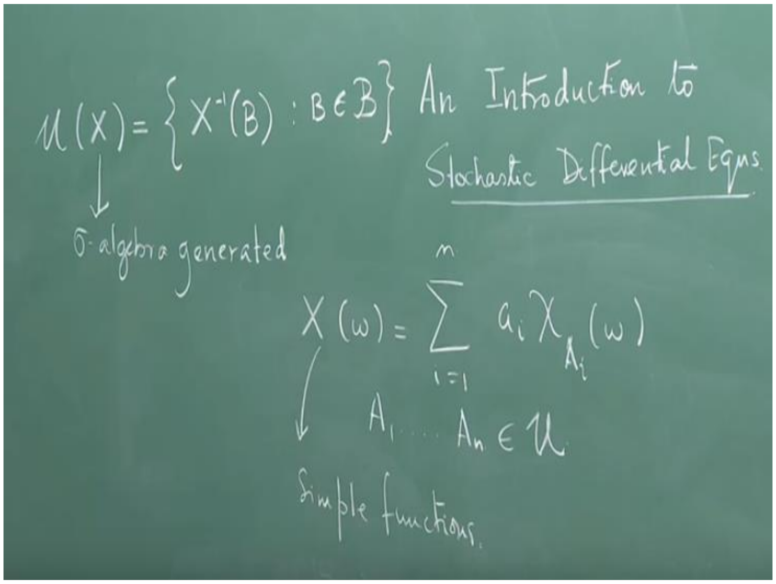

See what happens here is that for example I construct a random variable like this. Now this sort of thing, it is called a simple function. This also is random variable. So you have some set, say A 1 to A n, so all of these are in U. Of course, you can completely, it will completely make sense that this is actually only non 0, it will if you take the union of A i's and take omega from there then it will always give you one of the A i's.

Otherwise, it will become 0. So this sort of function, this sort of random variables are sometimes called simple functions. A key thing that is sometimes required is the notion of this Sigma-algebra. This is called a Sigma-algebra generated by the random variable. You can think these are the random vector, but largely I am talking about the random variable. Does not matter if you think of it as random vector, you can think x is from omega to R n where B is in the Borel Sigma-algebra.

This set is a Sigma-algebra itself and this is called a Sigma-algebra generated, so it is collecting all the inverse images of all elements in B. So you take the union of that set with collection of all such things would actually form a Sigma-algebra, which is a sub Sigma-algebra of U basically. So, this is a Sigma-algebra. This is called a Sigma-algebra generated by x. See, x inverts B, need not be always giving me every element in U, you need some element in U.

So those are have a special significance. Because they are the ones by which the whole game is being done because they are linked to random variables, but they are the stuff that is really required when you calculate stuff. So this is called the Sigma-algebra generated. So I will leave it in the question of exercise to prove that it is truly a Sigma-algebra.

Our question would now turn to a notion of stochastic processes. So I am just rubbing this idea. So random variables can be also lined up in the form of a sequence and these stochastic processes play a very very fundamental role in finance because everything that happens in financial markets you see changes of certain things over time and though all these changes are stochastic in the sense that they are essentially random experiments.

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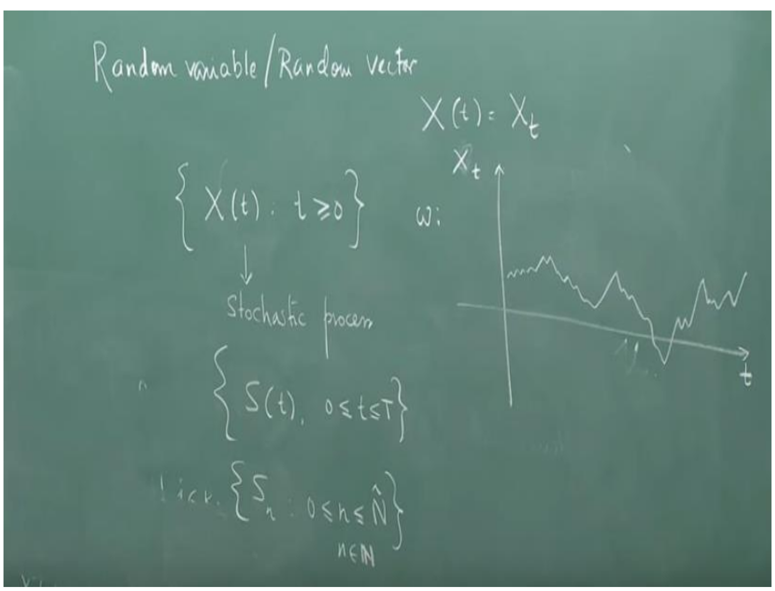

So you take a collection of random variables. For each t, t essentially denotes time, this X(t) is a random variable. Sometimes some authors write it as X(t) as x of t, but we will maintain the symbol X(t) here because it is much more helpful for handling certain things. So this collection, it could be infinite of course or it could be finite also. So this is called a stochastic process.

So for example, if you look at this stochastic process so that S(t) is the price of a given stock where 0 is the starting time where you know what is S(t) at this current moment and capital time T is the time when if your trading is over with that particular object and S(t) denotes the price of the stock of a company share, 1 share of a company, the unit price of the 1 single share of the company within this period.

Of course, the real issue that lies here is the following. The issue that lies here is that every S(t), s (0) you know but you do not know what is s (4) s of any t which is beyond 0. Of course, you could think this is a very continuous setup. You could think of a discrete setup, you can think of s (n) 0 to say capital N where each n belongs to capital N.

So basically then you have s (1), s (2), s (3) so now 0 time I know the price but s (1) I do not know, I do not know what would be s (2), what would be s (3) etc., so that is the stochastic process. So if you take a fixed omega and even calculate for each of the different, see as you change the time your structure of the function X(t) changes; they are different random variable not the same random variable. So s (1), s (0) is of course a fixed, nothing is known. S (1) and s (2) are different random variables, s (2) and s (3) are different random variables.

So if you take a fixed omega from the sample space and then calculate that s t omega for every random variable for every t right for every t this the different random variable you press the omega. Then what you get is a sample path of a stochastic process.

So here what you have done is you have t and here you have say S(t) or X(t) it does not matter anyway X(t). So what you do you have you have basically for a fixed omega you can see how the value of each random variable changes at omega.

So in a market setup, if this was again the stock price x t then I am trying to see, obviously the stock price would not be negative. What I am trying to see is that if I am assuming that for a given some time period, I would have a particular type of scenario and how the stock prices would change, that is very unpredictable.

Assuming that I had the same scenario, a scenario does not change, market conditions are say at a particular are in a particular way then how does my stock price changes. That is something economists want to know, that something the market people want to know, the brokers want to know.

If I change my, at every t if I change my scenario then it is a true truly troubled market, it is truly a bizarre market which you cannot really handle mathematically. So the real mathematics is done when you are really thinking that over if this is my particular scenario now, how would my prices look like, how would my prices look like. So if you finally look into, so you know given the particular scenario what would be the price at every time T but you do not know what scenario would actually come.

So this is called this is called so what you have done is you have computed X(t) and omega sometimes written as x (t, w). So this is called the sample path. A fixed omega a changing t, but changing, keeping t fixed if you change omega then it is bizarre basically. So this idea we have just introduced because they are linked to random variable which we will very soon come the use of stochastic process would increase in the next discussions.

So here now we are going to talk a little bit of what is integration and what is the meaning of expectation okay. So simple functions would play a very very fundamental role here. So suppose I have a simple function like this.

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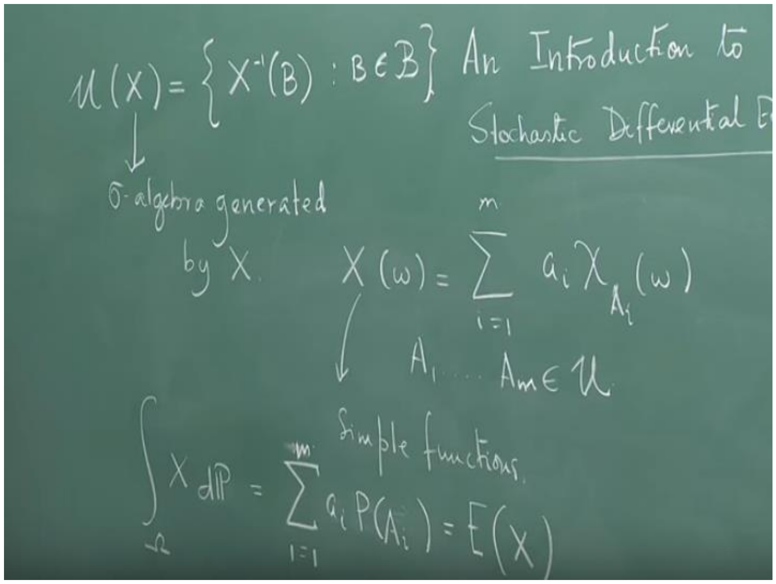

And I want to and I define that the integral of this simple function, every simple function of course is a random variable because you see actually I have defined a random variable. A random variable and also another random variable. So a simple function is, then we defined the integral of x with respect to the probability measure P as 1 to m, let me write this way. So this is the definition. Probability A i is know because these are the events so you know the probabilities of that.

So then the integral is nothing but this. What essentially is this. If anybody has learnt some probability they will immediately recognize that this is nothing but the expected value but how do I do for, this is for a simple case, very simple setup essentially is a some sort of a discrete situation that if you breakup the whole thing A 1 into A n then these are just taking the values A 1, A 2, rest of the values are 0.

So I do not need other parts except the parts which are within A 1 union A 2 union dot dot dot A m. So this is nothing but the values being taken over those path and the probability. This is nothing but an expectation.

So this is essentially is the expectation of x or the average of x of the random variable x, but how do I define this when I when it comes to an arbitrary random variable and not just a simple function. We define this by using the simple function. This is the way (25:40) integral is defined.

I would ask you to really look up because we had told in the beginning that we need some mathematical maturity to do this course but does not matter that you can still follow by looking at what I am writing is that.

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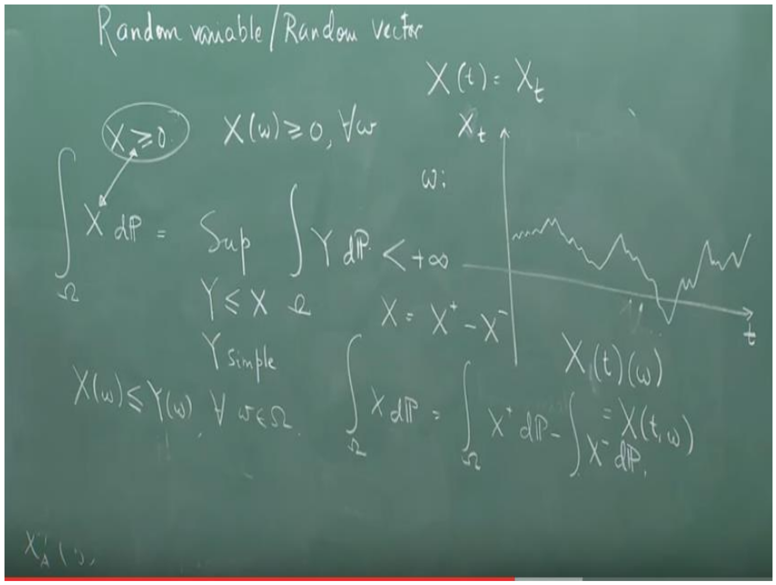

If you have any random variable and the integral of this random variable of course you can talk about a random vector I will not I am not getting, I will write to you at the end what one can do for a random vector but this is just for the random variable. So why I required simple functions? I am looking at all simple functions which are lying below x so this actually means I am writing it for all omega. It will be enough if it is for every if every omega other than a null event.

So let us not get into those technicalities; okay, assume this thing, so this is what is the definition that you can compute for all the simple cases and then take the supreme whatever be the value assuming it is finite so assuming that this is finite then we say that the integral exists. What this integral does is nothing but a generalization of this idea, you used the simple function idea and generalized it so this integral is called expected(()) 27:45 of a random variable x. This of course is defined for x greater than equal to 0 means a nonnegative random variable.

Here, I just want to mention that A i s that I have taken they, so these are simple functions if so if I am talking about nonnegative random variables some of the Y's can be with the situation where all the A i s are nonnegative. Some can be some nonnegative some negative. If all the A i s are negative of course such a Y is less than this.

So this definition actually works for nonnegative random variable that is x of omega. This writing for all omega is not really technically fine but does not matter for this situation. That is exactly what it means. So now if you want to talk about what happens for a case which does not have such sign then you just have to write x as x + minus x - so you can always express a real number as the difference of 2 nonnegative real numbers and then this is nothing but so that point I missed when I made the definition but you have to always remember that this has to satisfy this property is greater than equal to 0.

So for that we are defining this. Once you have nonnegative (())29:33 lot of good properties can be proved and once you have that then all those can be translated to the other things by this that is so much easier way of generating properties of integral. Of course here we assume that integrals are well known means this integrals you know you know what the properties of integrals.

You know how they add up. You know what happens if I multiply a function by a constant. The constant comes out of the integral sign and all those stuff. So that we will not get into such details here. What we are going to now talk about is that this is what we call expectation of a random variable.

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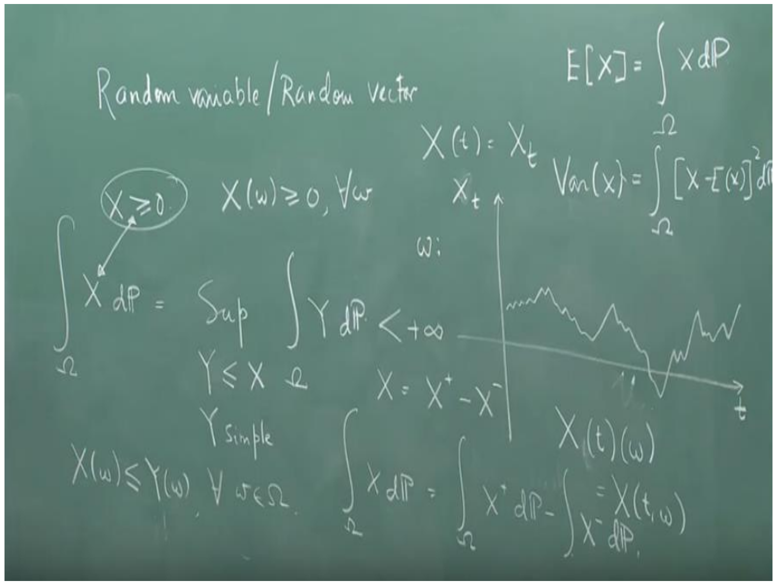

This is what is called the expectation of a random variable and this is what is called a variance. Of course these are all very abstract looking things, it does not make sense and those who have it probability might also start getting confused okay what is really going on but the key idea of really studying probability is to know the notion of a distribution function which tells you how a particular observed random variable would actually behave.

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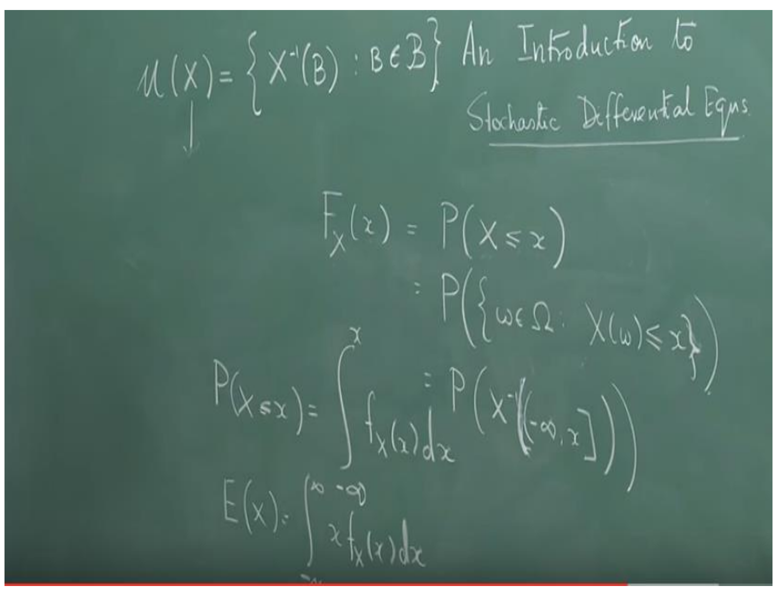

So, given a random variable, the distribution function is a real valued function which calculates the probability that once you want to evaluate this F (x) X this calculates the probability that X is less than equal to x which means calculating probability of the event a set of all omega in small omega in capital omega such that X of omega I leave it to you to confirm that these are actually events, these sets are actually new because if you look at this what is this? This set is nothing but x inwards minus infinity to capital X.

So basically you are finding probability of x inwards of minus infinity capital X and that is why that particular Sigma-algebra is important. Why I need to know the distribution function because that allows us to understand, it is a cumulative thing just like you have studied mean median mode you have talked about cumulative effect of certain observation means we are looking at the growth of x how much with what probability it can come under x.

So once I am able to tell you, I am able to give you some sort of an idea about the stochastic behavior of that particular random variable. So random experiment is giving me some information but that is translated in a nice way in this set of our distribution function. I will not talk about the properties here, but I would just like to tell you that associated with every random variable, it is characteristic.

Usually it is either something called discrete random variable which it can take only countable number of values or finite number of values or a continuous random variable which takes you know non-discrete set of values and then how do I calculate this? The probability is usually calculated by the means of a density function. This I am just recalling the thing, this is something which has to be known in the first year of probability course and once you known this then expectation of x becomes an easy thing to calculate.

So why do you need the density function. It is very important to know when you have a continuous random variable you just cannot say that probability of x is equal to x is something.Probability of a continuous random variable picks up just 1 value is 0. How do you look at it? What is the intuition behind it? I am not going to write down the detail description, I will just tell you an intuition behind it. The intuition is the following. You take the real number 0 1.

Now you are arbitrarily picking up a point from that set. So what is the probability that x equal to half. So what is my favourable outcome how many objects I have in that favourable outcome 1 but I have huge infinitely large all possible outcome, so you can immediately understand that probability is 0. This is not a mathematical explanation but this is just a rough intuitive idea but intuition is anyway right to mathematics so this density function plays a very very important role and now what I will do is I will introduce to you the notion of independence and by introducing to you the notion of independence I will actually stop my discussion.

So we will talk about conditional probability because we are going to talk about conditional expectation very soon so it is very important that we talk about conditional probability. I am writing, I have some notes written down which can be scanned and later on put into the web. So what is conditional probability?

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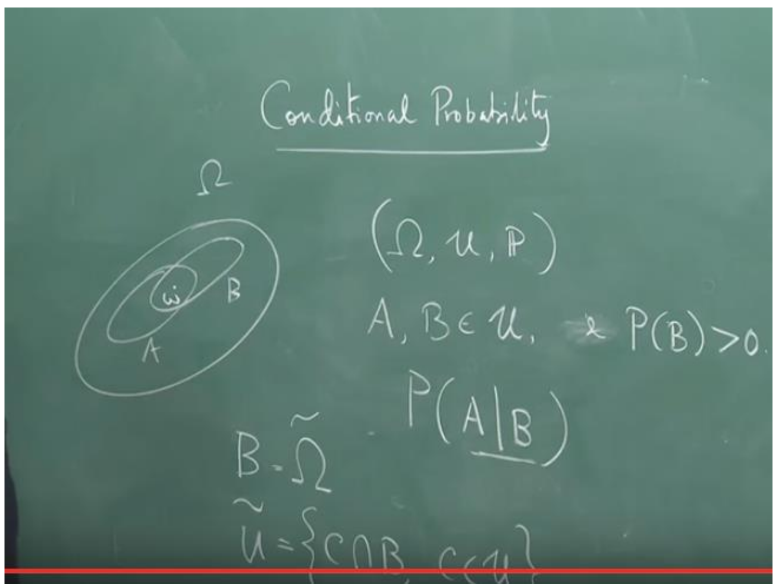

So let me now talk about this very important issue. Conditional probability asks how would be the probability of one given event A be if I know that some other associated event has occurred.
Mathematically it means the following. Mathematically it means that I have a probability space and I have 2 events and it implies that and also I know that probability of B is strictly greater than 0 that B can occur with a positive probability, right.

Now suppose B has actually occurred, so what is the probability of finding A knowing B so which is symbolized like this probability of finding the probability of A given that B has occurred and of course you have to assume this that B has a finite probability of occurrence sorry a positive probability of occurrence. So what does this actually mean?

It looks very fishy that you are just using language to talk about this. It means that if I chose an omega in B so B is a element in U. So it is a subset of the sample space so I can take some omega from B assuming they are non-empty of course, because U cannot be empty set (())38:53 probability would be 0.

So I am choosing an omega from B. The question what we ask here by asking this A probability A given B is what is the probability that omega is also in A, that is the question. So here you have your omega and here you have your A and here you have your B. So if omega is in A what is the probability that omega is in B.

So omega must be in an area where A and B both can occur. So we then have to think of joint occurrence of events, right. Now, if I want to define this probability, this is not the same probability as the one I have defined there because that only talks about probability of event of probability of events in U, but what does this, where does this event stand, where is the Sigma-algebra where is the sample space for this.

So when I am looking at, so my question again is very important and I am talking about if omega is in B, that I know that this omega has occurred, what is the probability that it is also in A? So then A given B is an event which is not really a part of this Sigma-algebra. These are the sequence A and B; A, B, C, D; but here I am talking about something else so which means I really now have to consider a new Sigma-algebra and a new sample space here.

So I am only bothered about those omegas which have occurred in B. If they have not occurred in B, I am not bothered about them. That is why the talk that I am looking for A given that B has occurred. So we are only looking at those omegas which have occurred in B, which have come up in B.

We are not looking up at other omegas and thus B is taken as my new sample space which I mark as this and what is my U tilda. U tilda is C intersection B where C is element of U. So B has to be related to all such events. So you see joint occurrence must now come if you are going to talk about condition probability, okay.

Now we define by P tilda the following probability measure. P tilda they have to put some event in by P(B) which I know. But A Intersection B is actually an event in U.
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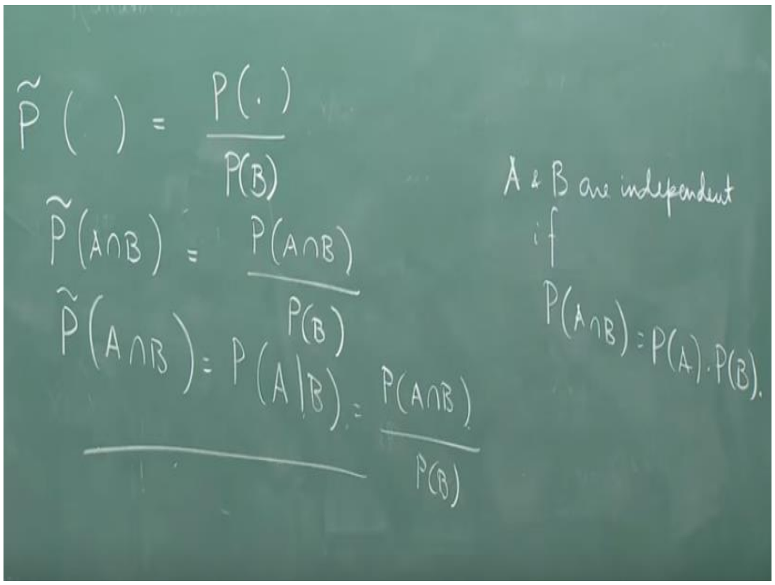

But here I am considering some sort of subset of the event U. Now if I want to consider P tilda of A intersection B that is under this new probability measure what is the joint occurrence of A and B. That is exactly what we are seeking, a new probability measure.

Now, my sample space has changed. It is no longer omega, but it is omega tilda, which is nothing but equal to B, the event B, right. So then by this definition equal to probability of A intersection B and we mathematicians define that P tilda of A intersection B. P tilda of A intersection B is defined as or said to be the probability of A event B and that is essentially the meaning of conditional probability.

So this is nothing but P of A intersection B by P(B). 2 events are said to be independent if it does not matter whether omega was in B or not. The fact that omega can be in A, that probability is independent of the fact that omega has come up in B. That is the meaning of independence of 2 events A and B. So if 2 events A and B are independent that is the probability of occurrence of one does not affect the probability of occurrence of the other.

Here the probability of occurrence of one affects the probability of the occurrence of other because the sample space gets completely changed. So A and B are independent if probability of A intersection B is probability of A into probability of B. So that is it and with this we stop our discussion today and next class we are going to talk about (())44:49, Borel–Cantelli Lemma and some surprisingly interesting facts, okay.
Thank you very much.