Probability and Stochastics for finance
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Lecture - 20
Black Scholes formula-II

So we continue from what we were doing in the last class. We were talking about how to get an hang on the option price at any time t at a for any stock price S t. So we will start to see what should I do when I get the premium because I had I had been taking the role of the person who has sold the option and you the viewer is the person who has bought the option from me. So such an option is called a call option where you are buying the option not selling the option.

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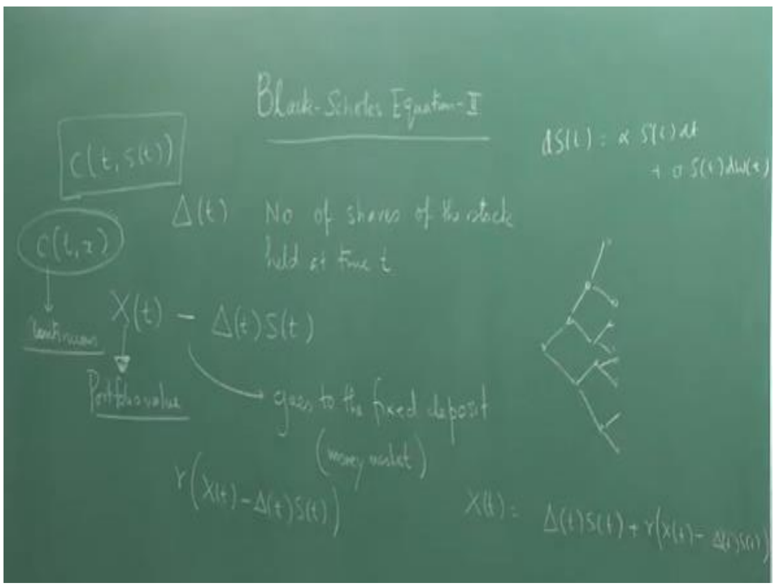

So now what I do let this is delta t. This is the number of shares I am holding at time t. So this is the number of shares I am buying of the stock that I have. There is 1 stock in the market and 1 bond 1 bank basically bank account. So number of shares of the stock held stock held at time t. So if S t is the cost of 1 stock at time t on 1 share of the stock then total worth of my stocks is delta t S t is the total worth of my stock.

If I sell the stock at that time this is what I will get. So my portfolio value so suppose now you have at time t I have X t amount of money and I now buy delta t amount of stocks paying delta t S t money. So this is the amount of money I have left. So I came. So at time t suppose I had X t amount of money I bought this amount of stocks shares of delta t shares I bought of the stock each share costing S t so this is the amount and this is the amount left and this is the amount I am going to send in the bank.

So this amount goes to the bank goes to the fixed deposit say fixed deposit or money market also this language is used in finance. Now at time t once I have invested how much would be my interest rate right. From the interest I would have the following earning. So on this amount of money this is the amount of interest I have been given. At time t this is exactly the amount of interest I will have because that is the when he is calculating the interest from time 0 so at time t this is the exact amount of interest I will have.

This is the amount of interest I will have. Of course if I take back the money then I will have this plus this but just the interest I have is this. So I am just taking the interest. So basically, what is my X t. X t consists of this plus this. So at any time X t is nothing but my any time my X t should consist of how much money I have spent for the stocks plus or at basically see the idea is this. I am buying selling instantaneously.

I am selling and buying at this very moment which is very strange thing does not happen but the time interval could be very small so that you can take it almost at the same time. So basically I am holding the thing for a very short time so I have employed this money and after that very same time again taking back the interest on that money and selling the stock that I had held. So then I sell the stocks at this and get this amount of money plus I add this money which I get from the interest.

So this is very strange idea that you have instantaneous selling buying. Instantaneous selling buying does not happen in the market. You could sell now buy now and sell after 2 minutes. But we assume that the gap is so small that that is what you are doing an instantaneous selling and buying. So these things take time to seep in. So when you do the finance course gradually you will see that why we are talking why the model is (()) 05:20. We will do it very slowly.

I think I will plan a much longer course slowly. So that is our better idea of the financial aspect. I am not talking about the financial aspect at this moment. I am just telling that that is what can happen. So if you write the differential of this whole thing then what is this?

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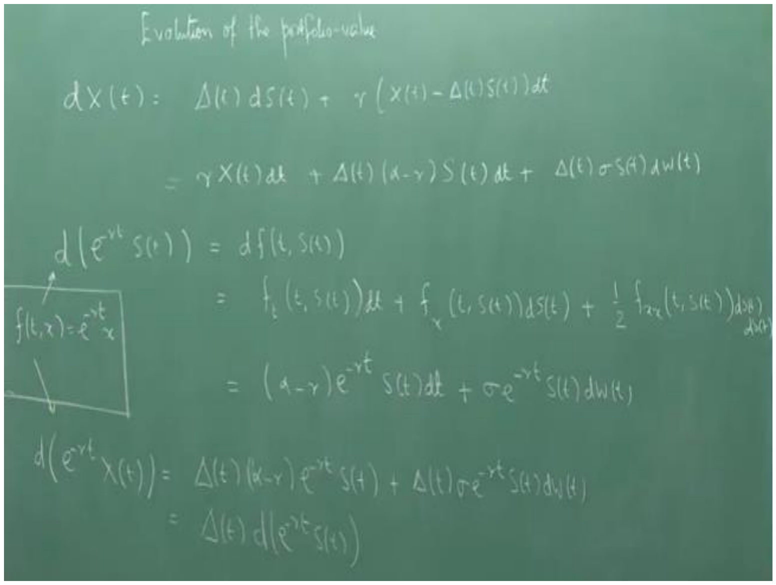

This is delta t d S t plus r into X t minus delta t S t into d t. That is the differential of the stock price because X t right X t itself is so writing the differential in that form basically. So at time d t what is the change in the interest price? At unit time the change in interest price is this. At time d t the change is this into d t. It might not be so easy to understand this unless we have seen discrete time analysis of the financial market.

That will be done in the next course. Discrete time is now you have time 0. At time 1 the price could either go up or go down. If it goes up from there it can again go up or go down like this. It is a binomial tree. So here here here and here here just like this. And actually, from there you could actually get a lot of ideas. So unless you do the binomial studies it is not really feasible to give anyone a better idea of what actually is happening.

But assume that this is the scenario basically. So then differential of this is given like this. So this is the amount of interest accumulated at time d t. At every unit time the accumulated interest is r into that and at time t is and at time d t del t is the fraction that is accumulated. So this can be now written as opening of the d s t actually which you know. If you have forgotten it I just make you remind you that ds(t) is alpha S(t)dt plus sigma S(t)dw t plus sorry dt plus delta t sigma S(t)dw(t).

So this is the differential when I open up S(t). Now you would be concerned not with just the portfolio value but by the portfolio the discounted portfolio value because I am taking decision at time 0 about what can happen at earlier times or at later times. So I want to look at everything from this point of view, current point of view. So what would be the price of my portfolio now so that after time say t the portfolio value is say X(t).

So we will look at the discounted portfolio value or we will also first look at the discounted stock price and write down its differential equation or stochastic differential form and then we will also look at the discounted portfolio value. So basically, we are going to look into the differential of this first. In order to do this we consider the function of the form f(t), x. Now we will again apply the Ito's formula. This will be nothing but d f(t), S(t) because f(t) if you put s t here it is this which is f t t, S(t) say again so again I am applying the Ito's formula.

So you will see how Ito's formula is being applied where it is the quadratic variation term. So d S(t) dS(t). So you see you start learning the use of of course you can find the quadratic variation term from here. So you know that will be nothing but sigma square S square t d t and then you analyse the whole thing when you analyse and put d S(t) here and write down the quadratic variation here and of course maybe here if you take the second the first derivative would give you e to the power minus r t with respect to x, the second derivative is 0.

So you can basically forget this part and just have this part and when you write down what is d S t from here you will finally get the following expression which is alpha minus r e to the power minus r t S(t) dt plus sigma e to the power minus t S(t)dw(t). This is what you are going to get. Now once I know that can I now because I have S t in this expression. Can I now know what is how my portfolio value is evolving? So, both for this and this the Ito's formula is applied using the same function.

Only instead of X in my case now when I am looking at the portfolio value I have to write X t in the case of stock price I wrote S t. Here S t is my portfolio value. Just if you have portfolio value so I have a portfolio. My portfolio consists of some shares number of shares of a given stock and the amount I put in the fixed deposit. That is my portfolio I am holding at every time t and I am instantaneously selling buying. That really does not happen.

The selling buying happens at various periods and when the periods become very very small they tend to the towards the continuous case. So we in mathematics we always try to look at the limiting case because that is much more easier to deal with. In real life there is no such thing. It has to be only done at discrete period. You do at time after 5 minutes also it is not really instantaneous but okay but for understanding the real the whole process continuous things are much more easier to handle.

So this is also called the portfolio value. So if I look at this I will go ahead and apply the same thing and then you can prove that this is nothing but delta into d of sorry delta t into alpha minus r e to the power minus r t S(t) plus delta t into sigma e to the power minus r t S(t) d omega t. See if you look at it very carefully if you take the delta t common outside and this is nothing but delta t and if you look at this expression so there is a beautiful relation between these 2 differentials.

So the differential of the portfolio value is nothing but of the discounted portfolio value is nothing but the number of shares I am holding into the differential of the discounted stock price. So this is exactly what it is okay. Now I have this fact. Now I have to see how my option value is evolving. Option price I have already mentioned in the last lecture is a function of the form c t, s of t okay.

At every time t that should be my that should be the premium you should pay me to buy the when I come into the contract at time t. The expiration time capital T remains the same whether you buy at time 0 or time when I capital T by 2 it does not matter. So this is so essentially I am going to look I am going to seek to find a continuous function c t x that should be my goal. That will and of course again I am writing continuous.

So what we now do is we now try to see how my option value evolves. So what we have here observed is the evolution of the portfolio value and then I am going to observe the evolution of the option value and what I had studied there was evolution of the portfolio value.

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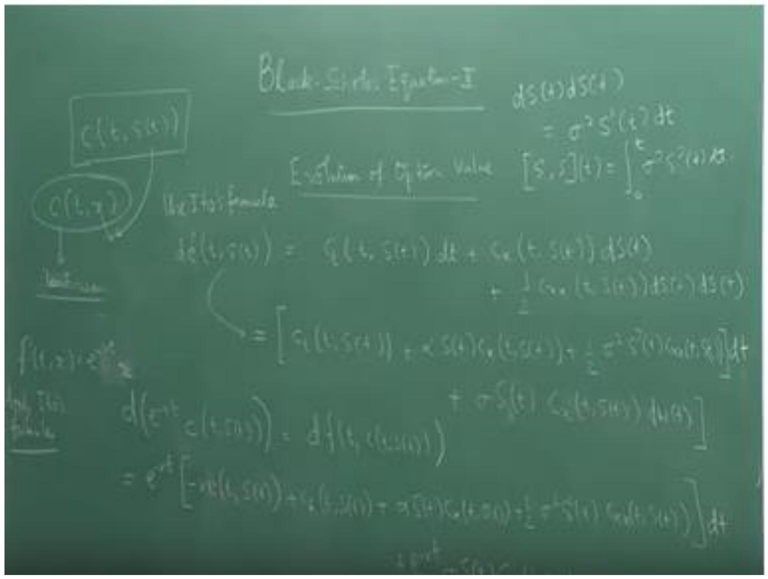

You might understand that why I am using the Shreve's book. Ya I could have run it or if I had to really do it it would have taken a long time and cannot be covered in the lecture. So I am just picking up certain parts from the book and trying to give you more explanations right. So this is to explaining more clearly which cannot be understood if you just for the first time go and read Shreve's book.

This book means to be read a lot of times to really understand what is going on. So the idea is this. If I can find the way my option value is evolving I want my portfolio value and option value to be evolved in the same way. Then only I can hedge the risk. So then I will equate the differential of these 2 and from there I will generate the equation by which c can be obtained. So actually, here I am looking at the function c t, x where x is playing the role of s t right.

So x is playing this role role of x t. So if I look at now so now again by Ito's formula so you will immediately understand so this so use Ito's formula so you see lot of finance is largely about how good or how well you are able to apply Ito's formula. So if you look at this thing this is of course c t t, s t c t means of course derivative with respect to t d t plus c x t, S t d S t plus half c x x t, S t d S t d S t.

Now you see here you cannot put this is equal to 0 because we do not know the form of the function. That is what we have to find out. We will be led to a some second order partial differential equation. We really have to find out what is this what is the c. So we do not know. So we cannot throw away this term. So you have to use the d the variation this quadratic variation. So quadratic variation what would be the quadratic variation of this? Sigma square S square t d t d W t d W t which is d t.

So this is the quadratic variation. Essentially if you want to write it more clearly or in a more proper fashion this is 0 to t sigma square S square d t t. So instead of so now if I put all those values all those things putting d s t and opening up d s t then finally I get a form like this. C t t S t plus alpha S t c x t S t plus half sigma square S square t c x x t S t this whole thing times d t this whole thing into d t maybe I should write it slightly on the side so that you get a get the writing clear. So I will write.

So this now equal to the same thing is now equal to so I have now after all the simplifications are done just plug in the d S t check at home and put this value here plus half sigma square S square t c x x t S t d of t. This is the d t part. So this is the drift part now of this Ito process. Everything is viewed as Ito process something that has to be remembered. Some sigma S t sorry sigma S t c x t S t d W t okay. That is what you will have.

Now we have essentially spoken about the discounted stock price. See discounted the stock price itself is not a Martingale. The discounted stock price is a Martingale. So if you have a discounted stock price you have much more stronger properties of the stock price. You can have you can have more information about it.

So we are always working with discounted stock price, discounted stock because we want to know at present what is the when a I am evaluating everything from the present scenario. So at present what is their value? So basically now I have to talk about so I am expecting the stock evaluation of the see the stock and money market evolves in the same way because the stock the option price depends on the stock price it will also evolve in the same rate.

The rate at which the stock grows the option price will vary accordingly. So we are going to look into, so without getting into, again applying, now the Ito's formula with now in this case you take f t x is equal to e to the power minus r t into x and instead of x you put c t S t. This now the formula that you have to use to apply Ito's formula the f that you have to use is of this form where x you put c t S t.

So basically, it is d of f t c t S(t) and then if you actually compute the whole thing out then you will have a very rather a very complex looking stuff. Of course, because we are at the end of the course some complex looking stuff would be a good thing but no mathematics is essentially a simple thing but okay this has its own thing so this is equal to if you apply the Ito's formula. So now apply Ito's formula apply Ito's formula and then you write the following thing sorry minus r c t S t.

So there will be a d t part so I am writing down the d t part. I am not doing this calculation because it will take too much then it will take I think 1 lecture would go more than 1-1/2 hours or 2 hours. Basically, when you do this you basically use again this formula here to open up the whole thing to get this final form. This thing into d t plus e to the power minus r t into sigma S t. Of course, you can ask me what is the real use of sigma.

These will come in the finance course where we really talk about these objects in a much more detailed way. Now what is the key idea. Let us now discuss the key idea. So we will keep this formula at hand which is this evolution will not require we will just remove this. So we want the portfolio the evolution of the portfolio value and the evolution of the option value that or rather the evolution of the discounted portfolio value and the evolution of the discounted option value are the same.

Their nature has to be same. So we can really say that at the end what we want. We need we need a discounted thing.

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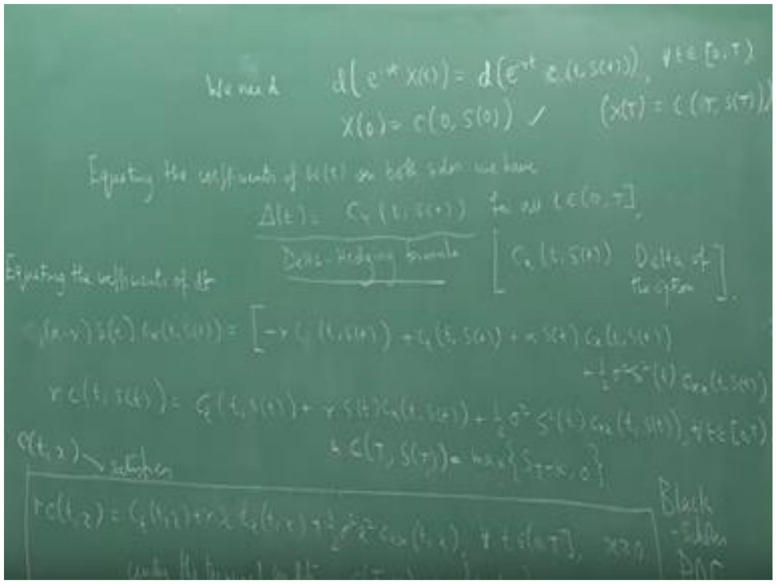

See what we are looking at is the intermediate values from time 0 to time capital T but capital T is not included there because at capital T I know what will happen because the market is complete I will be able to completely hedge. I will be able to have c capital T S capital T is equal to the max of that. So it will be exactly equal to x of t. At that time at time t I know exactly what is happening. So I do not have to bother about time t.

What I want that in between they should behave in a very nice fashion. They should exactly as the same behaviour. So that is some sense telling okay okay possibly if they when if it has to match at time t possibly it is a better idea to make the match at every time. So we need this idea. We need this idea. So basically, I want to equate this with you want to equate with this. How do you do that equation?

Of course, you know that x 0 is the price that you have x 0 is the amount of money that you had taken in the beginning from I have taken in the beginning from you which is this and x of capital T of course should be and x of capital T should be and this is already guaranteed by your complete market hypothesis. This has to happen. I should be able to completely hedge. Completely offset my risk. That is the main idea.

Now let us do one thing. See if I start hedging now if I start equating so I equate this with this. So what would I get. I would like to first equate the random part with the random part and a volatility part with the volatility part and the drift part with the drift part. So if equate this part with this part then what I get is so equating the coefficients of W t on both sides we have what do we have? We have the delta t.

We have delta t is equal to c x t S t. So here I want this to happen for all t element of starting time to capital T. This is already known. So what do you get? You get this. This is called the delta hedging formula. This is one of the Greeks of the option. This is called the delta hedging formula. So this formula is called the delta hedging formula. What does it say?

It tells you at every time cap T if you know the option price I can tell you how much stock shares of the stock you have to buy. So if I differentiate this function and at time 0 I will know what is the starting amount of stock I should buy. See if I am able to compute the function c t x or c t S t by differentiating c t x and putting the appropriate time in I will know how much stock I need to buy at time t.

So that is why this is a very important formula. This is actually used in practice. This is of course the implementation is slightly different. It is called the delta hedging formula. Now we this c x t this thing is sometimes called the delta of the option because this is the equal. Now we have to equate this part with this part and e to the power so equating the d t parts the coefficients of d t. So you know e to the power minus r t will get away from both the sides and you have to equate this part with this part. I am not writing it again in detail because then you can check it out.

So if I do that after doing some manipulations this part will get cancelled you see this part will get cancelled with some part you have S t here you have S t here e to the power minus r t will be taken off. So you will write alpha minus r S t and alpha minus r S t. So I am just equating this part the d t part sorry there should be a d t here. Please note there should be a d t here.

If I missed it in the earlier thing please make it sure. So I am equating this part. So alpha minus r S t into delta t should be equal to this whole thing. Now if I do that so what I would have but delta t is equal to c x t S t. So I can write c x alpha minus r S t into c x t S t is equal to this part that is equal to minus r c t t S t plus sorry minus r c t S t plus c t S t. So that is just the c and then this is the plus alpha S t c x t S t plus half sigma square x square t c x x now you have this term c x x second order term c x x t S t.

So we have equated the d t part by knowing that delta t here is this. So you have found out delta t using the other one and then you have done this. Now if you look at it very carefully alpha S t c x t here you also have alpha S t c x t here same thing. So when cancel that out and you can take minus r S t c x t to that side and keep this the other minus to this side so you will have an equation of this form c t, t S t plus r S t c x t S t plus half sigma square S square t c x x t S t.

And this should be true for all t between 0 to capital T and you know that with the fact and c T S capital T c T S capital T is max of S T minus k and 0 because that is exactly the hedging that I have to offset my risk. Now which means that the options price at every time t is can be found by trying to find a continuous function c t, x where c t, x satisfies the following partial differential equation. S t is playing the role of x.

So this is for all t element of 0 to T this equation must be satisfied and x must be greater than equal to 0 because the stock price must be nonnegative and there must be a terminal condition which is this and the and under the terminal condition under the terminal condition and under the terminal condition the following this which is c T, x T is there terminal time which must be equal to max of x minus k and 0, where x is actually S of capital T right.

So this thing the last this one is called the Black Scholes PDE or the Black Scholes equation. Of course what is this price. So we will show in our finance class finance lectures that the price at time 0 can be evaluated not just by solving this equation because solving this equation numerically is a huge piece of research.

A lot of work has been done how to solve the Black Scholes equation. It can be always written and converted into a type of heat equation and then try to solve it but lot of work has been done on the numeric of it. But there are other ways to forget the partial differential equation and using the risk neutral idea and doing something called the Girsanov’s using something called change of measure going from one type of Brownian motion to the other using Girsanov’s theorem.

And using basic probability theory one can actually compute what should be my price at any given time t c t x can be computed. So that can only come in a finance class where lot of lot more information about finance has been given to you. So it is called the Black Scholes partial differential equation PDE and with the as promised we have ended our discussion by coming to the Black Scholes equation and this ends the course and hoping that you have enjoyed it even if you have not enjoyed it you have got some information.

In a 10-hour course it is not really just it is not one cannot do too much justice to this huge fascinating subject called stochastic calculus. You think that you are sitting down there and you are really understanding what is going on no because possibly I also do not understand what is really going on. So you might ask me that this equation is in almost everywhere sense, yes but this equation when you do it is not really almost everywhere sense. It is the standard PDE.

So hope that you have enjoyed here and hope that many of you would join the finance class but which will be done in a much simpler straightforward way and which will be done in a with not coming to these equations, not too much bombarding you with Ito’s formula.

The Ito’s formula would require in some cases but essentially giving you the basics of finance. There finance would play a major role, the financial intuition will play a major role and also you will see how the mathematics can be naturally evolved out. That would be the course. But many tools from this course would be required in that course and that would not be repeated here.

We will come with the assumption that you have certain understanding. So with this I thank all of you for being a part of this course and I hope you had enjoyed as I had enjoyed in giving these lectures. Thank you very much.