Probability and Stochastics for finance
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Lecture - 02
Interesting problems in probability

So, we will start with whatever we had promised. We will start with something called the Bertrand’s paradox, which shows the inadequacy of the classical definition, which is a circular definition. So, here we will just try to solve out the Bertrand’s paradox and see that it will give two different answers for the same problem, two different probabilities.

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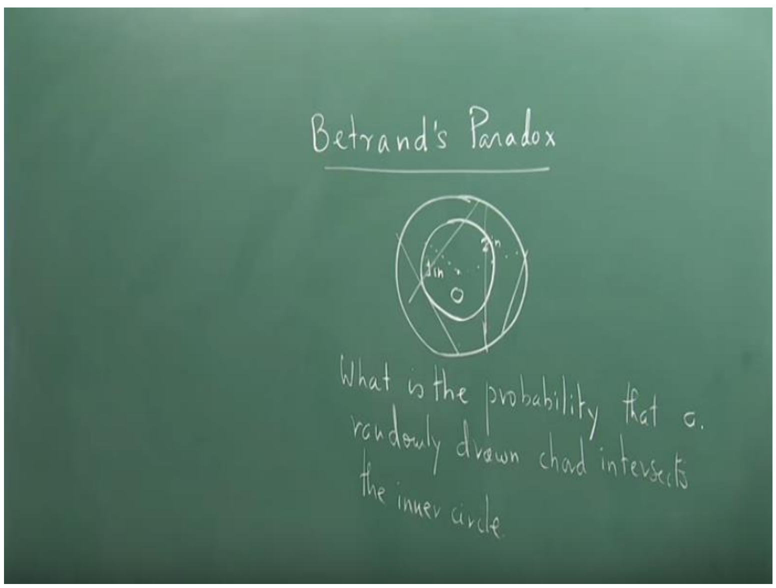

So, what does this Bertrand paradox says. The Bertrand paradox says the following; so, you have a circle, with a center say at o and a radius of 2 inches and then you have a concentric circle of radius 1 inch, 2 inch and 1 inch, the question is that, if I randomly draw a cord in the circle, if I randomly just draw a cord in the circle, what is the probability that it intersects the inner circle.

So, the question is, when does, what is the probability, what is the probability, that a randomly drawn cord intersects the inner circle.

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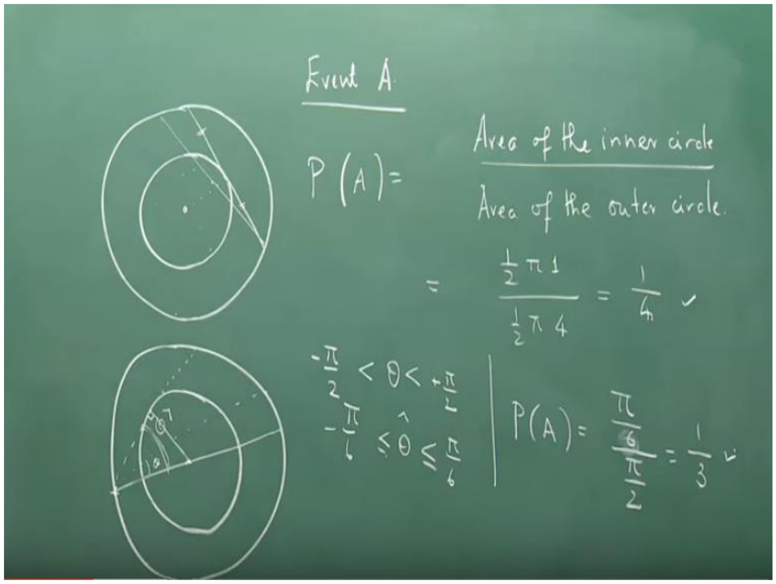

So, what does, if you look at it, let us look at it in the following way. So, by the fact that, it is intersecting the inner circle, whose radius is half of the original circle, when do I see a cord is intersecting the inner circle; if the midpoint of the cord lies in the inner circle. Where you see, even in the tangent one, if this is the tangent also, it does not matter because, if you join it by the radius of 1 inch that will be perpendicular and hence a very standard (()) 03:18 Euclidean geometry you will break it up into two equal parts, so the midpoint has to be inside the circle.

So, essentially the midpoint thus, can be here, here, here, here, anywhere in the circle and for any cord that is drawn the midpoint would be anywhere in the original circle. So basically, the probability, if I called that event as A, so let me call the event as A. Now you observe that essentially, I can say that my sample space is the space of all points in the bigger circle, all the points inside the circular disc and my possible position of midpoints are any point here on the inner circle.

So, the probability of an event A here we have an infinite sample space and here where the problem will come, the probability in the event A is thus, the area of the inner circle divided by the area of the outer circle.

So, it is the area of the inner circle divided by the area of the outer circle. So, area of the inner circle is 1/2 pi * 1 square which is 1 and this is 1/2 pi * 2 square which is 4 right, pi r 2, so this will just give me 1/4. So, that looks like to be a very feasible answer. Now I can look at the whole problem, in a very different way, which will also appear to you as logical.

So, here you again, take the outer circle and then draw the inner circle. So, what is the meaning of a cord; so take any point, I have taken any point. I can draw infinite cords from this point. Every cord need not hit the circle. So of course, you can draw the diameter, which is passing, a cord, which is passing through the center.

Now what would happen, this cord actually can vary, so from here, I can just sweep 90 degrees in this direction, 90 degrees in this direction and I can vary the cord. I can get all the possible cords; so basically, I can vary this thing by an angle theta right, so I can take any cord.

Suppose, this is the new position of the diameter, so I moved it by a theta angle but this theta angle can vary between - sorry + pi by 2 to - pi by 2 (()) 07:01 either here or it is here. Similarly, I can say that, if you look at the cord, what is the maximum amount theta can vary, if the cord has to touch the circle, as to cut the inner circle, the maximum it can vary is this amount, which let us call as theta head.

Now, what is theta head or what is the maximum amount theta head can go, so this is the maximum amount head can go. It starts from here and moves up and comes up to here, starts from here and moves and comes out to here; so, all the cords that are passing through this point.

Now, whatever probability I will have, for the cords cutting the circle from this point, this will have the same probability, all the points; so, that will be the probability of hitting, because you have to emanate from one of the points, start from one of the points. Now how much does this angle vary. So, this angle we have to observe that.

For example, if I draw again, by Euclidean geometry, if I draw the radius of 1 inch this would be perpendicular, so this is now 1 inch and this is 2 inch. So, sine of theta is ½, so theta head is actually varying from, sine 1/2 is 30 degrees, it has 30, so it is pi by 6 to okay. So, all the possible angles to get the cords coming out of this point is varying from pi by 2 to - pi by 2 and all the cords which has to be inside the circle, in the sense cutting the circle, has to, have an angle variation from theta head from pi by 6 to - pi by 6.

So, then so, pi by 2 - pi by 2 what is the total possible angles we have, essentially; so, if I just look at this part only, which is symmetrically it will give the same thing on the other side. So basically, I am having 0 to pi by 2 possibilities on this part, and 0 to pi by 6 possibilities.

So, probability of the same event A, sorry pi by 6 by pi by 2, so pi by 2 is the total possibilities here on this side and pi by 6 is the possibilities that are favorable to you and then hence this is giving me the answer 2/ 3 which is 1/3. So, these 2 answers are varying and both of them are equally logical, looks equally logical.

So, this is the Bertrand paradox and hence that sort of approach, that we had the way, we have defined probability, the circular fashion that you learn at schools, that cannot really work in general and hence probability itself has to be thought of as, some sort of function which takes an event and maps it on to a real number between 0 and 1, which follows certain property, the Kolmogorov axioms, that we had studied in the last lecture.

So, let us go back to the true founder, the problem of the true founder of probability, Chevalier de Mere, the great gambler, who gave his friend Blaise Pascal, the question of how many times you get a 6 if you throw a die, 1 throw of a die 6 times or throwing 2 dies 24 times, throwing them simultaneously. So, let us just go and solve that problem

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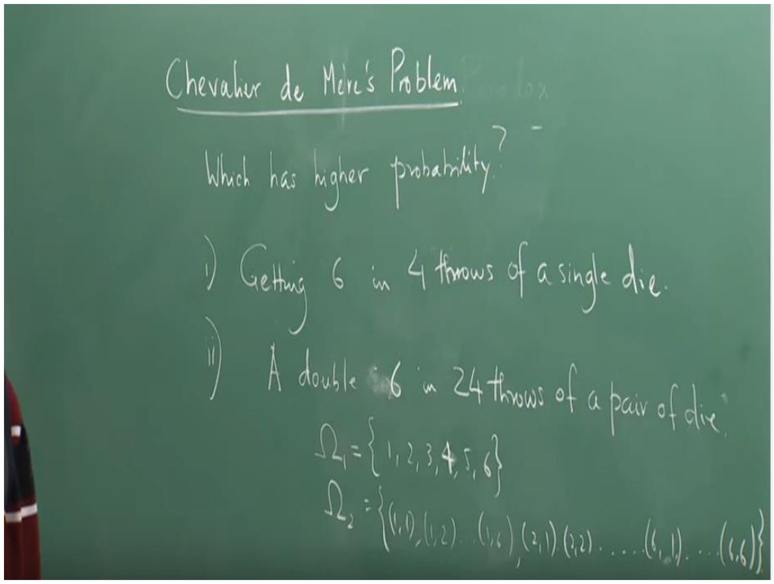

So, this is where stated the problem, we will state it again; of course, you will have practice problems with you, but this is something which you can enjoy, because that is what, how the history comes in so, which has higher probability, that is the question; getting 6 in 4 throws of a single die. So, first operation is, trying to getting 6 in 4 throws in 4 throws of a single die or No. 2, getting 2 sixes in 24 throws of a pair of die, a double 6 in 24 throws of a pair of die. So, which has a higher probability that is the question.

So, in order to solve this, we have to use the language of compliment events. So, if A is an event, A compliment is the event, that A has not offered. So, if A is an event in the Sigma-algebra, and here you can understand in both the cases, here my omega has 6 elements, so it is a finite set, now, and the second one, this omega ones and omega 2 has 36 elements they have the form (1, 1) (1, 2) (1, 6) (2, 1) goes like this (2, 2) (6, 1) (6, 6), so it has 36 elements. So, these are having finite sample spaces. So, they are Sigma-algebra associated with this u1 and u2, would have, would be nothing but subsets of this right.

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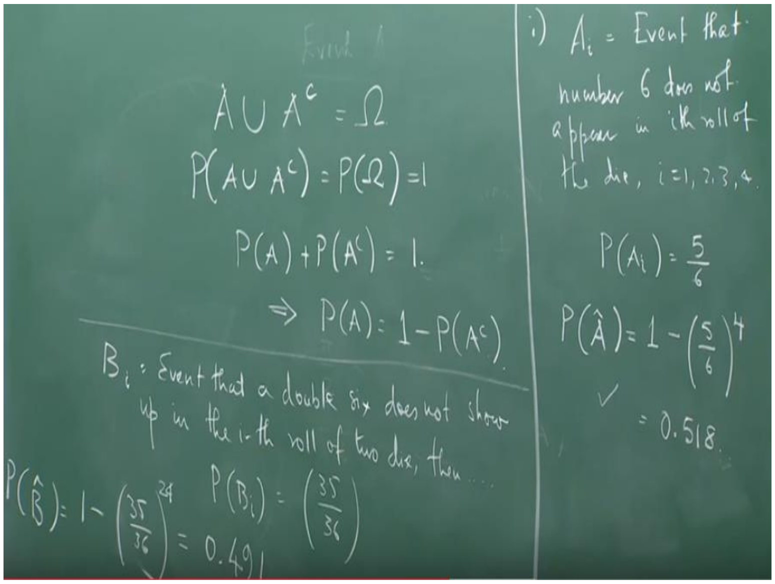

Now, here we will rely on this fact, that if A is a given event and if you consider A compliment and A union, A compliment where A is the subset of omega, this will give me back the whole sample space omega, then you know, why, now hence probability of A union A compliment is equal to the probability of omega and that is equal to 1, because that you know from the Kolmogorov’s axiom and hence now here again we will apply the Kolmogorov’s axiom here, because these are disjoint events, so this will immediately say that probability of A is 1 - probability of A compliment.

So, what I am now going to do is to, use this very idea that first I will talk about finding the probability of A compliment and then immediately you will know what is the probability of A by subtracting it from 1.

So, let us see where do we start; so, let us take the first case, case 1. Let us go by the first one. First one let us denote by Ai at that the number 6 does not appear, under rolling the die for the i th time, so Ai event that number 6 does not appear in the i th roll over i. Of course, i has only 4 possibilities and we will be allowed to roll the die 4 times.

Now what does it mean that 6 has not occurred, which means any of the remaining 5 has occurred? so probability of their occurrence is 5/6; so, probability of a i is 5/6. So, probability of

6 not occurring in the first place, first throw, and the probability that it does not also appear in the second throw are two independent events, they are not connected with each other.

Now we have not yet introduced what are independent events. So, what I am going to say, you look at it in this form that, I throw a die so what is the probability of this not occurring 5/ 6. Again, I throw a die, so if the probability of success is 1/ 6, the probability of failure is 5/6. So, I am just looking at the failures. So, if it fails at each and every time, so which means in all the 4 times, I have not got a 6. So, when shall I get a 6, is just the complimentary event.

So, this whole idea of independence of two events we have not yet discussed, but if two events are independent in the way as we can see it, because throwing a die now and not getting a 6 and throwing a die after two minutes and not getting a 6 are not connected anyway, then you multiply the probabilities. So basically, now what you do, the probability of my occurrence of the event say A had, that a 6 occurs, in throw of a fourth die is, 1 – 5/6 to the power of 4 and that turns out to be 0.518. So, you see, that is how you do it.

Now you come to the case of, where you have, you are throwing a pair of die with a 36 possibility, 6 showing up in both the case is 6 is 6 is there; so which means if B i is the event that a double 6 does not show up in the i th roll of 2 die then, what is probability of Bi, so if (6, 6) has not occurred, (6, 6) is just 1 of the occurrences, so there are 35 other occurrence which can come, so they are 35/36 a number of very near 1, but it is still less than 1.

Now this, suppose it happens for each of the 24 cases, 6 does not appear, so then probability of B head, that is a double 6 appears in a throw of 2 die, is nothing but 1 – 35/36 * 24 because, each of these 24 cases is independent. So, when you have independent events you multiply probabilities, which is something we have not told formally, which we will tell in the next class, but I did this, problem, because this is the historical problem so is good, histories are always a good guide to pedagogy and then what should be the answer.

The answer turns out to be, because of this power 24, which actually pulls down the value of this, not very far off, but this is obviously bigger and that is what the Chevalier de Mere had actually observed in the actual gaming. When they actually played the die, actually threw the die in those casinos, he actually had seen that that was that 6 was appearing much more rapidly.

Now we will talk about a very interesting and important problem called the Buffon's needle problem and the answer, we are going to get a, fantastic answer to this problem, the probability is linked with the number pi and that is what we are going to show. So, let us see what is Buffon's needle problem.

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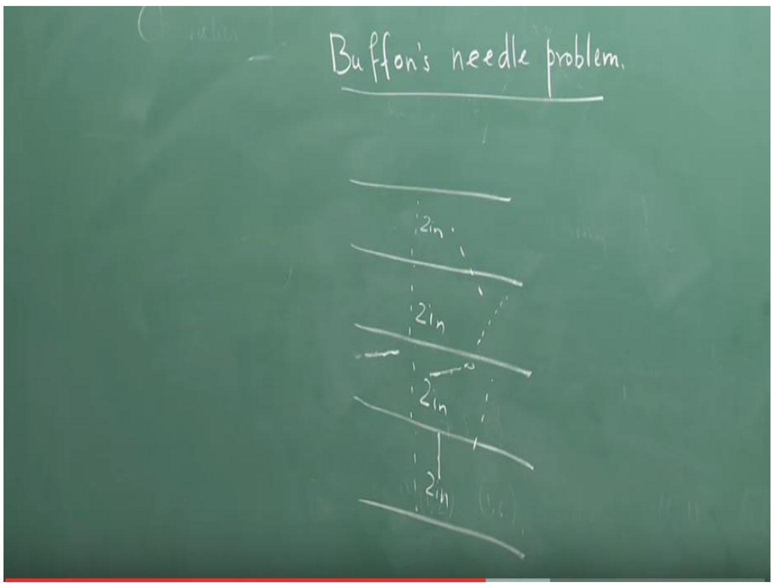

There is a lot of writing on Buffon's needle problem, but, not buffoon, but Buffon's. So, this would be the last problem. So here, you have seen in both the cases, we have been using the Kolmogorov axioms. In the first case, where you have finite number of outcomes, total outcomes, you see, you can use the so-called idea proportion, the other definition, but here, we need to we will use the similar sort of definition, but here we will see how to first, when the problem comes and when you have infinite possibilities you need to be very carefully construct the probability space. That is very fundamental

So, Buffon's needle problem is like this; it says that okay, let us take a plane, that is plane of the board and you are drawing parallel lines each say 2 inches apart, 2 cm apart does not matter, each 2 inches apart. Now, you are throwing on this so you have done it on the floor and you are throwing a 1-inch long needle on the floor randomly. So, the needle could land here, the needle could land like this, it could land like this, could land like this, could land like this, could land like this, could land like this straight touching at 1 point.

What is the probability that, the needle will cut or touch 1 of the parallel lines? What is that probability? How do we think about it? So, here our first step would be to know, how to, the answer is very beautiful, I do not want to tell you now, you will see when the answer comes out. So, first I have to realize how do I construct my probability space

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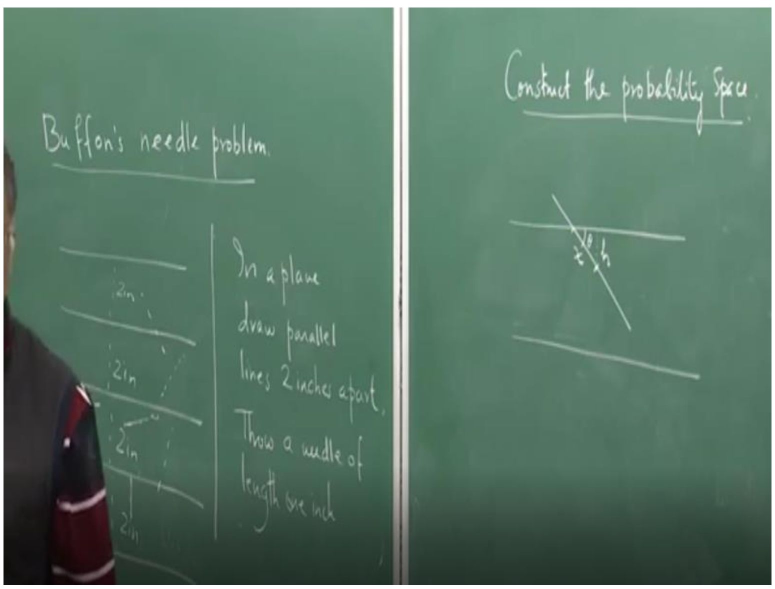

So, my first step is to construct the probability space and then how to use it, so let us construct the probability space. So, here is the line and suppose here is the needle, now take the midpoint of the needle, say here and find the distance of that needle from the nearest parallel line, so of course, there could be another parallel line here and then of course we will just look at the nearest one and the angle that it makes with the horizontal is the theta.

Now, this distance let me call x, this distance, so x, so this whole length is 1 inch, so what you have done, let me write down the problem clearly. In a plane draw parallel lines 2 inches apart, throw a needle of length 1 inch on the plane randomly. What is the probability that it touches one of the lines, parallel lines, one of the parallel lines?

My writing might not be very clear, but we will understand the problem. So, I have been telling, I have been reading it every time, that you have to throw the needle on the floor and see when does it touch one of these lines, so this is the situation. So how do I construct.

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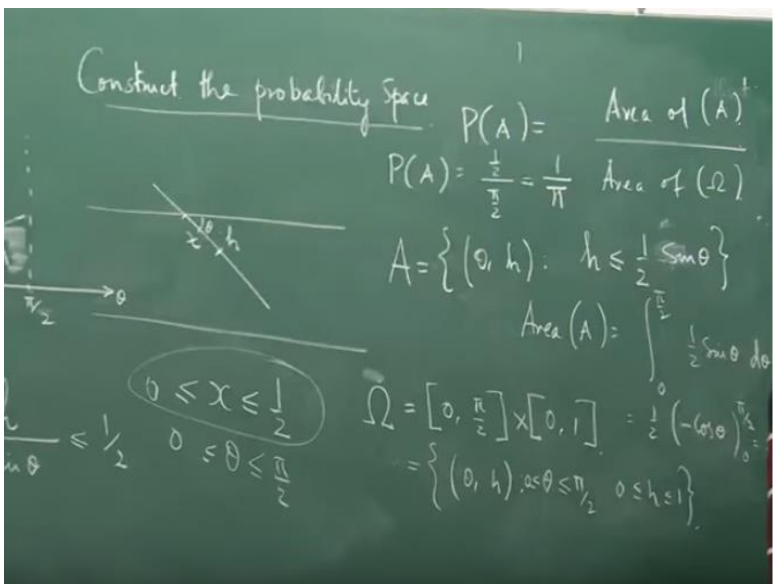

So, you see this x has to be less than or equal to half to touch, because it can be in this situation, this is the midpoint and this is (()) 27:48. So, the angle and this angle is of course 90 degree. So, the angle can vary from 0 to pi/2, while this can vary from 0 to half. So, this is possible if x is less than equal to half these are then equal to 0, because the midpoint can be exactly on this line where h becomes 0 and theta is theta and theta is of course varying from 0 to pi/2. So, I can define my space as, so I am not throwing it randomly. So, this is what happens, under this condition on x, it will touch the line, but I am just throwing it randomly.

So, my all possible outcomes need not, this is my favorable outcome. Angle will anyway remain from 0 to whatever you throw, it is either with the horizontal, it is either 0 or it has to vary between 0 or pi/2, either it is straight or something like that you know. You can always measure the angle acutely, but the length can be anywhere. The length x, the length of the needle is between, that x can be between 0 and 1 that, sorry this x can be between 0 and half. So, this is the possibly these are the favorable situations for hitting the line.

But this length x so I can construct this as the distance of h, so the length x has to be between 0 and half this is by possible possibilities, there cannot be any other possibilities right, but the length h where can it vary. It can vary from 0 to 1 because it could be like this, the 2 inches' lines and it is just lying in the middle, this is where the needle comes and hits it, it stops and this is h and this is 1 inch.

So, though the condition on x is 0 to half, my actual gameplay is happening with the 2 variables theta and h. So, theta can always vary between 0 and pi/2, but h will vary between 0 and 1. Now, so I can, but the interesting part is that, because I have theta and h and x is this line the hypotenuse, I can obviously convert x in terms of h and theta. So, the favorable situation is the following, that h/h and x is nothing but h, h/ x is sine theta right, so h sine theta is equal to x, so I can write so my favorable situation is when h and theta is linked with this (()) 31:25, this is the favorable situation.

So, I essentially my favorable event, my event that, I want to find probability of in this particular problem, is the collection of all theta h, such that h is less than equal to half of sine theta. So, you are just translating this x less than equal to half into this, h is obviously greater than equal to 0, which we are not writing down, which is obvious anyway, it is the length it is varying from 0 to 1. Now this is my event, so if I write down, so what I want can do? Now when you have infinite sample space, how do you really measure?

One way, of measuring or assigning probability, is assigning the standard measure of the sets, because you see, I can now find an area of the sample space sorry, I can write just omega; so I should be able to find the area of the sample space right and so basically, now what I am doing is

I want to find the area of this and the area of this. the standard measures, so I want to imply. Let us see if I take the area of this, if I want to compute the area of this. Can I use it as a probability?

Of course, the area of this is positive and area of this thing, so now how do I actually. know that total number of elements here and total number of elements here. that the only way of knowing is to calculate the areas, whether infinite elements, you want to capture the essence of total number is essentially, you have to capture it through the idea of an area. So, I find the area of this and I divide by the area of this, area of this is of course pi/ 2.

So, my probability of A is area of A/ area of omega; so how do I calculate the area of A. The area of A can be easily calculated in the following way, is that you draw a line put h and theta here, theta is along this line measured in radiance, so 0 to pi/2. So, h is equal to half sine theta, h equal to sign theta you known would be something, that pi/2 sine pi/ 2 is 1, so this is my 1, so half is half, this is half. So, this is your half sine theta, so h has to be below that.

So, essentially my event is consisting of all possible points this one, this is my A; so how do I calculate the area of A? The area of A is simply, area of A is nothing but, 0 to pi/2 because theta was varying from 0 to pi/2, half sine theta d theta, okay, so that is exactly it.

So that would give me, what would be this one half of, if I integrate sine theta the answer that would come out is, - cos theta, which I cos pi/2 is 0, no it is 1/2; just a moment, so it is 0 to pi by 2 - cos theta, so it is cos pi /2 is 0 and cos 0 is 1 - will cancel so it will become 1 so 1/2 so basically, half by pi, half pi pi by 2, probability of A is, half pi pi by 2. So, the answer is 1/pi. So, we have a very beautiful answer, the answer is 1/pi.

So, Buffon did a very interesting experiment. He took a plane like this, he drew those parallel lines and kept on dropping, say for 1000 times he kept on dropping a needle of length 1 inch and see, how many times it was hitting a line and then by doing it and increasing the number he could get better and better estimations of pi.

So, with this very beautiful example, I stop our discussion here and in the next class we are going to talk about random variables and random vectors, their distributions, what is the meaning of independence and some relative issues.

Thank you very much.