Probability and Stochastics for finance
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Lecture - 19
Black Scholes formula-I

So here we are ending this course with the last 2 lectures but at the same time we are also initiating you into the world of finance. So this last 2 lectures is also an invitation to the world of amazing world of finance and when we do the course in finance we will do it very stepwise. We will do it do it in a way that a larger audience can come except the audience that you have here but we would all do these things.

These things would come and you really have to know about Brownian motion and all those things. So this course actually becomes a prerequisite or even if not a prerequisite you should be having access to this course.

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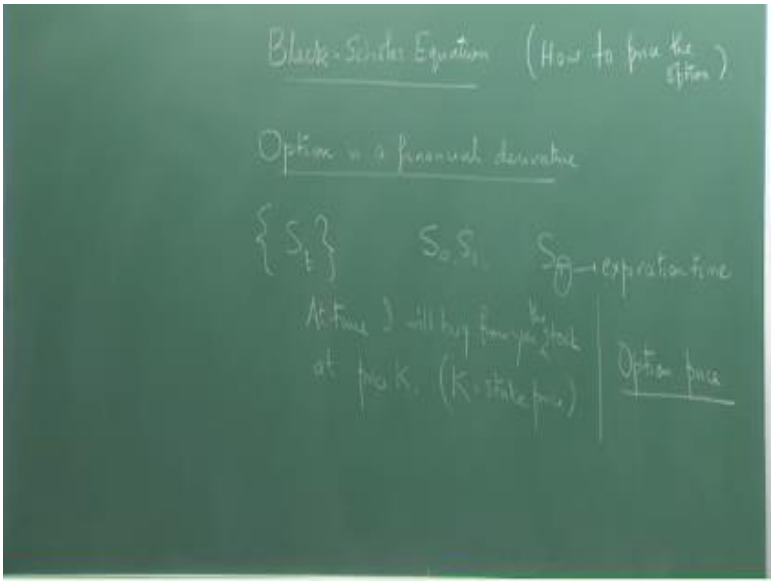

So everybody who did finance obviously wants to know about the Black Scholes formula or the Black Scholes equation. So this is Fischer Black and this is Myron Scholes. So we are going to derive a partial differential equation which has to be satisfied by the something called an option price but let us let the first lecture let me introduce to you what is the story behind all these things.

The story is the following that in a market there are lot of instruments people sell and buy right. For example I buy a stock of a company and then I say okay the stock price is increasing I sell it and make some money. This is 1 way. But I know when I buy a stock of the company I am also exposed to the risk that the stock price can fall.

There is always a time I can hold the stock too or the stock price can fall very badly and then some people want to get rid of that and get whatever little money they have. Nowadays banks almost force you to do some sort of investments where a partly or money would be kept in the fixed deposit and partly it will be traded in the market because in essentially the growth of the value of a stock if it grows is faster than the growth of a fixed deposit that you keep in a bank.

Now one of the most important derivative financial derivative is called an option. So a financial derivative is a financial instrument but its price depends on the primary instrument. For example the price of an option depends on the price of a stock. So option that is why option is called a is a secondary financial instrument it is called its price is derived from the price of a primary financial instrument like a stock.

So option and we will tell you what option is, option is a financial derivative. Remember option is such a strange financial derivative which people with not a great amount of money should ever think of actually going and using it. 95% of the option that are traded in the market are actually not realized, they are not operated and then options are used by big companies and cash rich speculators. If you are not cash rich do not bother about options.

The option is a thing like this. Suppose you have a stock and its price is S t, so the stock you observe the stock price S 1 S 0, S 0 is the stock price now. So you are speculating about some stock price at a time capital T. So you come to somebody and get into an option contract with him, the seller of the option. You tell him that okay at time t the contract is at time t I will buy from you a stock buy from you the stock at price K, K is called a strike price.

Means I agree in the contract that at time t I will come to you. Suppose I have a stock so you get into a contract and you tell me at time t I will come to you and buy a stock the share 1 share at a

price K say. I said I take 1 share only do not bother about me. But there is an interesting feature of this option contract. You are not obligated to buy from me at time K at time capital T which is called expiration time this T is called the expiration time of the contract.

You are not obligated to come and buy from me the stock. Suppose the price of the stock so the interesting part is that you might see that okay I have got into a contract but now the stock has lesser price in the market so I will simply go the market and buy the stock at a price which is less than K, why should I come to you and pay a higher price.

See if I had been obligated then I would have to come at time T and pay you the higher price and buy the stock but since I am not obligated and you are not obligated to me and I am ready with the stock because I have to keep my stock price stock with me till time capital T to sell it to you. Suppose if the price is higher in the market than K the strike price K is lesser than the market price of the stock you definitely come to me and buy the stock.

Then what is me as a option seller who is selling this option this option is called a call option where I am selling the option and somebody is buying the option. There could be some reverse swing also but we are just bothered about call option at this moment. So suppose I am the seller of this option what is my use. So I have kept a stock in hold and now where because of the stock price is low you buy the market stock price stock from the market and you do not come to me.

So I would have no other option but to take a loss and sell the stock in the market at the market price. I was thinking I will get a money K. Now but suppose the stock price money is higher now because I am in contract with you I am obligated to give you the stock at price K at time capital T. So you come to me because you know the market price is higher you come to me and ask for the stock and I have to give it to you.

So in both cases I am at a loss. Then I could have sold the stock in the market at a higher price which I cannot because I am obligated to sell you at a lower price. So when I am so that is why when I enter into this contract I ask you a premium that you have to pay in order to enter into

that contract. That premium that you pay is called the option price. So this is not a free contract that you make and okay you are the boss and you can do whatever you want.

So but but when you make the contract you have to pay an upfront price which is called the option price. Now how do you determine that option price is the issue of Black Scholes formula. How to determine an option price in a very simplistic setting. So the whole game is to how to determine the option price.

So now at time T equal to 0 when you have come to me to buy and you are getting into a contract and your expiration day is time say capital T equal to t maybe out of 6 months then at time T equal to 0 what what price I should charge you. How to price the option, that is the question. So the question is what sort of price I should charge you. Now what should I do as a person. My goal would be that I should not be at a loss.

If I am selling you the instrument I am getting into the option contract I should not be at a loss. Then what should I do. The basic idea is the following. That I should ask you for some money. In this given period 0 to T I will keep on trading with that money in the market buying and selling stocks, could be the same stock and then at the end of the time period T.

The worth of my portfolio that I create of buying and selling stocks the value of that portfolio would be exactly equal to the loss that I would have by selling the thing to you or the gain that you would have by taking the thing from me teek hai.

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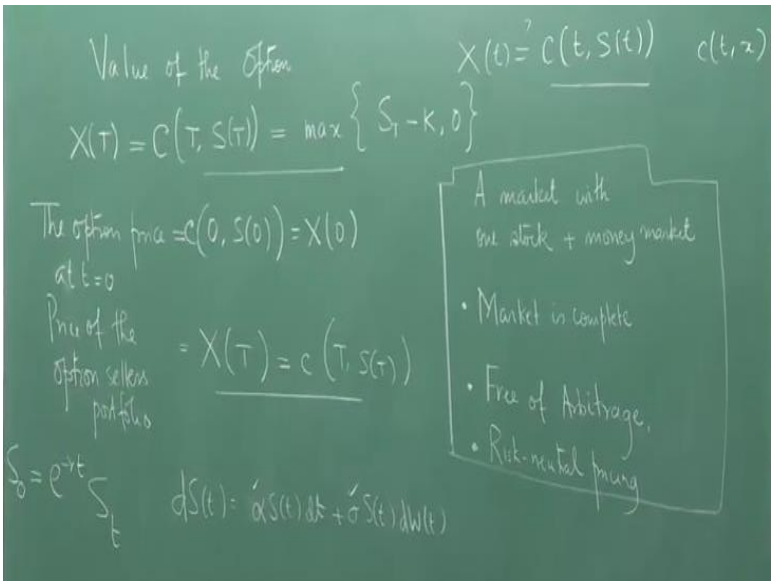

So there is something called the value of the option and that value depends on the stock price at the time T capital T. So what happens. So if at time capital T the value of the stock price is S T I should write S 0, S 1 because that is way I was writing the stochastic process. Please note that I should write it like this. T is the expression. So now what would be that value? So if the stock price is bigger than K then what would I do?

I will simply buy you, you will, if the stock price is K stock price is S of T and S of T is strictly bigger than K what you will do. You will simply come to me buy it at K and sell it at the market at market price S T which is higher than K and make a gain of S T minus K and if the stock price is lesser than K you will not even come to me. So if S T is strictly less than K or equal to K you will not come to me. So the value, the worth of the option is the maximum between S T minus K and 0.

The option price option price at t equal to 0 is nothing but the value of the or the worth of the option at time equal to 0 and that is C of 0 S of 0. Our goal is to actually compute a function like this c t x which would satisfy these sorts of properties. Under what condition can we do such computation. That is the question and it is very important to know what would I do by this C 0 S 0. Suppose you pay me that C 0 S 0 I would trade in the market.

Some money I will put in the risky stocks some money I will put in the bank and then I will so I have a portfolio of some stocks and bonds and stocks and fixed deposit and then I will see how my value grows I keep on trading in such a way that if X t is my X t is my the X determines the price of my portfolio as a seller I am doing that price of my price of the seller's portfolio, the option sellers portfolio.

So this price if I do not want any loss must be equal to your gain. In what sort of situation did of course a person can buy option on 30 stocks but here we will take a very simplistic view of 1 stock. Now in what sort of scenario you can actually find this price. You have to understand that looking at all the complex interactions of various stock prices etc., and trying to find various option values for various options.

The basket of options that you have bought is a very complicated thing. So when you model you cannot model the exact reality but you try to do some modeling try to do some sort of thinking in a way that you take the most simplistic situation and look at it as which is a as if it is a part of the more complex situation and then try to interpret that more complex situation using that simplistic idea.

So what these guys had done is they had considered a market a financial market with 1 stock plus the money market is bank or a bond 1 stock and 1 bond market. Money market means you have kept it and there will be a fixed interest rate constant and at after certain time T you will get that interest rate and in the Black Scholes world you have to assume that the market is complete. I will tell you what is the meaning of this.

And the second point is that the market is free of arbitrage and we will come to tell you what is that. In the finance course we will deal much much more details about these things, free of arbitrage. So what is arbitrage? So arbitrage is a very interesting thing. Arbitrage is making money without spending a penny, making money without spending a penny is called arbitrage.

Example, example is that suppose I go to a shop buy 10 pens of 8 Rs. each and then I come and sell among my friends each pen with 12 Rs. each. So how much money I get from my friends

120 Rs. So I have recovered my 100 Rs. I have I can take put it back in the same place so I have not made any investment actually and I have just generated 20 Rs.

But the markets immediately notice in real markets immediately notice that there is some difference of price. So that is what mainly arbitragers do in the market. They keep on for example the they keep on looking at the stock prices of some particular stock at BSE and NSE National Stock Exchange and the Bombay Stock Exchange and if 1 price is lesser they will buy it there and sell it somewhere at a higher price.

But this arbitrage condition remains for a very short time because other people start noticing it and the prices are leveled. So in a (()) 17:59 situation of the market the market is actually in no arbitrage situation. So the modeling has to be done in a no arbitrage condition. I and I have seen this arbitrary business even in doing in railway platforms in 1 platform I was standing and I wanted to buy cold water and someone the guy said okay this railneer bottle should cost you 12 Rs. I said but the official cost is 10.

I did not do anything I went to the other platform and I got it at 10 Rs. I came to him and asked him for a bottle. He said 12 Rs. I said I told everybody surrounding there that you see I have got his bottle at 10 Rs. I want more 1 more and he is asking for 12 Rs. You go to the other side you will get it at 10 Rs. People started leaving. So he said okay okay dus rupiah. So the price gets leveled like that.

So that is that is the sort of thing and market is complete is the means the following. See when you have taken money the option price and you are now starting to actually trade in the market so that you make up your loss and equate it to my gain. A market where this can always be done where you can always hedge, hedge means safeguarding yourself. You are exposed to the risk now, you could lose money right because you have taken the contract.

So you have to have that money to use it in the market so that you can completely hedge off your risk which is exactly this. So a market which where you can do this which allows this is called a complete market and markets are (()) 19:47 not complete. So that is why it is not always the fact

that whatever you compute in mathematical finance should be actually used in the market. You really have to think a bit and do it.

And when we do the option pricing very slowly step by step we will talk about these things more clearly and you will see where the arbitrage condition actually been used. Here it is not no time to really use that here. We are essentially trying to use Ito-Doeblin formula. So if this can be done always such a market is called a complete market and a Black Scholes formula is computed in this setting and his setting.

The stock has a stock price has satisfies the following differential equation, is called the drift and sigma is how much is the stock price varying captures the variation basically. So these are all constant numbers alpha and sigma. Sigma is always greater than 0. Now, how will you suppose you have taken this money from me which I say the initial value of my portfolio. So I have taken this money from you right. I am the seller of the option thing and you have come in.

So you have paid me this money which I now say they are at 0. What I am supposed to do now is the following. I have supposed to start using this money X 0 and start trading in the market, but my market now has 1 stock in the money market the bank. So what I do a part of the money I invest in the same stock because I have to buy that stock. I do not really have that stock. I have to buy that stock from the market.

So I am interested in buying that stock and some part I put in the bank in fixed deposit. Then I at some point I sell that stock again so I see how much money I will have the interest from the fixed deposit at that time and add it and with that I will again buy partly that money would be invested again into buying the stock and again sending back some money to the money market.

So this will go at every step so that at the end my X T my price should be this should equal to should be exactly equal to the option value. So this is exactly what is the meaning of a complete market. So the bank will give I will assume will give me a fixed interest rate R. I will assume that during the time 0 to T it will not, 0 to capital T it will not change its interest rate which is not

really true but okay. So these are the you know you are making the idealizations which do not actually happen in real world and it is not always possible to hedge it.

Hedging is this is this is a very once you are talking about hedging you are essentially this sort of pricing is called risk neutral pricing. In general when you are exposed when you have invested in a stock you are exposed to more risk than you are than you are investing in a fixed deposit. That is the fixed deposit interest rates are low because that is guaranteed because that is a sure shot income that you would have.

While this interest rates on the stock are higher they grow faster because you are exposed to the risk of having a loss. There is a downside risk which is not there in the case of fixed deposit.

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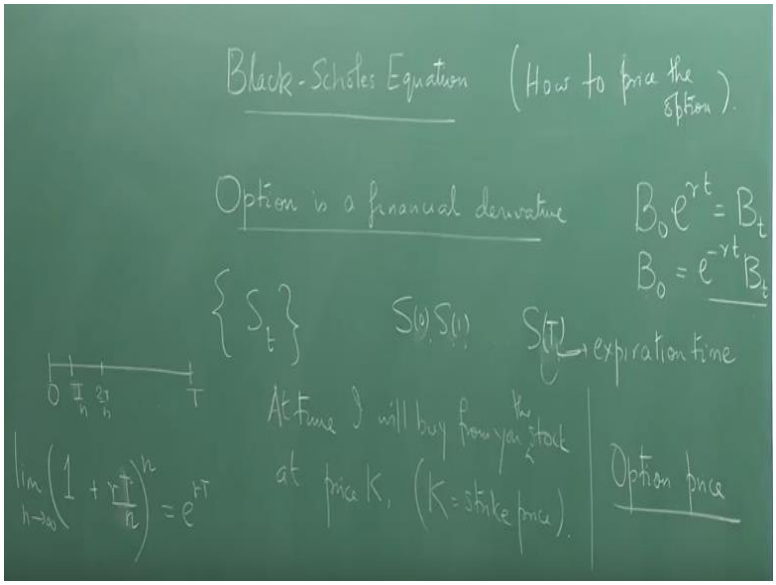

So we would start trying to learn that suppose I have this time interval 0 to T divided into small small intervals each of size T by n okay and I have invested 1 rupee at time 0 which I will take you at that at the end at time capital T I will take it from I will get something from you. So each I have divided into the interval T by n 2 T by n and so on, the partitioning of the interval. So if I want to get some money at T by n how much money I will get.

I have invested 1 rupee. My interest rate is R and my time duration so how much is the time duration. My time duration is. So if you repeat it for the next this is called the compound interest.

If you repeat it for the nth interval you will have this because you had n partitions. Now because so R into 1 rupee so and T by n so R is the interest rate over the whole interval. So in the interest in the interval T by n you will get a interest rate not the full interest rate. R is the interest rate I will give you only if take the money at time capital T.

But if you want to take off the money at time capital T by n then you I will not give you the full interest rate. Of course, why should I give you the full money because I take the money from you to invest somewhere else. That is the whole game hmm. So what do you do. You give a money proportion to the interval of time. So you give R T by n. So 1 plus R T by n into 1 rupee. So that if you repeat over n intervals you will have this.

So when you have a continuous system like watch this scenario that we can trade at each and every movement of time. Actually we do not do that. But in reality, we can trade at every movement theoretically. So this is nothing but e to the power r t. So this is actually my instantaneous interest rate. So if I put B 0 amount of money in the stock market in the bank now then after time small t I am supposed to get a money of this amount.

So what is B 0. If you look at B 0 which is the price when amount of money I have kept now right. This is nothing but e to the power minus r t into B t. So this sort of pricing is called the discounted price of the fixed deposit at time t at time 0. So if B t is the amount I should get at time small t what is the amount I should deposit now. Suppose I want to make this amount of money after 5 years.

What should I money I should now put in the fixed deposit given my interest rate is fixed R. This is the and I can means it is a continuous time interval. Is not that okay 0 to t fixed interval, ithna diya gaya and 1 plus R into R t R into 1 that is that is all. It is not that it is 1 plus R just it is not like that. So then this will tell me that if this is the amount of money I require B t is the amount of money I require this is the amount I should invest.

So this is called the discounted price at time t. So we will be interested in the discounted price. The interesting fact is that the discounted stock price if you talk about the S t is the stock price at

time t. So e to the power minus R t into S t is the discounted stock price. So what should be the stock price at time t so that I have current I have the current stock price is maybe if my current stock price is S 0 S 0 should be this.

Now you might ask me why are you putting R here. The stock price growth is higher. Stock grows at a faster rate. The interesting thing is that in the pricing structure of Black and Scholes we also do something called risk neutral pricing. So essentially you are not bothering whether you are investing in a stock or investing in a bond. So there is a huge amount of simplification which does not work in the market.

Actually because you want to get a understanding of how to price. So even with that the thing looks very complicated. You see so you are telling that the stock and bond you are is growing in the same rate. So you are neutral to that investment whether investing in stock or whether you are investing in a bond. Our next talk, so you have some very basic idea of what we are going to price.

Our next talk in the next and the last talk we are going to see that how we are going to get the discounted values stock prices and the discounted portfolio values and see how they evolve and we will see that we will say that every time t my value of the option is C t S t means if you do not buy between time 0 to capital T, if you do not come to me at time 0 but come to me at time small t which is lying between 0 and capital T then C t S t should be the money you should pay me because that money I will use to and that should be at every time should be X t essentially.

The question is in the next class we are going to figure out a differential equation, a partial differential equation which would actually solve whose solution would give me this function. Basically we have to find some function C t x where x is taking the place of the stock price where C t x would solve some partial differential equation.

Of course you might ask me why do you not give me the chance to write down the exact Black Scholes formula. I will not do that now because then the fun would be gone then we would still wait for starting the actual course in finance which will give you much better which will have

more siding with finance. We will do the math. We will also have a lot of pep talk on finance. So with this we end the 4th lecture of the 4th week and then in the next class we are going to actually compute and get the Black Scholes differential equations whose solution is actually the option price at any time t.

See essentially we need to know the option at any time t and knowing that this function is continuous we will get the option at time 0. Thank you. We will start quite soon.