Probability and Stochastics for finance
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Lecture - 18
An Application of Ito Integrals-2

Hello, so it is our third lecture in the last week, in which we are going to talk about Ito's formula in higher dimension. What we mean by that. See, we have been talking about only for example stock price of one stock, interest rate just one interest rate. Suppose, I want to talk about two stock prices. I want to monitor two stocks, at the same time and their prices and see how they are correlated, how their behavior is related to each other or not.

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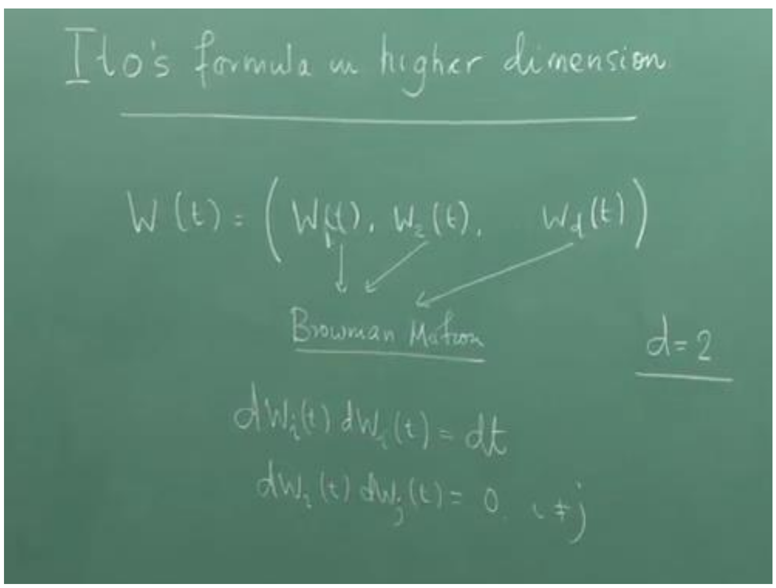

What sort of approach I should need to analyze them and then you have to talk about Ito's formula in higher dimension? In doing so, we need to talk about Brownian motion in higher dimension, that is the Brownian motion vector, that at every time t, the Brownian motion is not just a random variable, but a random vector. We had spoken about random vectors very briefly during our very first discussions of probability. So, this is what we are going to talk about.

W 1 t, W 2 t so it is the d dimension of a vector if you assume. At every time t, so this is a higher dimensional Brownian process or Brownian motion and this consisting of d, say d components of Brownian motion, where each of them are random variables, each of them. So, now the question would be, why we need to think of such processes and what is the utility. See you can still define, a filtration adapted with ft in the same way.

Of course, when you are talking about Brownian motion, we will write W t - W s for example. This increments are independent. So, W t 1 – t 2, W t 2 – t 3, t 2 – t 1 so these would be having independent increments, in the sense that, we really do not bother much about the vector. We really bother about its components.

One important thing that you have to understand about the components is the following, that if we are talking about a component i and taking its quadratic variation then, this is just d t and if you are talking about a cross variation of 1 then it is 0 okay. So, that is what you have to keep in mind.

So, when i is not equal to j, then the cross quadratic variation is 0 and quadratic variation anyway is d t is known to you. So, what we are looking at. We are essentially looking at, not only one Ito process at the same Ito process, but we are looking at two Ito processes at the same time. For example, if you are looking at the price of two stocks they are two stocks are two different Ito processes. So, we are trying to find relations between them.

So, in general we are looking at Ito process. Suppose now you take d is equal to 2. So, we will now discuss, just look at scenario at d is equal to 2. So basically, we have two different Brownian motions right W 1 W 2. So, two different scenarios.

See the Brownian motion, the W function is not a unique thing. Wt is not unique. So, W1t and W2t are two different Brownian motions right. W 1 t and W 2 t are two different Brownian motions. Only that they have to satisfy the properties of Brownian motion. Their functional form need not be same. That is something, which you have to always keep in mind that, these are do not have unique functional forms, when it can be it depends on a what sort of a it just it is a stochastic process which would follow certain pattern.

So, if it just follows that pattern it is a Brownian motion not that it so there can be 2 different Brownian motions and has very different sample paths for a given realization omega but they are following all the properties of a Brownian motion and hence they are Brownian motions.

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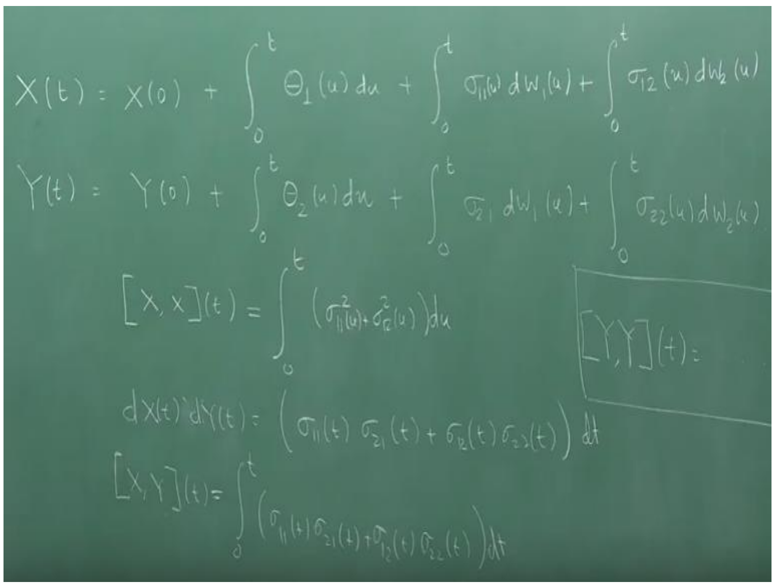

So, usually an Ito process, so for example, we are looking at two different Ito processes. The first one is given by this formula, theta1 u du plus 0 to t sigma 11, essentially these are trying to recollect the, some sort of a core variance type thing, some sort of matrix. We will come to it very soon. You will see soon see why I am writing like this. So, this every process xt that we are taking here, is involving both the processes. It is not just involving one process.

So, it is some sort of a core relation coefficient sort of thing and it is a function where, you are essentially describing the first Ito process, but you are talking about the second Brownian motion. Similarly, you can write Yt is equal to Y0 plus 0 to t theta2 u du plus 0 to t sigma 21 and 0 to t sigma 22.

So, when you have two Brownian motions, so you can define this sort of processes. And then, for example, I leave it to you to figure out, what is xxt that is, what is the quadratic variation of this. So of course, you can write the simple formula, d xt is equal to theta 1 t d u d t plus this W 1 t plus this W 2 t. You can write down the shorthand's. So, using the shorthand's you can come to this conclusion, that if you take xxt what will happen.

See every time, we are using, what these sorts of facts and whatever we had learnt earlier right, these shorthand notations. For example, if I want to write d xt the cross variation, do not write the cross variation immediately as 0. Basically, the multiplying of all the things. So of course, I would not write to you, what is d xt and what is d yt. You can write it out yourself, now because we have been doing it so much repeatedly.

Then this would turn out to be sigma 11 t, sigma 21 t plus sigma 12 t plus sigma 22 t, this whole thing dt. Basically, if you want to write it like this. So, it is not 0 you see. See if you want to write it like this so this cross quadratic process, is a stochastic process given like this.

So, I leave it to you to find, figure out what is the quadratic variation YYt. I am sure, you can just immediately write it down, without me telling you what to write. They are exactly the same thing with this, these things are replacing these things, that is all square of these things.

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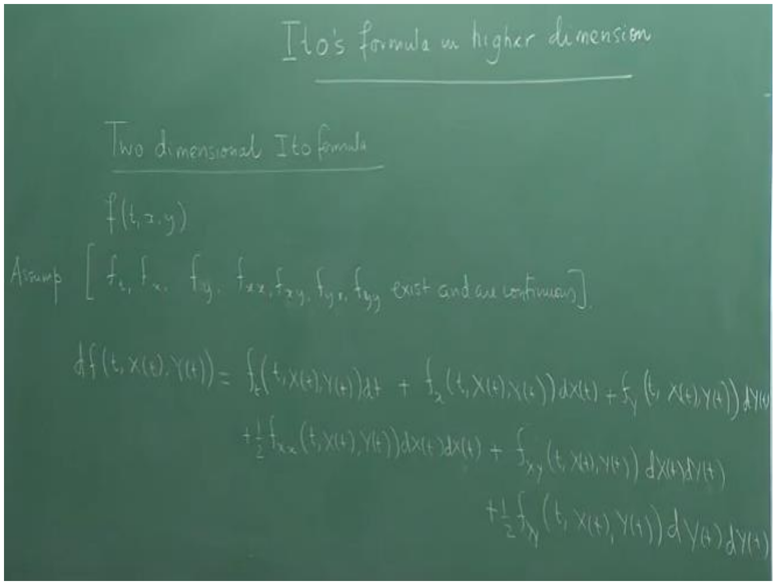

So, once I know about this, I can write a two-dimensional Ito formula. So, we are going to now talk about a two-dimensional Ito formula and then we will use it to study stock prices, two stock prices at the same time. In the two dimension Ito formula case, we do not use the function just ftx but ftxy because, here also we have to account for the process 2nd Ito process Y.

So, we are basically now considering a function of this form and once I consider a function of this form what do I expect? I expect that ft, fx, fxx sorry ft, fx, fy, fxx, fxy, fyx, fyy exist and are continuous. So, that is the assumption. This is an assumption okay and I have made this assumption, so assumption. So, once I make this assumption, you know from basic analysis that f xy and fyx are the same thing.

Now the formula is like this. The Ito formulas are basically stochastic Taylor's formula, Taylor's expansion. D ft, xt, Yt is equal to ft t, xt, Yt dt plus, with fx you will have the dx t term f xt, xt, Y t, d xt with fY you will have the dYt term. So, when if I had put Wt here W1 W2 say, then it will have W1 W2.

So, f Y again I should tell you all this inequality that I am writing are in the almost surely sense, but these are random variables, but do not think that, when I am writing f tx ty t it essentially becomes a random variable. It is no longer just an ordinary function. You are writing ordinary function but you are then looking at its random avatar, basically or looking at the random form of that.

For example, if I take the function fx equal to half x square you can write half capital X whole square, where x is a random variable. The random variable at the end function, but overall behavior is slightly different. So, this is that is what I feel about random variables. This equality, I understood that okay this equality will hold with probability one, so we do not bother much about talking about this every time, so is just an equality plus half fx xt, xt, Yt with this you will have dxt dxt.

Now because fy xy and fyx are same so we just write one of them. So, half of each will have contribute a half term, so you will have for full term so if I write fxy t, xt, Yt dxt dYt so the cross one it is this one, plus half fyy. I again tell you that fxy equal to fyy so basically, I should have a term here half xyt xt yt dxt dyt plus half yxt xt yt dyt dxt, but both the terms are same so that half half would give me one so, I do not write anything here.

So, half of txt yt that into dyt. So, this is you see, there is a pattern in the formula, with t there is d t with x there is dxt with y there is dyt with xx there is dxt dxt with xy there is dx y d x t with y y there is d y t d y t. There is a pattern. So, once you recognize the pattern applying Ito’s formula is become a child’s play it becomes fun. You can just keep on applying and getting your results.

So, you have to keep in your mind this pattern that Ito’s formula gives you. Once you know this pattern then everything is fine. Of course, proving this is a very different game but it is better to first learn to use the pattern and learn to apply them. We will when we do the financial stuff we will actually go and do that we will see more of these applications.

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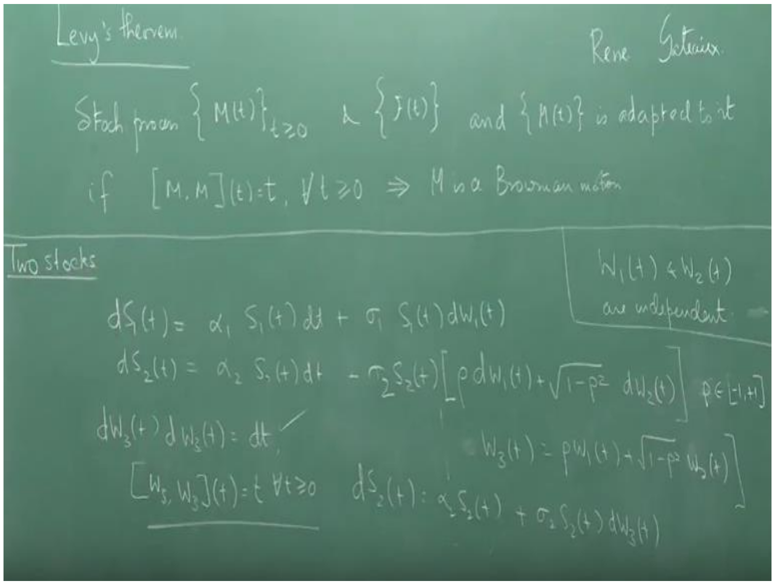

Now let us look at the stock, two stocks, price of two stocks, that will be important, to know the price of two stocks and their behavior. But it is very important to know, one of the theorems. It is called Levy's theorem. So, Levy's theorem tells us, how to recognize a Brownian motion if there is one. It tells you that if you have a stochastic process Mt with t greater than 0. Suppose you have a stochastic process Mt it is given to you. So, this is a stochastic process M t with t greater than equal to 0. That is given to you.

Now once you know that, suppose you have more additional information that, you that, there is a filtration Ft, to which this stochastic process is adapted and Mt is adapted to it. And then additionally if, you have the quadratic variation to be t at time t, it accommodates the quadratic variation of the amount t, which is path independent right. If you have this then, it implies M is a Brownian motion.

So, this theorem is due to Paul Levy. It is always better to tell you some one or two stories about this guy called Paul Levy. Let us understand that Levy was really not a probabilist. He was analyst and in fact it was Rene Gateaux on whose name we have the Gateaux derivative.

Rene Gateaux, who started working on this sort of stochastic processes to look at them in a more detailed mathematical way but he was killed in the World War II, sorry World War I, and one or two, I am not, so one of the World Wars Rene Gateaux was killed. And then Paul Levy was told to see that his papers what he has written should be published. So, while reading what he has written he got interested in probability and he himself became one of the leading probabilist of the 20th century, Paul Levy.

So, the modern stuff of the Brownian motion that we see the modern theory and modern definition are essentially due to Paul Levy. So, it is very important to know the guy who had actually done stuff for us.

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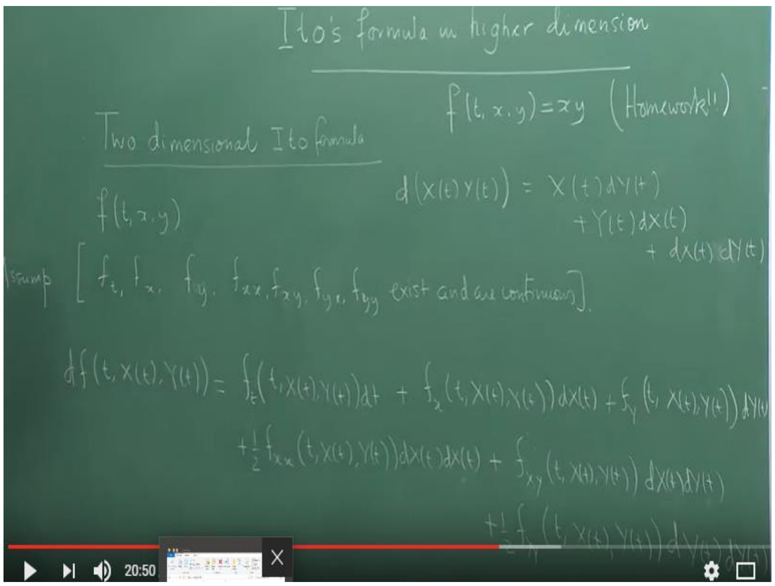

Now I give you a small homework. Thing is the following. Take a specific f txy equal to xy and then see what will you get. If you look at it, ftxy equal to xy gives you the feel of a product rule, that you are talking about x t into y t when you are talking about d of f t x you are talking about d of x t y t. You are talking about the product. So, what would the product be, how do the product rule look like. I would like you to finish that as a homework to prove the following.

So, you see there is always an additional term. So, it is not the standard calculus, so it is always there is an additional term. So, you can apply this formula, to this function to prove this and this is a homework I think all of you are, so even if you do not give the exam does not matter but you should really try out this with your own, to see that, you are developing a gradual feel for applying the Ito’s formula. Not that you really have to prove things etc., but just applying that.

So now I have 2 stock prices, I am analyzing two stocks. So, if I am analyzing two stocks, what should I look into. So, let us observe, let us just take, correlated stock prices. So, let us just observe or let us just say that, I have two stocks of this particular type of behavior which satisfies the following stochastic differential equation. So, they are Ito processes of this type.

So, this process only depends on the first Brownian motion, but now we will define a second stock price. This is an example, this is just example from Shreve, it is not that it is exactly everything has to follow like that. The second price, the second one, depends on not only one Brownian motion, but of course, this 1 minus rho square has to be positive, so rho has to lie between minus 1 and plus 1, with rho in minus 1 plus 1.

So, this rho is some sort of a correlation coefficient. When there is an exact correlation, rho is 1 then this becomes 0. So, it just depends on W1t and when there is rho is 0, there is no correlation between W 1 and W 2 t that W 1 and W 2 then this S 2 only depends on W 2. So, that is what it is trying to capture. So, it is so of course, you can have a more feel if you have a more knowledge about statistics.

Of course, so depending on the rho, you know whether the processes are connected or not. So here it seems to be that, this process is depended on W 1 t and W 2 t but, this W 1 W 2 there is when you write this vector, W t is equal to W 1 to W 2 to W t this, each of this W 1s they are independent processes. W 1 is independent from W 2. W 2 is independent from W t. So, here the important thing to remember that W 1 and W 2 would always remain. This W 1 t and W 2 t are independent processes. They are not correlated here okay.

So, you are getting the W t as W 1 to W t, that is independent process, but it does not mean that, but this process, this whole this process could be correlated to this, but they might not be independent. So, let me figure out okay, let me, if I write this as some other Brownian motion right, then this new Brownian motion, this whole Brownian motion, may not be independent from this, but just W 1 and W 2 t are independent.

Then if that is the case, let us put W3t is this, so I am making a new process, a new Brownian motion and because all other properties of the Brownian motion would be satisfied, because you just have linear sums does not matter linear per combination.

So, this is my new Brownian motion, so new process, again it is the Ito process, but it is a Brownian motion fine. Basically, I want to say that okay, I want to have W3t, this is nothing but rho W1t plus rho 1 minus rho square W2t. This is what I want. So, I had made this new process, new Brownian motion. So, this is the differential form of that okay.

So, what I will now show essentially, technically I am writing that this is a differential part, but one has to show that, see sorry sorry now, one has to show that, if I take W 3 t like this the d W 3 would actually be like this. I had just written it down, because that is what will come that, if W 3 t is chosen like this then d W 3 t would actually be this and there will be a correlation then between W 1 t and W 3 t. You can compute the covariance and that is why, we just start looking at how to do the stuff.

So, if you look at the covariance, sorry, look at dW3t into dW3t then, of course dW3t is this one basically. So, what I am trying to say is that, if you look at it. So, what I have done. So basically, I have now replaced this by dW3t. So, my S 2 now depend on some other Brownian motion which is a combination of W1 and W2.

So, here you have W3t, W3t if you take, it will come out to be dt and I want you to really do this calculation. It is just basically, if you look at it, it is nothing but rho square dt plus, 1 minus rho square dt. So, it turns out to be dt because this is the, if you do, if you do cross when a d W1, dW2t is 0 so that is why the cross variation becomes 0, so you have to take only their quadratic variation.

So, their quadratic variation is rho square d t plus 1 by 1 minus rho square d t. So, this turns out to be so the quadratic variation is t for this and hence here W0 is 0, Wt is greater than equal to 0. It is adapted, it can be adapted to the same filtration because if I have full knowledge of W1 and W2, I have full knowledge of W3 that can be adapted to the same filtration as W1 W2 the vector W and then W3 is satisfying this for all t. So, the quadratic variation accumulated up to time t is t and hence by Levy's theorem W3 is a Brownian motion.

So, though we had been telling that okay W3 would be a Brownian motion it is very intuitively clear that things can just be transferred. But you see to mathematically to be rigorous we have to apply Levy's theorem here to show that it is truly a Brownian motion.

So, that is why I have written the Levy's theorem up there, to show that it is truly a Brownian motion. So basically, what you have now is that, you have written dS2t is equal to alpha 2S2t plus sigma 2S2 dW3t. So are these processes now independent, when are the movement of stock price 1 and stock price 2 are independent of each other, they do not depend on the price of the other.

In fact, they are independent to the extent they are not independent because we will show you can show that there is positive correlation there is correlation the covariance between W3 and W1 is not 0.

Though their means are 0, but because covariance is expectation of xy minus exy ex into ey, you can show that the covariance, which I am not going to calculate actually, you can show that expectation of W1t into W3t is actually equal to rho t. So, you have a nonnegative for every t unless t is greater than equal to for all t greater than equal to 0 right. You have a nonnegative correlation coefficient. You have a non-0 covariance.

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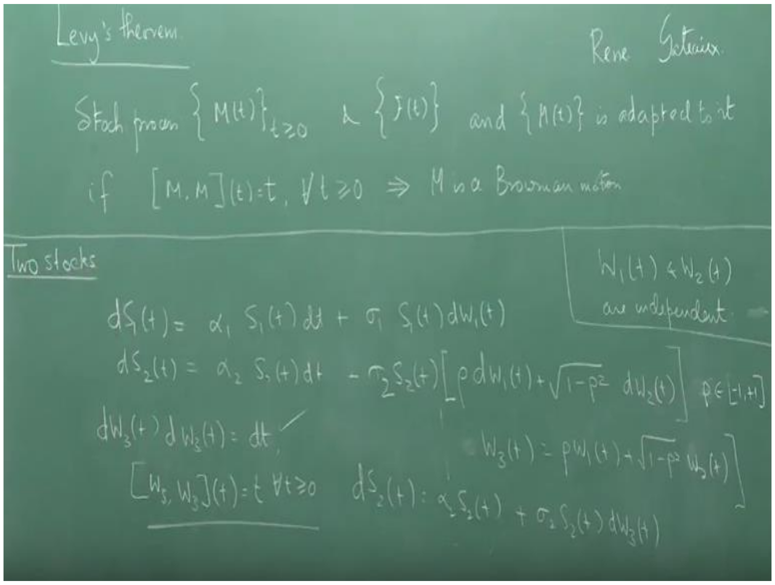

So, there is a relation between this and this. So, when you have two stocks you are essentially looking into a scenario where the stock price movements can be related to each other correlated to each other. It is not that one price is going somewhere another is going somewhere they are independent of each other.

They could be independent of each other if you possibly had just W2 here, then they are independent but once you involve say this sort of when I if both of them are multidimensional Ito processes like what we had done then stock price can get correlated. So, that is a much more difficult situation and in finance, we try that is, that is not, that is something is slightly difficult to study and we will study it when we will talk about finance separately, separate course in finance so okay.

So, thank you for listening to the third lecture and then we are ending culminating in the last two lectures which is on the Black Scholes equation for which most of you must be waiting and I had myself learnt all these things, in the Black Scholes equation, and I would in this lecture I would ask my TAs to come up and join me to at least they should be able one should be able to see who are their TAs. Please come up for a minute before we start the Black Scholes lecture. Come on.

So here is Prashanth Jha, who is a PhD student in mathematics, who is working in statistics actually and she is Poonam Kesarwani, who is also a PhD student in mathematics, working in optimisation theory and both seems to have some good idea about statistics.

Maybe at the end of the course, after fifth lecture they can say, their own opinion about how they liked or enjoyed the course and we can end that. So, at least the viewer should be able to see with whom they are interacting. So, that is, that is quite important. So, it is very important to see the person with whom you are interacting, then things are things get much better.

So, thank you, go ahead. So, we will start the fourth lecture after some time. Thank you.