Probability and Stochastics for finance
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Lecture - 17
An Application of Ito Integrals-I

So, I am going to now speak about the use of Ito calculus, to understand interest rate models. So, do not get into this issue of why this model was built like this, and why it was built like that. So, we are going to talk about the Vasicek's model. The Vasicek's model of, so this is, we are what, we are so this is what we are going to now discuss. So, this is again the second version of application of Ito's calculus.

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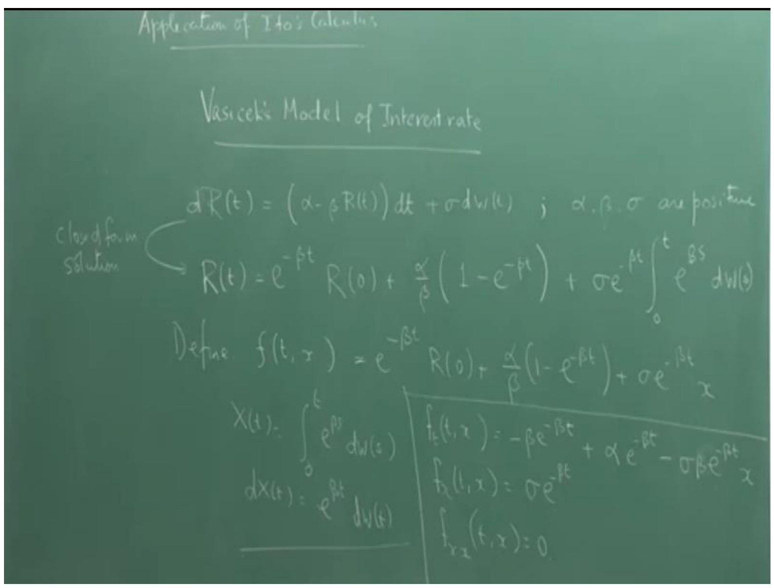

So here we will see two aspects of Ito's calculus. The first Vasicek define, the Rt to be an Ito process, which is governed by the stochastic differential equation. Of course, you know that these are only shorthand's but okay these are helpful to a certain extent. So, you observe that if I put Rt equal to alpha by beta, there will be no drift term. It will be completely purely a random term.

Now interestingly enough, if you assume that, we will assume naturally, that alpha beta gamma are positive. Let us note that, every stochastic differential equation, cannot be given a closed form solution. We will not go in to show how to get a closed form solution, but it cannot be given a closed form solution every time. But luckily in the Vasicek's case, you can give it a closed form solution and the closed form solution is given in the following way.

So, you will immediately see that it will tell you that this is an Ito process. So, do not think that okay this is something why this is outside the integral. This was anyway outside the integral always. The integral was this was into this integral. So, this is actually a, closed form solution. Our first step of the use of Ito calculus here would be to show that this is indeed this Ito process is indeed a solution of this.

The question would be that is, Ito calculus only helpful to show that something is a solution of this, but and I do not know how to solve it, but I am telling you that, many of these equations do not have a closed form solution and when we will go to the Cox-Rand-Ingersoll model, another model of interest rate. We will see that it does not have a solution, but Ito calculus would help us, even if it does not have a solution can compute its mean and variance.

Now the interesting part you will very soon see that we can show that Rt will satisfy a normal distribution and hence by doing so we will come to the following conclusion that with a mean 0 in fact this Rt is a random variable which can also take negative values because this is a normal distribution with mean 0. So, that something is not favorable you do not have negative interest rates. The interest rates have to be positive or 0.

So, but though it is a initially good model to understand the behavior of interest rate. Of course, I am not going to the detail of how this model was built and why it was built. That will take us too much of a field into finance. So, when we are going to talk about finance, in our next course, then we are really going to talk about these sorts of models right. That is something that we will build up. But anybody, who would like to do this course in finance, will have an access to these lectures which I gave you in the very basics.

So, here we are trying to see how Ito calculus can be applied, what are the techniques. For example, you know when we do Ito calculus, our first idea would be to know how to choose my ft a f or f tx. So, that choosing the function f is of fundamental importance to know how to apply Ito calculus. Here of course, you want to use this as some sort of your ft f tx.

So, how will you do it. So here, we define f tx as, basically here, we are bringing a process within a process. This is my, so I am getting a smaller Ito process. So, I am having Xt, so your shorthand d xt, this is the shorthand right. So, shorthand is very useful. It is like game playing so you can have some fun with it always.

Now on this, I would start applying the Ito's calculus. To do this, we must first know, so you know that, I need to have the notion of three derivatives. The derivative of f with respect to x with respect to t and the second derivative with respect to x both the times. So, partial derivative second derivative with respect to x del to f del x 2.

So, what is my f t x. So, f tx would have a larger say here, because we have terms in t, so this will become minus beta e to the power minus beta t plus alpha into e to the power minus beta t minus minus will cancel and give you a plus. Here you will have minus sigma e to the power minus sorry, beta e to the power minus beta t into x.

Now if you do f of x the derivative, partial derivative with respect to x all these parts would be 0 and we will simply have, sigma e to the power minus beta t and if you want to do it, f xx of course, it is 0 because there is no x term anymore here. So, you know one of the terms of the Ito's calculus goes away.

So, all these two classes, we are trying to see how Ito's calculus is useful. So, do not bother much about the financial implications of the model. That only would come in the next course, as why this model is important what is the issue etc., etc., etc.

So, the finance game, is a very different game, and here we are trying to learn the math associated with it, with examples coming from finance of course. So, here let me start writing the Ito's calculus. So hence by Ito's formula we have let us see what we have by Ito's formula.

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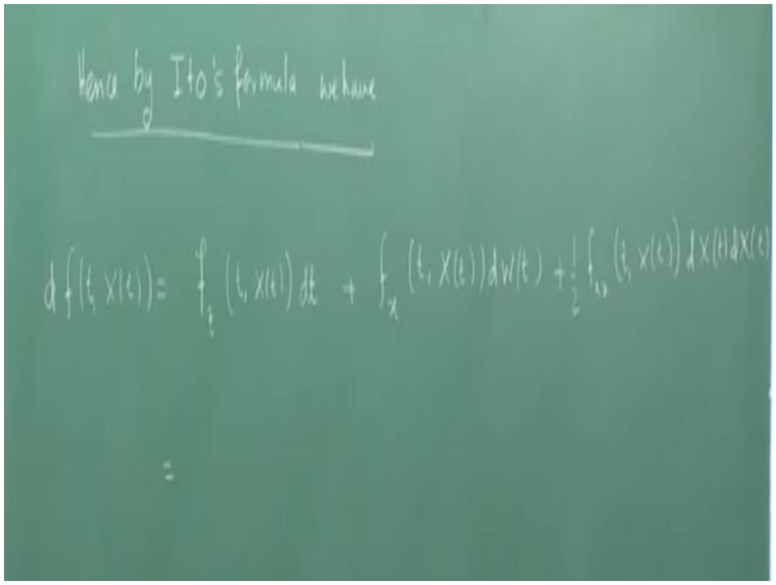

So, d of f t xt is equal to ft t xt dt plus that is now I am just writing the Ito's formula. F xt xt d Wt because you know only with this variable, this partial derivative where you have the d Wt term, plus half f x xt xt d xt d xt which is the quadratic variation term and you that this is 0. so it just goes there is nothing. So, the interesting part. which we have not told is that. all these equations that you write here. is all are in the form of almost everywhere.

These terms of Ito's inequality, what we are writing and we are not bothering to say, these are all almost everywhere type inequalities. When I am writing equality in a stochastic process setup, it is always almost everywhere type. There could be some omegas for which this does not hold because, we are taking limits and we are taking quadratic variations and all these things are all in almost in everywhere sense. So, all these inequalities it is very important.

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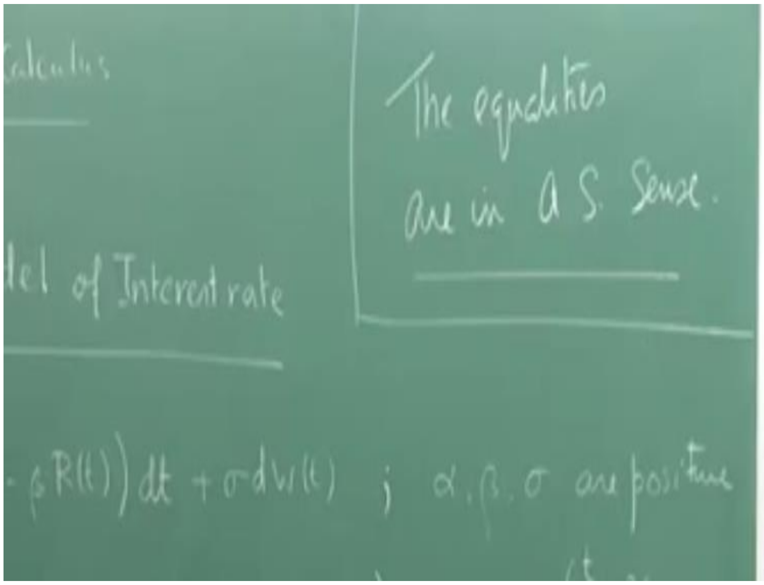

So, all inequalities are in almost everywhere sense sorry, equalities. The equalities are in almost everywhere sense. Please note this though it is very nice, so because of it only fails on a null event, we do not bother about it, we just write equality, so while doing just as a working tool, I can just write equality, but all equalities are in almost everywhere sense or it is same as they probably say almost surely sense, and this occurs with probability one and equality occurs with probability one.

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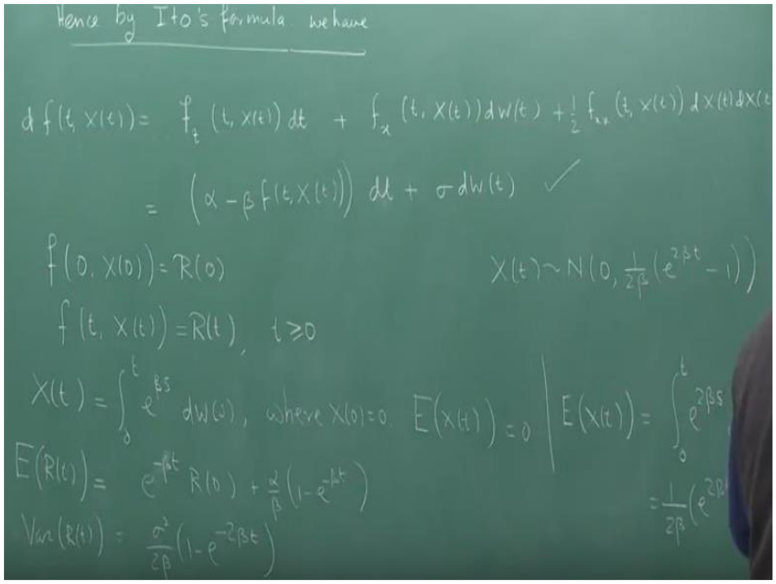

It is a random phenomenon. Do not think that it is a nonrandom phenomenon. So, then if I go and write down this and do the calculations here, what you will get is, alpha minus beta f this is 0 that part f t x t and you know what is this f x. So, once you write everything the whole thing once you put everything and do the simplifications you will get this formula.

I leave it for you to do the simplifications. I am not writing the simplifications down because that will take too much of time.

So, you see here you see this is exactly the same thing. So, this we have verified by Ito's calculus rule what is the that this is actually the closed form solution of this stochastic differential equation. So, let us put f0 x0 and f0 x0 is equal to R0. So, f0 x0 is r0. So, the initial condition of this process and f t xt xt is now this one so if you put f t xt this is Rt.

Actually, what happens ft xt has the same behavior. Essentially if you look at it if I put xt here, it is exactly your rt. So essentially, we are if you once we are trying to apply Ito's formula, we have to extract from the given Ito process, the form of the function. That is the key thing that you have to observe when we do the job.

Of course, there are certain issues. So, what about, of course you know that, what about this xt, what is the nature of this xt? xt is also Ito process. So of course, xt is a process where x0 is 0. So, xt is a process where x0 is 0 naturally. So, expectation of xt, what is expectation of xt is 0. So, what is expectation of Rt? Expectation of Rt, is nothing but sigma e to the power minus beta t expectation of x t right. So, over these constants. So, expectation of this plus expectation of this. This part is it will come out.

So, let us see what is the expectation of Rt. Now if you look at it very carefully. You can find also the variance of this. So, what would be the variance of this? Again, by the Ito isometry, the variance of this would be, which is equal to 1 by 2 beta e to the power 2 beta t minus 1.

So, what we have done, what we had shown, that this x t so W is following normal distribution right. So, xt, so scaling of a normal distribution also gives me a normal distribution. So, xt also follows a normal distribution with mean 0 and variance this. So, xt follows normal with mean 0 and variance. So, that shows from here this is just a constant so this shows that expectation of Rt is.

So, Rt is also normal distribution, with mean this and variance which I am not writing in detail. You can calculate out this thing. So basically, when you take the variance, the variance of this constant part is 0. You do not have variance of a constant, where the constant term does not vary. So, your Rt also follows normal distribution with mean this and okay.

So, what is the lesson here. The lesson is that if this is following normal distribution, and this might give me negative values, so what it cannot have negative values. So how do I modify this? The question is how do I modify this.

This modification was done by Cox-Ross-Ingersoll model. To study the Cox-Ross-Ingersoll model, we will see they have just changed the drift term slightly, sorry, not Cox-Ross-Ingersoll but Cox-Ingersoll-Rand model. I will just CIR, this is called the CIR model, or the Cox-Ingersoll-Rand Ross model CIR. These are all big names in financial economics. Cox-Ingersoll-Ross CIR model.

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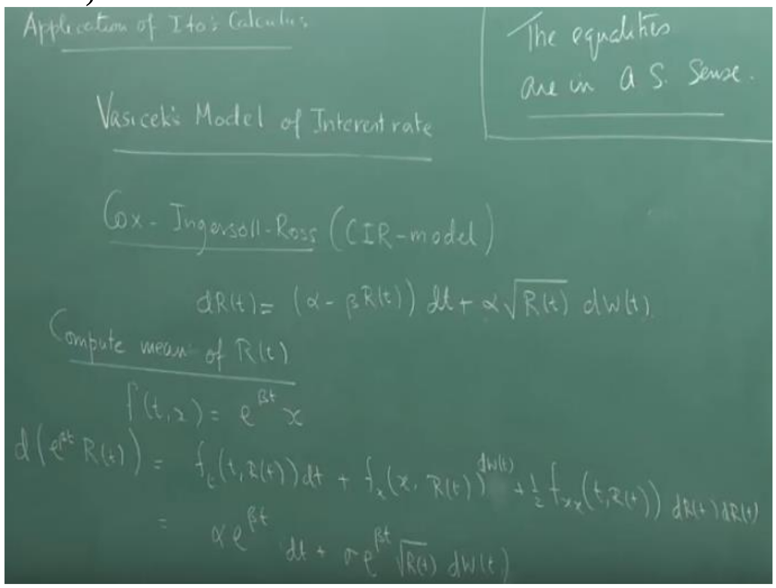

So, the CIR model, they assume that it should be of this form. The interesting part is that this model does not have this drawback of this becoming nonnegative. Rt would never become nonnegative, but this is the example of a stochastic differential equation, which does not have a closed form solution. Even if does not have a closed form solution, you can still get some idea of its mean and variance.

If you want to compute that okay what sort of distribution Rt satisfies. Then it will be quite a difficult thing to compute the distribution, but you can at least compute its important parameters, mean and variance.

So, let us just give a hint, as how to compute mean of Rt and you will see how the Ito's calculus would be helpful here. See Ito's calculus is not just to verify some given solution our SDE. It is helpful in many-many states. So, to do this, you have to choose. Now this choice of f tx it must look very arbitrary to you, but it is done because you have done some work. It is done from guess and test. So, if you always bring in exponential things because, you want to have things nonnegative. So, exponential thing always comes up usually.

So, your xt is your Rt. So, your so d of e to the power beta t Rt. So, that is again ft t rt dt plus fx x Rt dt plus half f x x t Rt d Rt d Rt. So, I will ask you to calculate this thing through okay. If you calculate this thing, as you can have some fun computing d Rt d Rt because, it will become alpha square R t d t. So, if you do this calculation this will turn out to be the final form would be of this form. Alpha e to the power beta t dt plus sigma e to the power beta t root Rt d Wt sorry here I will have d Wt.

So, when you write put in all the values, you know what is e tx ft is beta e to the power beta tx. So, you know what it is and what is x. It is e to the power beta t. What is f xx. F xx is, what would be f xx here. Here f xx would be 0. So, once you know that fact, so you do not have to bother about the quadratic variation, and then you write down everything, alpha e to the power beta t for example here ft would be beta e to the power beta t x which is Rt.

So, it will be alpha beta e to the power beta t R t into d t. So, write the full form Rt again this one. So, once you write, sorry, you do not have to write the full form R t. So here you have f x so x e to the power beta t x e to the power beta t x means 1 so f t R t e to the power beta t into d omega t. So, once you write this you will get back this formula. You see we are just essentially getting that.

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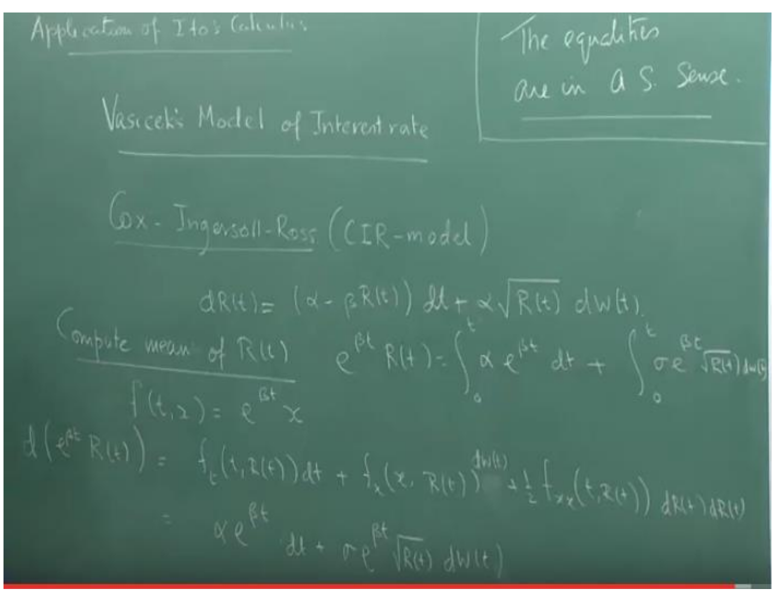

So, if you now know this formula, I would similarly ask you. You can now integrate both sides. So, you can integrate both sides take off the differential and you can write e to the power beta t R t is, what you have alpha e to the power beta t dt 0 to t plus 0 to t sigma e to the power bt root R t d Wt.

So, once you know this you can immediately use your other ideas about Ito's integral and this is a simple case. To really figure out, what is the mean, you just take the beta t on the other side compute this out and then take them in. So, I will leave that to you as homework. You should try it out something yourself. If you do not try it out yourself things might not always be fine.

So, with this I end my talk to give you some idea about how things can be used. Please check up these calculations. Do not completely rely on what I have written at the end. Just checkup that you come to that conclusion.

So, thank you very much.