Probability and Stochastics for finance
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Lecture - 16
Ito Integrals in Higher Dimension

So we are in the last week of our course. So we will today start about applications of the Ito's calculus. So you have learnt about Ito's calculus now you should learn about applications right? How can I do something with that Ito's calculus. One of the most important issues in finance is to express most financial processes as Ito processes. So one wants to express financial processes, all financial processes are obviously stochastic processes, processes as Ito processes.

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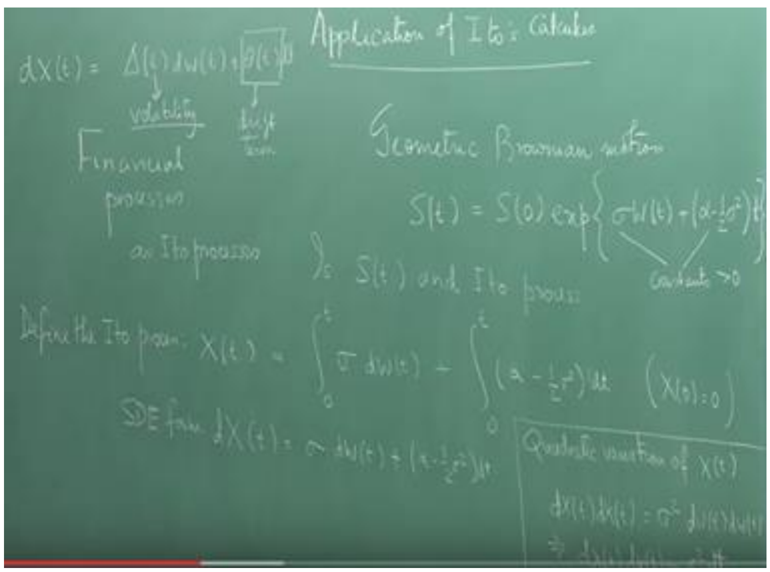

So Ito processes means you have to write down it everything in for more stochastic differential equation that is we have seen that the Ito process can be written down in this SDE shorthand as say what sort of symbol that we gave del t d W t plus theta t d t. So this term the coefficient of the d t term in the stochastic differential equation language is called the drift term essentially term similar to that in finance and this is called the volatility term which measures the unpredictability of the whole thing.

So the first question that one arises that can I model the movement of the prices of a stock as a Ito process. I have to model that. If I can model that then I can bring in the power of Ito calculus to draw conclusions in finance. So in general you have to understand just having Brownian motion would not do. You cannot expect to model Brownian make Brownian model as the model for your stock prices though they may look similarly the jiggling type.

But the issue here is that you have to know that the stock price always has to remain non negative. So that brings us to the notion of geometric Brownian motion. So geometric Brownian motion is described as follows. Basically the stock price at time t is the stock price at time 0 which is non random which is fixed which you know because you are at time 0.

E to the power of course the Brownian motion captures the Brownian motion captures the randomness. The question is this process S t satisfying an Ito process. Is S t an Ito process? So we will ask this question to ourselves. Is S t an Ito process? How do we go about doing anything about it right. So let us define an Ito process of this form. X t so define an Ito process. So define the Ito process.

So we are constructing a Ito process. Our aim would be to show that this is the Ito process but let us construct the Ito process. Suppose X t which is which we write as 0 to t sigma these are constant this sigma alpha these are all positive constant okay. This is something we need to keep in mind. So this sigma and alpha are positive constants. These are constants sigma and alpha. Sigma is called the volatility of course. So it is sigma d W t plus alpha minus half sigma square t sorry d t. This is what I have.

Let me define this Ito process. So if I write down this Ito process in the SDE form then we get d of X t so here I have not taken x 0, I have taken x 0 is 0 right. So basically, I am taking you must ask me what about the term X 0 right because in Ito process there is a time term X 0. So here we are basically assuming that X 0 is 0. So then I can write this as d x t is equal to sigma d W t plus alpha minus half sigma square d t. This is the SDE form.

Now if you look at the quadratic variation of this, so the quadratic variation of the process X the quadratic variation note that this process S t is equal to S 0 like this is called log Brownian motion or the geometric Brownian motion. You see because of the exponential term this of course we are guarantying that a price can never be negative.

So if you want to write the shorthand form then keeping in mind the formulas that we have given d W t d W t is d t d W t d t is 0 d t d t is 0 you can then write this as sigma square d W t d W t implying that d x t d x t is equal to sigma square d t. So basically this Ito process has the same same sort of it is also path independent quadratic variation. So x x t is sigma square t.

So this is the quadratic variation and if I look at it how my how do I look at my price right asset price. Now look at an asset price S t.

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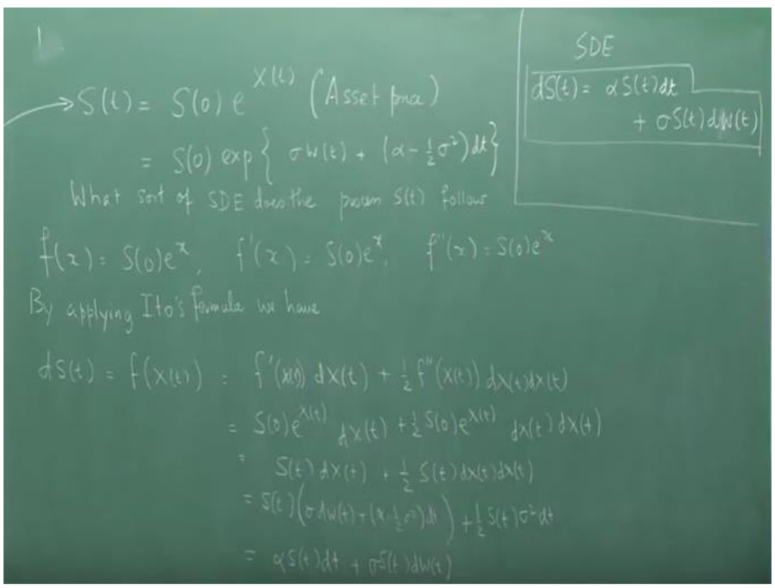

Suppose I define the asset price let the asset price we have written it in this form. Actually if you look at it I will define the asset price as now if I write like this we will come to the asset price. If I write like this what do I get? Let me write it down more carefully. I will tell you what do I get.

Let me define the asset price as S 0 into e to the power X t okay. So S 0 see you might wonder why I am writing why I am not why I am writing in the integral form because this is nothing but if you look at it this X t is actually sigma omega t + alpha - half sigma square t.

Actually X t is this process. Because if this sigma is constant of course you can start working with sigma and alpha and S where sigma and alpha not constant but since they are constant I can write it like this. So I can write this thing as exponential sigma W t minus sorry plus alpha minus half sigma square d t.

So though I wrote a Ito process first just like the way I should write a Ito process but noting that these functions are constant I can now write from here this fact. See integral 0 to t over d W t integral of 1 if you just go and use the of course do not think that 0 to t d W t is W t just by because this is a differential no. You have to apply the Ito's formula to come to this conclusion okay.

So that is the way you have written it. Now let us see what sort of stochastic differential equation this will follow S t will follow. So this is the definition of an asset price. So I want to I know that there is a volatility term associated with the Brownian motion and some drift term. So we will talk about it when we explain things in fiancé in our next course in much more detail but assume that this is called the drift term.

And now my question is what sort of SDE does the what sort of SDE does the process S t follow. Now to do this we will have to apply Ito's calculus. Now let us take f x is equal to S 0 into e to the power x. So we do not have the t here X only t is not separately taken or f t x. So f dash x is S 0 into e to the power x and so is f double dash x by applying Ito's formula and if I apply Ito's formula what will have what will have we have the following.

Let us see what we have. We have d s t is nothing but f of x is S 0 e to the power f of x t so it is f of x t which by Ito's formula is f dash X d X t + half f double dash d of X t not d of X t sorry d of X t half f double dash X t d X t d X t. So what is this? This is s 0 d to the power x t and into d X t + half S 0 into e to the power X t d X t d X t.

So let us observe the next step. This thing is your S t. So S t into d x t plus half S t into d X t d X t. So d X t into d X t jo hai this is sigma square d t. So we will write again S t. Now we will write down this whole thing. So I am writing now d X t. So d X t the S d form of X t is sigma d W t plus alpha minus half sigma square d t. Now here d x t d x t is sigma square d t that we have already got. So we know that I can now replace this by sigma square d t.

So S t sigma square d t sigma square d t cancels so to give you alpha into S t d t plus sigma into S t d W t. So the price process of a asset so if you look at this this is exactly the geometric Brownian motion. So the geometric Brownian motion is following the price process of the stochastic differential equation just like so it is a Ito process basically. Alpha S t d t the drift term plus the volatility term sorry.

So this is the stochastic differential equation followed by a price process. So what so see the clever way we have applied the Ito calculus. We have seen this thing. So this function, this thing we have considered as X t the process. So in general if you write a Ito process you have to write it down integral but we know that this is constant.

So this is so if I do the integration this is what I will get so basically this is what is X t. This is what I have taken as X t. So then so from here then from here we have found that X t is expressed in this form and then by applying Ito's formula you come and show that the geometric Brownian motion actually satisfies the following stochastic differential equation and in all the books and papers etc.,

You would see in finance this is the form of the stock price movement which is usually taken if you do not add things like jumps and all those things (())17:36 reality a bit more better in the sense that there could be a sudden jump in the prices because of some problem in the market or sudden drop in the prices.

So this is 1 aspect. So we will go to some other application. I would expect you to work it out yourself at home when you take this as alpha t and this as sorry this as sigma t and this as alpha t. This is of course sigma square t. So when they are time dependent then what sort of thing a geometric Brownian motion what sort of a process would the price process look like.

If you have a random volatility and random drift term what sort of thing it would look like okay. So this is our first application of the Ito's calculus which shows that our stock price actually satisfies the Ito process satisfies this stochastic differential equation. Not let us look at some more 1 or 2 more applications. So this series of lectures the 5 lectures that we are supposed to give.

This week we will end with the computation of the or rather derivation of the partial differential equation whose solution would give us the Black Scholes price of a European call option which we will come very soon but now suppose I define an Ito integral.

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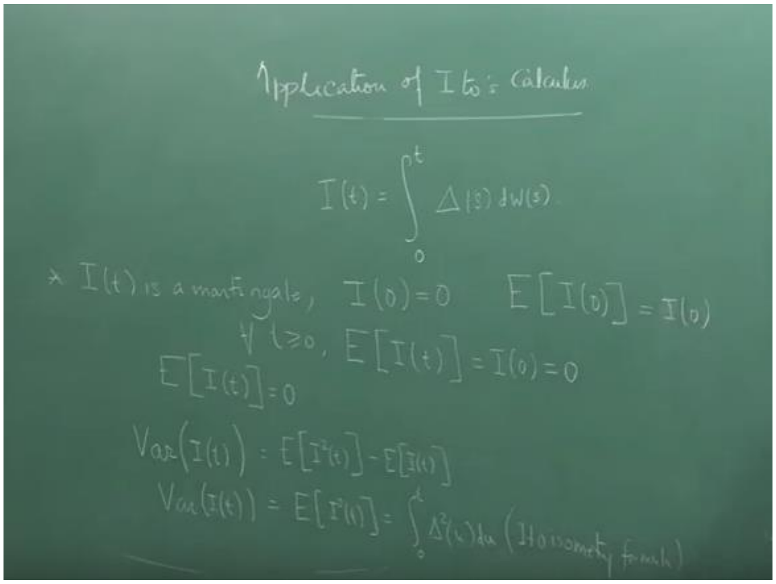

So once I define an Ito integral then what is my mean and variance? So these ideas would be required as we do more of our studies. So how do you recognize that what is the mean of a Ito integral. The key notion to remember is that the Ito integral I t is a Martingale. Martingale means the first random variable of the process or X 0 whatever be its mean that the same mean would be for all the other remaining random variables.

Now I 0 is equal to 0 because you put equal to 0 it is 0. I 0 is 0. So the expected value of I 0 and we know for any t you have this and this is 0 because expected value of I t I 0 is I 0 because this is constant it is 0 so is 0. So expected value of I t of a Ito integral is 0. Now the question would be what is the variance of the Ito integral.

The variance of the Ito integral is nothing but expectation of I square t minus expectation of I t and expectation of I t is 0 so basically variance of I t is equal to expectation of I square t and this by the Ito isometry formula this is called this is by the Ito isometry formula. Now these ideas would be helpful as we do our next applications. We will talk about how to model interest rates in the next class.

So we will just introduce to you what are interest rates. We will not get into tremendous amount of details about finance because there is lot of issues when discussing very basic concepts of finance. So what is the interest rate?

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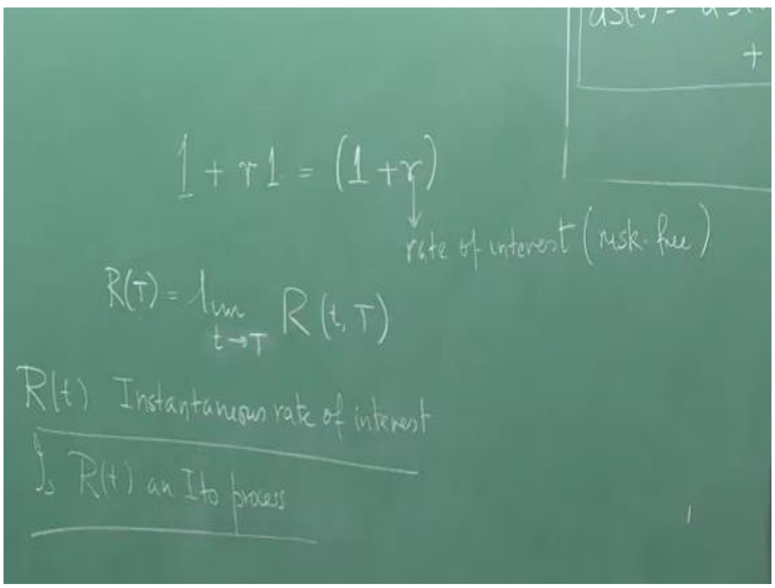

Suppose I have 1 rupee, I invest in a bank and the bank tells me that after time t you are going to after time small t you are going to get back or some money with an interest rate R. Which means he is going to pay me after time t not only rupees 1 but the interest rate which is the percentage rupees R into 1 so not only the principle but the fraction of my principle would be added to my original money.

So this is what I will get 1+r. So after given period I will get 1+r money. So this r this fraction is called the rate of interest. Usually what happens is that banks suppose you have kept money in a bank in a fixed deposit or you have taken a loan. So you are paying back money to the bank. So usually for a given period say between time small t to capital T for a very small period the bank interest rates are usually kept fixed.

This is called the risk period of interest. This rate of interest is called risk free because I guarantee you that if you take money from me now after say 1 month I will give you 1 plus if you give me 1 rupee I will give you 1 plus r rupee. So that is that is what is a for example if I say if you give me 100 rupees now I will give you back after say 1 month 110 rupees. So my interest rate is 1 by 10th.

So r is 1 by 10th. So that sort of a idea. So the interest rate usually in a bank is kept at a hold between it is held for very short time. Usually the interest rate keeps on changing in the bank. So in general just like in physics we want to know what is the instantaneous velocity is very difficult a concept. It is not so easy to understand that though we do lot of problems and do a lot of things with it.

But instantaneous velocity is one of the most deepest concepts. It is not so easy to fathom it immediately. Similarly we are talking about we will talk about instantaneous interest rates, what is the interest rate exactly at time t. So just like we did in physics that we look at the difference of the distance travelled by difference of the time taken and take the ratio and then take the limit as the time goes towards 0 and that is what we call instantaneous velocity in mechanics in physics.

Similarly, we call or rather I should say that is called the instantaneous rate of velocity at the time sorry instantaneous rate of interest at the time capital T. So here R t or in general R t is a instantaneous rate of interest or called sometimes called short-term interest, instantaneous rate of interest. So but this rate of interest is random. It changes.

This itself is random. You cannot say that today's interest rate is this next day the interest rate will be as per as you have decided it will it will be just based on the market, it will change. So R t is called the instantaneous rate of interest. Our next question would be can this R t this instantaneous rate of interest which is now essentially a stochastic process. It is a stochastic process is we are asking the question is R t an Ito process? Is R t an Ito process.

So with this we end our first lecture of the last week and in the next class we are going to answer whether R t is an Ito process or not and what sort of models had been there for R t and how Ito's calculus help us to understand those models.