Probability and Stochastics for finance
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Lecture - 15
Ito Calculus-II

So this is Ito calculus 2 the last version of Ito calculus that is needed for finance and that ends your third week's lecture. So we have 1 more week to go, 5 more lectures to go okay and then first let us just go back and let us see how Ito's formula can help you to compute the Ito integral. So we have computed the Ito integral if you remember in our last to last class we have actually computed this integral and we have written this as half.

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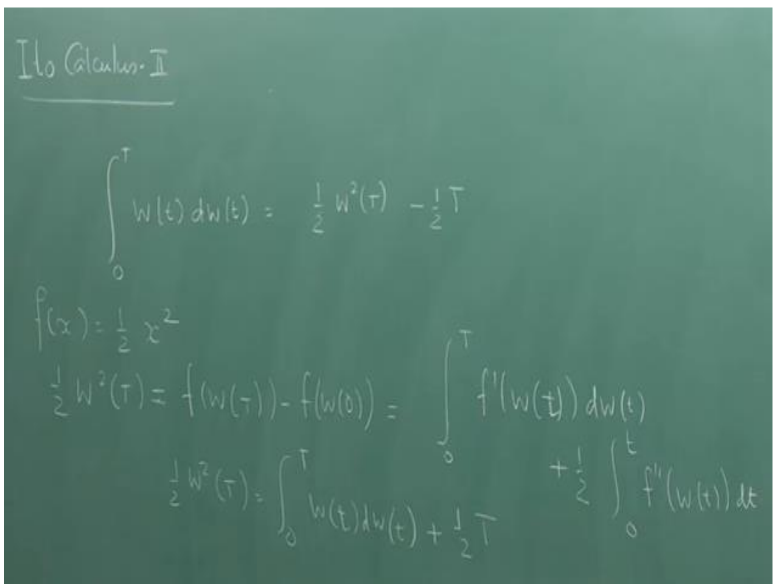

But you know this can come out just right out of the Ito's formula because if you take f x to be equal to half x square then half of W square T is f of W T - f of W 0 which is 0 and f of 0 is 0 so when a here x square is and this by our formula should be 0 to t f dash W T d W t plus half of 0 to t f double dash W T d t. So what is happening here. So we will just we know that this is nothing but W T and we know this is 1.

So we will have half W square T is equal to 0 to T W t d W t plus half of 0 to this 1 so this half of T. So you immediately know that if I take bring it this side I get this formula. So it comes out of Ito's formula that we have studied.So now we are going to define something called a Ito process. We have just mentioned in the last class what an Ito process would look like.

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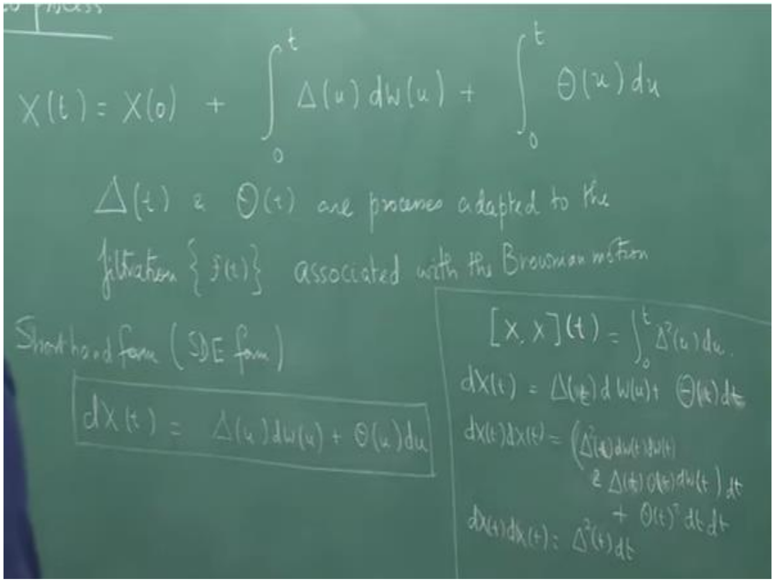

So an Ito process would like the following. So an Ito process X(t) a stochastic process consists of X(0) which is the 0th value of this which is non random plus an Ito integral so you have a stochastic integral and you have an ordinary integral. So this will be non random a number. This will be 0 I mean if you have W. This will be a number plus a random variable so basically it represents a stochastic process so x t is a stochastic process.

Where theta t and sorry delta t and theta t are processes of stochastic we are not writing the term stochastic process every time just write process are processes adapted to the filtration f t associated with the Brownian motion W t. They are adapted to the filtration f t associated with the Brownian motion okay. Now you know you can write this in a shorthand form. The shorthand form is let me write shorthand form or the stochastic differential equation form.

SDE form a shorthand form or the SDE form would be of this type. You can write this as d(xt) is equal to delta u d W plus theta u d u. Most of the models that you will use in finance would be represented through these sorts of shorthands. But please understand that this is nothing but a shorthand to this because we essentially talk about Ito integrals. We do not talk about differentials of Brownian motion okay.

One of the key formula that one now needs to talk about what is the quadratic variation of this process? So in this Ito world or the stochastic world quadratic variation is the major player. So, you really have to talk about quadratic variation. What is the quadratic variation of this process? That is something that we should know. So we will take 2 approaches to know about this quadratic process.

First, I will give you the easy approach. Then I will give you the actual approach. So let me give you the easy approach first. So we claimed this so we write x x t and let us see what we do. Let us put the magic. The magic is we write the short form. Then you do d x t and you use these things that we have written earlier the last class. So what do you have if you do this?

You have delta square u d W t d W t plus delta u theta u d W sorry d W t here sorry t here t t d W t plus so basically it is 2 delta t theta t d W t just you are doing squaring a plus b the whole square and then you are writing this as theta t square d t d t. So this d W t d W t this this this and you know d W t into d t is 0 cross variation is 0.

Second variation of an ordinary function is 0 and but a second variation of quadratic function is t. So you have d x t d x t is equal to (()) 09:13 square t d t and this is nothing but 0 to t so this is immediately so this is nothing but the short form of 0 to t (()) 09:25 square u d u. Okay I could possibly go by the more deeper thing.
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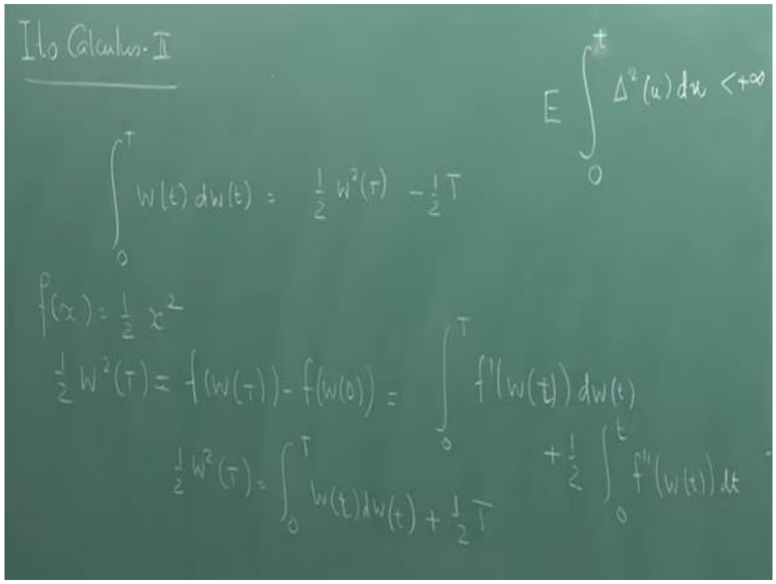

Always I know that when you are talking about Ito integral of a general function you should always assume sorry this is always little less than plus infinity because the expected value of the you know of this integral of the basically L2 norm of this integral this is a process so it will depend on t so this will also become a process. So the expected value of that random variable depending on t has to be strictly less than infinity.

That is the requirement because that requirement will actually help us write that the difference between the simple process and the continuous process when a delta n minus delta t whole square 0 to the expectation that limit would go to 0. So that sort of thing is there. So we do not get into all these things. We will assume that all these requirements are taken care when we are talking about this Ito business.

Suppose we are now going to write down formally this process. This is the shorthand process. You might say this is some sort of a backhand stuff. So maybe you will enjoy doing this backhand stuff lot more later on but we will do this backhand stuff later on. We will not get into writing too many things but if I do the whole thing then what I should do. Let me write down I t the Ito integral as and R t.

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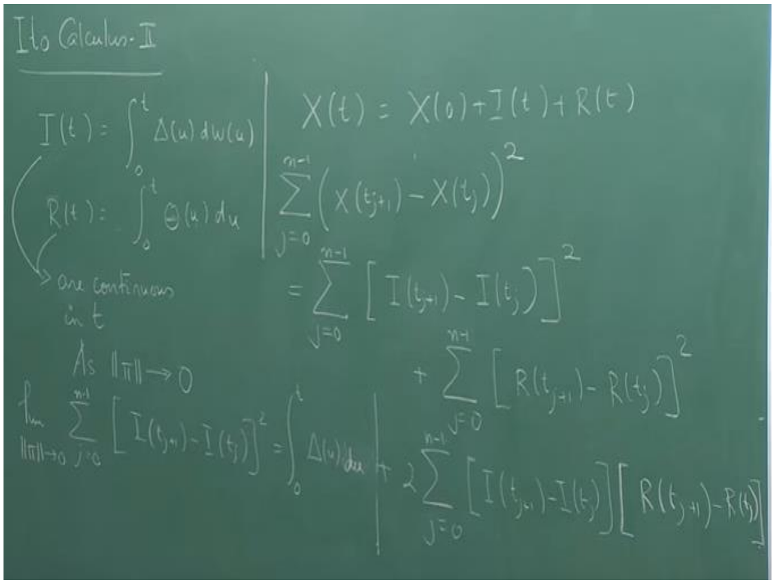

So if this processes are continuous in terms of u then from simple calculus from basic calculus you know both of these not these this is already separately known to be continuous but this you know would be continuous. So I t and R t are continuous functions in t. So these both are continuous and then you can write the Ito process X t as X0 plus I t plus R t. So if you are going to write about the quadratic variation of this thing then you are going to actually write I will write in more detail possibly.

You are essentially going to compute the following. The sample quadratic differences or which I should I do not know I think this is called just give me a second I will just tell you the name I tend to forget this name. It is called the sample quadratic variation depending on the path. So this I will write as x of t j+1 - x t j where the partition is as before. I am not writing the partition every time because that is boring to write down the same thing over and over and over again so now you know what does this mean.

This means X0 plus I t I t j+1 plus R t j+1. So X0 gets cancelled. So essentially when you expand it a whole square you essentially get and so now this is just this one. Now if I want to do something here what would I get? That is the whole point. See this if I as and this will be the quadratic variation of the Ito integral and that would give you this formula. So this thing so limit pi going to 0 nomination j equal to 0 to n-1 I t j+1 - I t j whole square actually gives you.

The so the Ito quadratic variation of the Ito process is nothing but the quadratic variation associated with the Ito integral. That is that is the key idea. So sorry (()) 16:01 d u. So that that is exactly so this is nothing but the quadratic variation. So what would be these parts. These parts would actually be 0. That is something you really have to just I will prove for this part and then you can prove it similarly for this part by looking at the continuity of the processes I t and R t.

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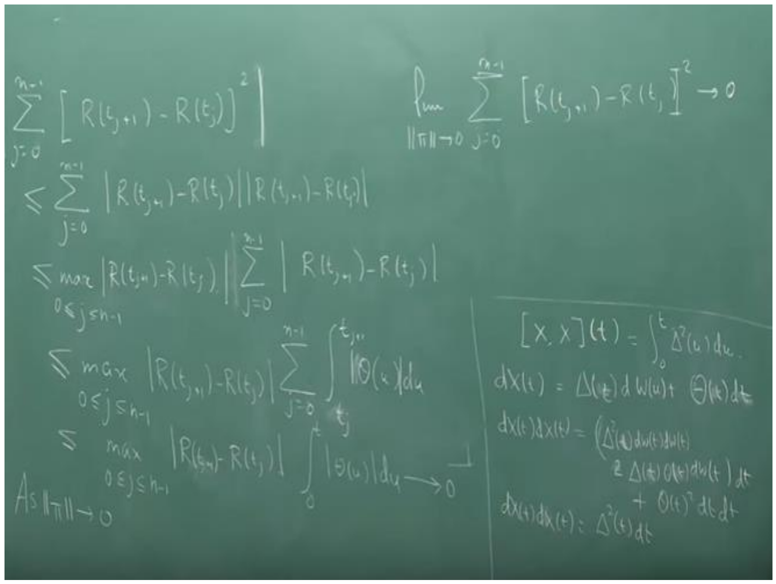

So let me just prove for this part. So what you can do that this value rather than the absolute value of this term I think there is no need of talking about absolute value because this itself okay does not matter. So I can write that this is less than equal to because this is a continuous function so this difference would have a maximum value of what the possible j s.

There will be finite number of such differences and hence I can write this to be less than equal to this thing into this summation this. Summation this actually means I will write the whole thing as this one the summation n-1 sum this sum goes out and you have sum of R t j+1 - r t j. So then I can write this term as 0 to t j to j+1 t j to t j+1 theta d d t or theta u d u. So which I can write again as because mod of the integral is less than the integral of the mod.

Using that idea a very simple fact about integration theory I can write this as and let me look at this part so I will write this as summation j is equal to 0 to n-1 0 to t theta u d u sorry t j to t j+1 mod theta u d u. So that is less than equal to max of and we have assumed that let us assume that this is always finite. Then as norm pi goes to 0 sorry here I am making a mistake as the norm of the partitions goes to 0 so the distance between t j and t j+1 vanishes comes down very significantly.

This mod goes to 0 but this is the finite number so this will be finally as goes to 0. This quantity whole quantity will go to 0 which will immediately mean that the norm of this whole square goes to 0 which will mean that the limit of this as norm pi goes to 0 is 0. So finally we will get that limit of norm pi going to 0.

When you take the third term out you take the I t plus I t j+1 - I t j out and do the same thing. How do you integrate if suppose you are given a adapted process, process is adapted to a filtration. How will you integrate it with respect to a Ito process means we are asking?

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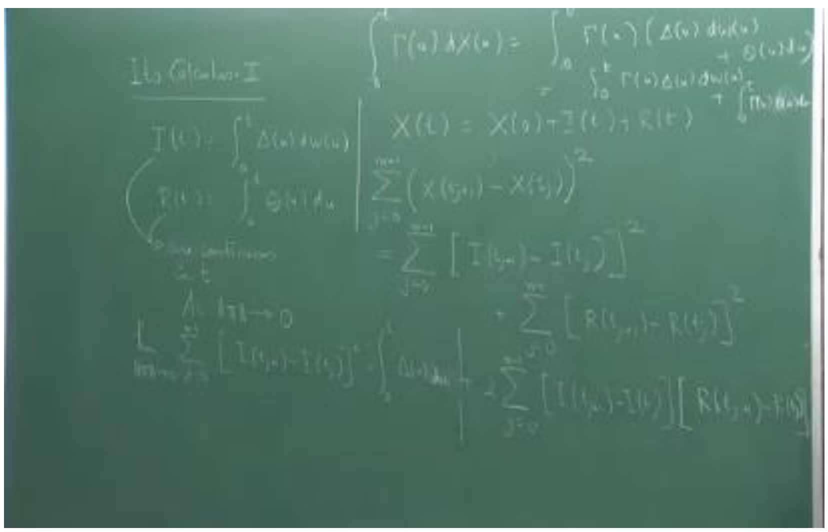

Suppose I give you a process what is the meaning of this? This simply means instead of d x u say 0 to t so instead of what you do is 0 to t you know what is d x u it is gamma u delta u d W plus theta u d u. That is it.

So if you want to take a process adapted to the filtration of the stochastic process W t there is a Brownian motion then the integration with respect to an Ito process is nothing but this 0 to t gamma u delta u d W plus 0 to t theta u d u sorry theta u gamma u d u that is a mistake. Theta u gamma u d u, u theta u.

Now of course the question comes what if I then start talking about the Ito formula for an Ito process. Let us see, what I can do. What is the Ito formula for an Ito process. That is what we so I am just going to write the formula not write the details. So Ito formula for an Ito process. With this we will end our discussion on the Ito calculus and then show you applications in the next coming classes and then end up with the Black Scholes formula. That which many people must be wanting to see.

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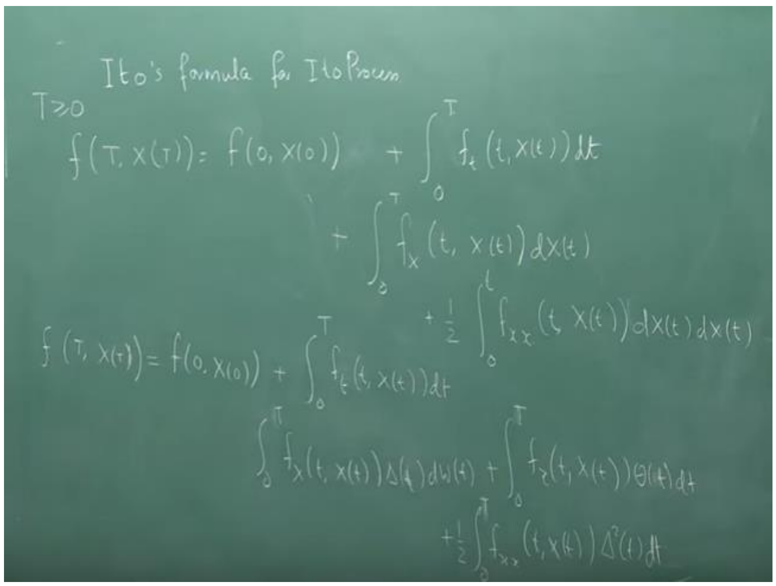

Ito process Ito's formula for the Ito process. So assume you take the a f with the same sort of assumptions like the last class f x and f t are continuous and f x x are all continuous functions the partial derivative and then Ito's formula for Ito process. Once we know this see this is just a repetition of what we have done.

The key formula is that you should know the whole thing is that the the key factor is that you just should remember this and also remember that most models in finance are Ito processes Ito's formula for Ito process. So in the Ito's formula for Ito process it is just same thing. So for any t greater than equal to 0 if t is greater than equal to 0 then same thing. Now instead of W t it is X t here observe that.

In the same formula I am just replacing. Instead of d W t I will now have d X t. Now I had d t in the other case. My d W t the quadratic variation of d W t d W t d W t which became d t but here I will have d X X t which is nothing but d x t d x t okay. So basically I can write this instead of writing like this I can write this as d X t d X t.

Now you know you can still simplify the formula by knowing how to putting the stuff of the Ito process. Do not worry, examples will now flow in as we will show that so many things in finance are actually Ito processes. You can you can model as Ito process sorry no d t here that is a mistake plus.

So here I will break it up into 2 parts in the same way. So I will have 0 to t f x t x t delta t d W t plus 0 to t sorry 0 to capital T f x t x t theta u theta t d t plus half of 0 to t f x x t 0 to capital T t x x t t f x x t x t and d x t d x t is nothing but delta square t d t.

You see so this is the Ito's formula for the Ito process and with this we have covered a huge ground of stochastic calculus. Of course we have not been completely rigorous because if you need to be completely rigorous then you need lot of lectures to do it step by step and be very very rigorous at every step.

So we have not gone into rigour but just had some fun but at the end getting the formula that you would require to actually study finance. So with this I will end today's lecture. Thank you very much and we will start our next lecture very soon which is the last week of the course.