Probability and Stochastics for finance
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Lecture - 14
Ito Calculus-1

Today we are going to learn something about Ito calculus, that is how do you write, a Taylor expansion for Ito's integral. Now you see quadratic variation will always add some extra term.

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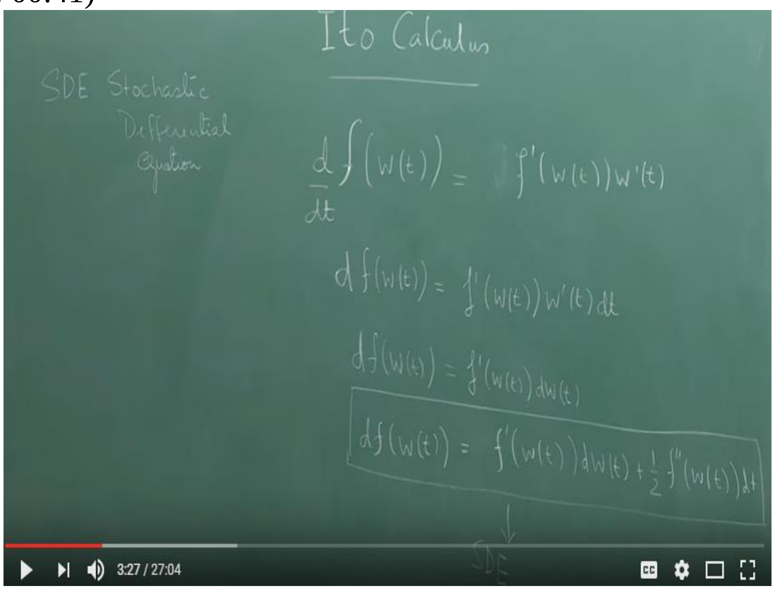

For example, for the moment, just think, that you have function f Wt and assume that Wt is differentiable, which you know that it is not, but just for the heck of it, so just to gain some insight. So, if you take a derivative it is exactly equal to, if you use the chain rule then it will be f dash of Wt into W dash t okay.

Now you know that, this is not really true, you cannot have W dash t. So, you can write d of f Wt is f dash Wt into W dash t dt, if you go by the standard way of doing differential calculus, this actually symbolizes the differential of t and so you will write. Though we know that we have been able to give some meaning to d Wt, but we really have not, this is not really the job.

But as we will show that, if we can give meaning to this, because we have given some meaning to this d Wt, in actual case this expression does not end here, but a term is added because of the quadratic variation. So, in the Ito's sense, this actually means, assuming that f is nice enough to be twice continuously differentiable, this is what you will get. Instead of getting this, which you get, if you are thinking in terms of standard calculus and this is what you get as a stochastic calculus.

This equation is often called a stochastic differential equation or SDE. So, I will write down what is the full form of SDE. It is called the stochastic differential equation. Now I would just want you to recollect a few facts.

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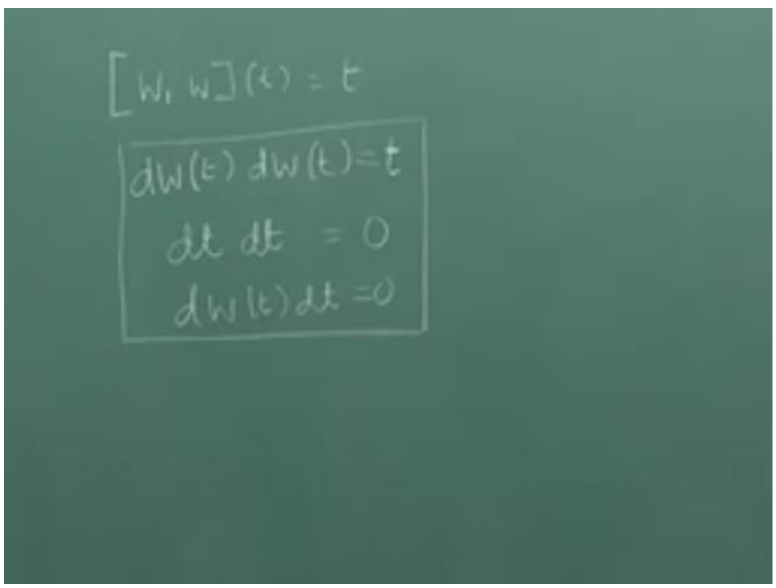

One of them you know that, the quadratic variation of Brownian motion is path independent. A shorthand short form of writing this is, so this is the actual process, but we are writing it in a short manner this is a shorthand. Do not think that do not immediately try to attach meaning to this. So, this is just a way of doing the math. It is not.

Similarly, if you take d t time d t quadratic variation of t it been a normal function you know that will be 0 and then if you take a cross variation between Wt and dt that is dt t Wt into dt that is also 0. So, this is a basic shorthand that you should remember when you are going to study finance or do derivative pricing. This is a very basic shorthand, very basic shorthand.

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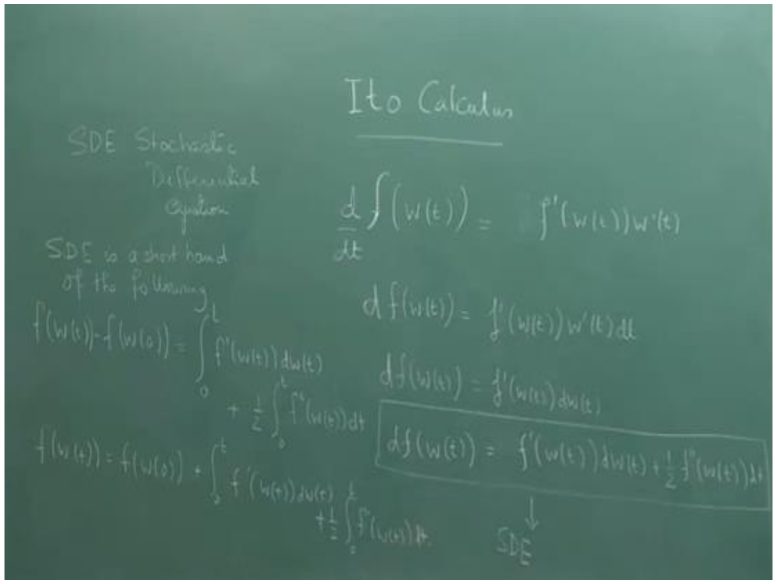

Now this SDE, what you see here, is actually a shorthand. This SDE is a shorthand of the following. Suppose, I integrate from 0 to capital T, I can integrate it out, if you think in terms of standard integration, then it is a shorthand of the following, or 0 to small t, so this one this SDE is a shorthand of f Wt - f W0 which is essentially giving you this, is equal to integral 0 to t f dash Wt d Wt which an Ito integral for which we have clearly shown how to think about it, plus this additional normal integral. So, this becomes an Ito's expansion, Ito's calculus.

So basically, you can rewrite it as, f of Wt is equal to f of W0 is equal to sorry, is equal to W0 plus 0 to t f dash Wt d Wt plus half of 0 to t f double dash Wt dt. Now a, Ito's formula which is also called the Ito Doeblin formula in Shreve's book, can be done at a slightly general level, where you take a function of two variables.

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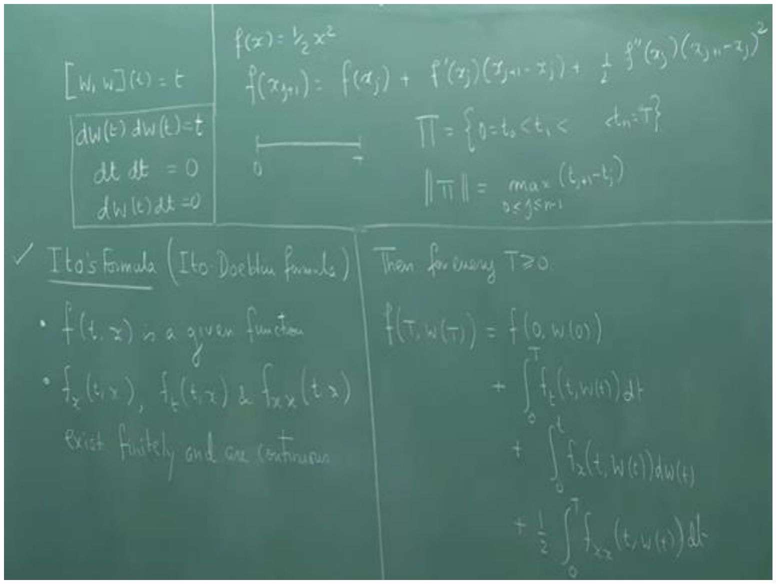

So, I will now write down Ito’s formula for two variables. So, Shreve also calls it Ito Doeblin formula in the name of another young researcher, who possibly died in the world war, in the World War II, but he also had almost got the same, came to the same idea. So, whichever you want to call it. I think I found this Ito Doeblin thing only in Shreve’s book, everywhere it is being called Ito’s formula. So, we will just go by the traditional name Ito’s formula. So, let us see what does Ito’s formula tell us.

Ito’s formula tells us the following. It considers not a function just of t just of Wt, f of some function of a process, but t and Wt where t and a process. So, f tx is a given function. X is in r, t is in r right, is a given function okay. Then to fx tx that is the partial derivative with respect to x ft tx, that is partial derivative with respect to t, and partial derivative with respect to, everything else, partial derivative with respect to xx, second order partial derivative. These exists and are continuous. Exists means, exists finitely.

Now, once these things are done, what formula we have. Once these things are given, we get the following formula. So, Ito says then, for every t greater than equal to 0, we have the formula f T WT is equal to f 0 W0 plus 0 to t ft t Wt dt sorry Wt dt.

See what happens, only when I take the partial derivative with respect to x I will consider Ito integral because the random process is at the x in the place of the x variable, plus 0 to t f of x, t Wt d Wt. Then there has to be the correction term, to be a quadratic variation okay. So, that will be 0 to t fxx t Wt and dt. So, it will not this will not be an Ito integral. This will be an Ito integral. So finally, you are talking about a stochastic process.

Now, what we are going to do, right now is to, instead of trying to prove this whole thing in a very general setup, we are going to look into the very simple case of f in half x x square. We will forget the fact that the t variable is also included.

We will not take the t variable. We will just leave the t variable and try to deduce a formula like this when I have something f is half xx square. So, our job now would be to consider a function fx is equal to half x square and show that this actually holds.

So, that will give you a fairly good idea about how to proceed. So, let us fix t greater than 0 and then do a partition and proceed. But before that you realize, that for a quadratic case, you have an exact second order Taylor expansion. So, if you have f of x j plus 1 it is f of x j plus f dash x j x j plus 1 minus x j plus half of f double dash x j. The higher order terms will all vanish. They will be 0 higher order derivatives. That will be the Taylor's expansion. So, can we use this idea now to proceed, that is the idea.

So again, you will have we will start with the interval 0 to t and I will consider a partition, given partition, where 0 equal to t0 less than t1 less than, strictly less than, tn equal to T. And as before we will consider the norm of the partition, is the maximum value of the length of the intervals.

Now, it is nth interval, you could have large right, I am just writing like this. I am not giving n possibly should have given n. Okay I will just take the partition norm goes to 0. So, this partition norm is max t j plus 1 minus t j as j goes from 0 to n minus 1.

So, this is the basic preparation that you take and then once you take this preparation, you start working on it, trying to prove this formula.

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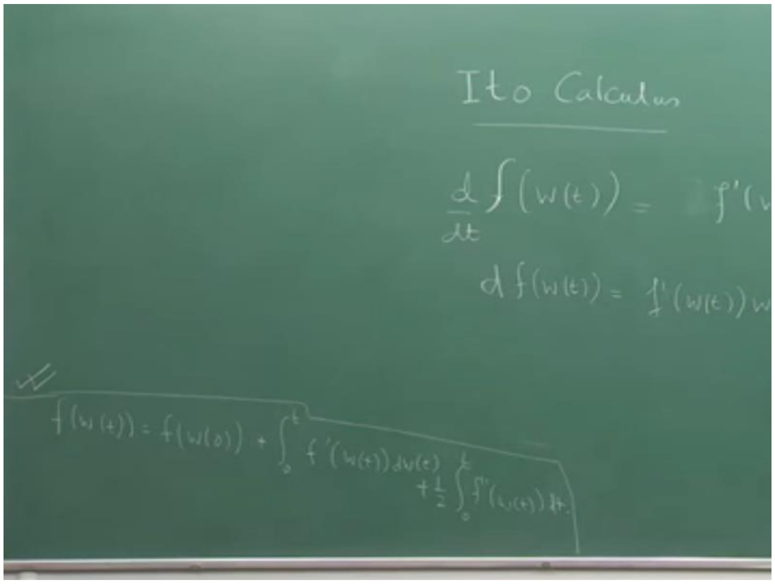

So, let us keep this formula on the side and try to see how do we arrive at it. But then so let me just remove everything and then start to prove this. So, once I prove this formula you just accept this formula as correct because it will be too complex to go and work it out. It is better to get a feel of it because this is many people for this is the first course in stochastic calculus rather than going into every detailed rigorous step.

Math can be sometimes understood with too much of rigor built in for the first time and then you increase your levels.

So now we take the partition and let me look at this. This is nothing but the collapsing sum W0 is 0 anyway. The collapsing sum, so I have forgotten the t, I am not taken t, f is half x square the summation j is equal to 0 to n minus 1 f of W t j plus 1 minus f of W t j. This is what you have. Now if I think that this is just this expression f x j plus 1 minus f of x j is a quadratic expression half x x square. So, f of W t is half W t square.

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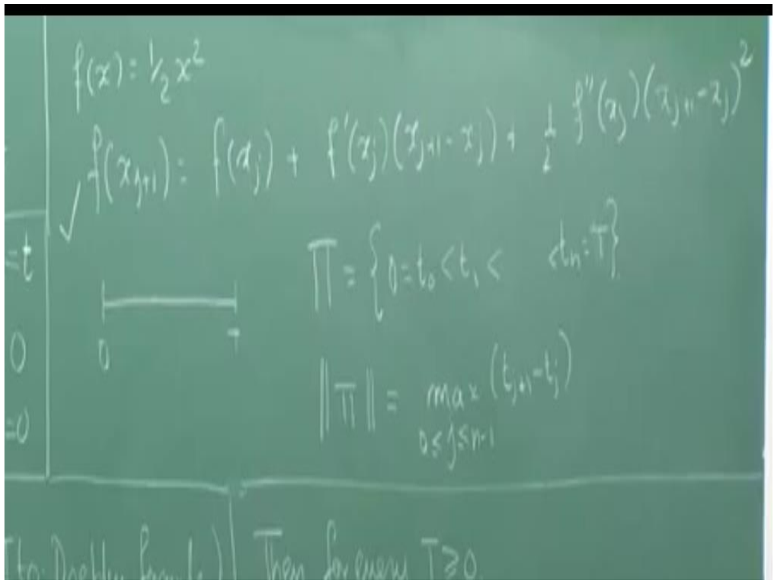

So, I am just using the Taylor's expansion at this point nothing else, the Taylor's expansion which you have known here. So, you know because of the higher derivatives going to 0 this is the exact Taylor's form Taylor's expansion.

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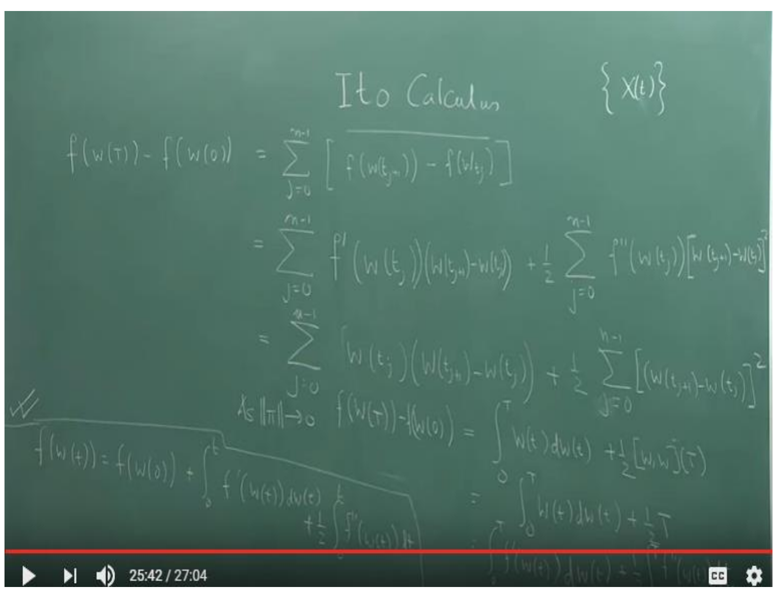

Thus, I can write just for each of them, I am writing so, I am finally writing the sum also. So, f is a continuous function. Now if I go by f dash of x, which is x, so this is nothing but W tj sorry I made a mistake because this W tj is the argument of the function f now. So, that is playing the role of xj plus 1 W tj plus 1, please correct it.

Now what is f dash of x? F dash of x is x. So, f dash of W tj is W tj where W tj is a real number so does not matter. This W tj into W tj plus 1 minus W tj and when you take the second derivative it is 1 so it is half of summation sorry j is equal to 0 to n minus 1 f dash W tj okay. Sorry f dash W tj is just 1 f double dash W tj because that is the expression of th that is the value of the second derivative.

It is 1, so 1 into W you can immediately realize, what has happened see, if I go and take the limit as j as the norm pi goes to 0, this would represent an Ito integral of W and this would represent the quadratic variation right and so what does this represents so as norm pi goes to 0, f of Wt because this is not depending on that partition this is independent of it this is the two end points. This will be nothing but 0 to t W t d W t the Ito integral, plus half of the quadratic variation. So, it will be W W of t. So basically, it will be 0 to t Wt d Wt plus half of t okay.

Now how do I semblance with this? What is Wt? Wt is nothing x, which is f dash Wt. So, it is 0 to t f dash Wt d Wt. This is one thing and then you have to realize that, what I can write? Half of 1 into t. What is 1? It is f double dash Wt. So, I can write this as, half of 0 to capital T, f double dash Wt dt sorry dt small t.

So, you see finally I have got this formula, by using the f x. So, if I choose a particular f I can get a particular formula. So, I chose f equal to something and then you see it matches this formula, f x equal to xx square and it matches this formula. So, under this f x equal to xx square we have proved that I can write it like this. Or else in general when you take a function f tx you write you have this formula.

You can ask what about f xt, f, f xt and f tx. So because you see when you do f xt and f tx you are doing this cross variation. That will become 0 when you take the limit when you actually write for this particular case which will be pretty complex, which I plan not really to get along with.

Secondly is that, if you consider the case, where you want to, consider the case f tt but when you have f tt that will be corresponding to dt gt, tt when I quadratic variation of this function t and that will be equal to 0 so that is why those terms really do not appear in this Ito's formula.

So, here the idea is not to give you something very, you know some very tricky thing, but really to give you some, very basic idea what Ito's calculus is like and see these formulas would be very-very useful and please keep in mind, this set of shorthand rules. A very important idea is to understand that this d Wt, what I can write from here I can take it to this side and write it as d Wt is equal to f dash Wt d Wt plus half f double dash Wt dt.

So, that thing is just a shorthand of this. You do not really have stochastic differentials. What you have is stochastic integrals. It is of fundamental importance to remember, that there is nothing like a stochastic differential. There is nothing called a differential of a Brownian motion because Brownian motion is not a differentiable function, so you cannot define the differential of a Brownian motion. So, since like W t into d W t. So, you cannot define that.

What you can define is a stochastic integral and that is the key. We are essentially talking about stochastic integral. So, Ito's formula what you see in this Taylor's expansion they are not written really in terms of derivatives but in terms of integrals and that is the key idea that you really have to remember, if you want to make a progress.

In the next class, what we are going to talk about? We are essentially going to talk about, a very general process. So, we are going to talk about a stochastic process, a very general process. So, we are going to talk about a stochastic process, a very general stochastic process called an Ito process. So here we end this class by describing you what a Ito process is.

So, Ito process is a process, which consists of the constant x so there is a random variable x and xt a stochastic process xt and then this x t sorry, x of t write x of t so this xt can be decomposed into x0 plus a Ito integral plus a normal integral.

S any Ito process is Ito process is a stochastic process. S any stochastic process is called an Ito process. It can be decomposed into three different parts. One is x 0, the 0th value of the process plus an Ito integral plus a ordinary integral. If we can do that then essentially, we are talking about something called Ito process, and what should be the Ito's formula for an Ito process. What should be the quadratic variation of an Ito process. All these things will come in the next class which you will see very soon.

Thank you very much.