Probability and Stochastics for finance
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Lecture - 13
Ito Integrals-II

So we are gradually getting deeper and deeper into finance and looking at the type of people who have really enrolled for the course getting some them 1 or 2 mails from them I have decided to if you look at the course format I have decided to change it a little bit and take you slightly more into finance so that you get the finance flavor.

So here we are going to start getting more deeper into the tools for mathematical finance and then we will end our show with 2 last lectures which would be on the Black Scholes formula for pricing a European call option. Of course, then we will try to describe what is a call option and etc., etc. and what are the things, how to go ahead with the things. So we are now going to talk about a Ito integral for a general integral.

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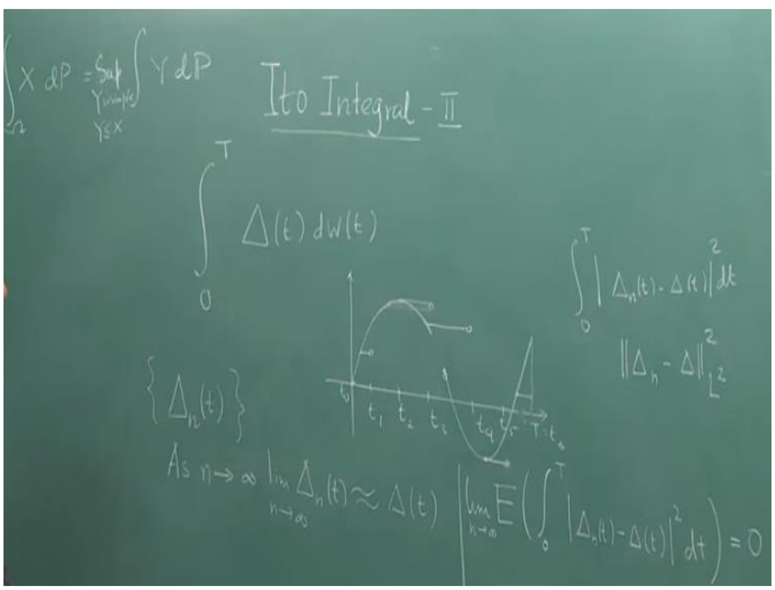

So this is no longer a simple process right. It is a it is essentially a general one. If you look back and go back to the sections on expectation, then you will see we have tried to define an integral in the following way that if you want to define an integral of a random variable defined over a probability space then this has been defined as a limit okay of these integrals.

Limit Y is simple and Y is less than = x. So it is a limit but sorry it is essentially not the limit but supremo because you are taking Y less than = x so the integral of Y is less than integral of x and then you are taking the supremo but it is a limit in some sense you can also put it in a limit form. So this is what you have learnt. So the same idea should come here.

So as you keep on changing the partition of the interval 0 to t for each partition you can define 1 particular simple process and each as you make the partition smaller and smaller the simple process will actually go closer and closer to the suppose this is your actual process up to T. Now you have divided into say this t0 which is 0 t1, t2, t3, t4, with t5 and so on and this is t6.

So what you do is you define 1 value here maybe then you define another value here in this way sorry from t2 maybe like this, from t3 might be like this t3 to t4 something like this, from t4 it might be just like this and from t5 it might be just like this. So this is very crude approximations very bad approximations but as you make the thing smaller and smaller and smaller these partitions your approximations would start becoming better and better and better.

So what do I mean by that that a simple process delta n t so I will have a sequence of simple process delta n t each process is simple. So if I take this sequence of simple processes as n goes to infinity we expect that this simple process simple process should somehow be same. Of course you have to define what is the meaning of this sameness, same as this continuously varying process delta t.

So these are discrete processes whose limiting form is this continuous process delta t. Continuous not a continuous process this continuous process here but okay continuous more continuous than the things that we are talking about the parts are continuous as they are continuous parts. So there is nothing like every step is discrete. So it may not be just a constant function on the interval, it could be just varying.

So we expect that this is somehow what we will get but what is the meaning of this sameness how do we talk about this sameness. In probability theory, the sameness is spoken in a slightly different way. It tells you I expect that the distance between these can be made smaller and smaller and smaller. So it tells me, now each of these so what we expect is the following. I will tell you what it means.

So Shreve tells us that if by these functions simple functions coming to delta t it means that it what it means that over all the sample paths if you average the square of the distance. So this is nothing but the L2 norm. If you look at this integral. So in the terms of the function this is nothing but delta n - delta whole square and those who know about the L2 spaces the space of all square integrable measurable square integrable functions.

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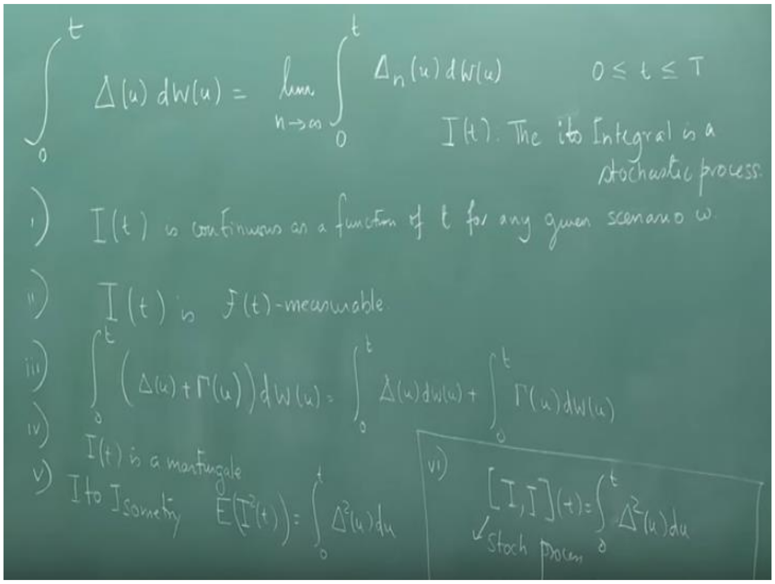

So once you have and you expect so let us forget this thing and let us not bother because people might be not so comfortable with it. Let us just look into this. Shreve tells us that when I am expecting the fact that I expect that when n grows this simple processes must somehow manage to imitate or will be almost same as the given process.

By this statement we are actually meaning this statement, we are meaning that if we take over all the paths we take the distance between these 2 by the square of the distance then the expected value of the distance between these 2 points should be 0 because remember in a random setting we cannot be telling that it is exactly = 0, this will be exactly 0. We have to talk in terms of expectation because we do not know the outcome. So we have to only speak in terms of expectation.

So the expected distance so this is this distance so as n grows the expected distance should be going to 0 because the expected distance itself is a random variable so the sorry the distance itself is a random variable so the expected distance between these 2 has to go to 0 that is the idea. So by this we actually mean this.

So once we mean this the definition of a standard integral standard Ito integral from any point so would be limit. So it can happen when you take any t between 0 to t this will happen. This is the way I define. Now why such a limit would exist. Remember that this is a stochastic process. I am taking a limit of another stochastic process. So why should such a limit exist right. For every given t why should such a limit exist.

That can be answered by using the Ito isometric theorem and showing that this actually forms a Cauchy sequence but we are not going to get into that sort of argument at all though I assume that you must be understanding this what Cauchy sequence and those who do not just forget it for the time being. Just assume that okay I can calculate it in some procedure.

Now what sort of properties would such an integral satisfy. You will be amused that all the properties that we learnt about integrating the simple process would be applicable here. For example that I t is continuous as a function of t for a given sample path t for a given scenario omega t for any given scenario omega. Number 2. I t is F t measurable that an I t is a stochastic process adapted to the filtration F t which is the filtration associated with the Brownian motion.

From now on whatever filtration we are talking about the filtration would be associated with the Brownian motion. So it is F t measurable. The third property is linearity. So if we take any 2 processes gamma and u sorry delta and gamma it can be proved that this is same as we are not going to prove these things just to announce you that these are the important properties. Another fourth most important property is that I t is a Martingale.

Then of course we should talk about the Ito isometry property. So Ito isometry you know what is Ito isometry, it is exactly the same. You basically copy what you know for that to this part. The Ito isometry says a second movement of this and the last property which is one of the most seminal property so I separate it, the quadratic variation of the Ito integral I t.

Remember Ito integral I t itself I t the Ito integral again I this is the point I want to state again and again the Ito integral the Ito integral is a stochastic process. So the major property that I want to state is that a quadratic variation accumulated up to time t is t that is independent of the path quite a Brownian motion type behaviour sorry not t I am making a mistake here.

This is not Brownian motion type behaviour sorry it is the same as same as the isometry. So here you see so it gives you the standard. This is the normal integration in terms of the variable u. So this is a stochastic process. For every omega, you take you will get something. So as omega changes this will change sorry I made a mistake in my it is path dependent actually.

So this is itself this quadratic variation itself is a stochastic process just writing in short and this is path dependent of course not path independent I made a mistake sorry. So these are the important properties that you learn about the Ito integral. So these are the properties that you have to keep in mind. Of course, because of the time we cannot be proving each and every step. Now what we will do we will show you a simple example of how to compute the Ito integral and (()) 16:52 will make a difference between computing the stochastic integral and the standard form of computing this integral.

We will just see. So the next part remaining part of the class would actually consist of computing this integral. Please understand quadratic variation is a very very fundamental thing here. So our goal would be to compute.

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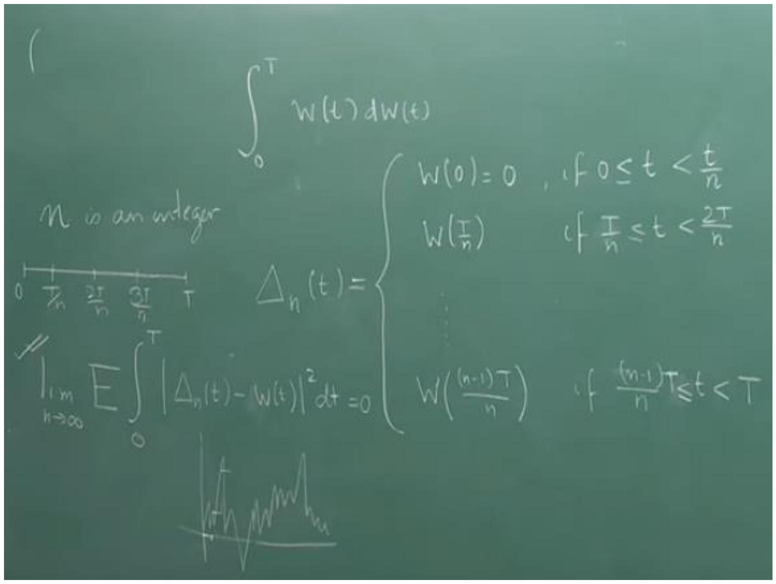

So delta omega t we have replaced by W t. So what would be this? That is basically 0 to capital T x d x that will be x x square by 2 that will be t square by 2 would be the answer if I just take it as a standard calculus integral but you will soon see that t square by 2 is not the answer right. We are just putting x here and dx will be x x square by 2 would be the answer. Xx square by 2 0 to t so t square by 2.

But here you will see is the quadratic variation which will get you an additional term with a term of that type and that makes stochastic integrals very different from the standard integrals of calculus. So here we do it start doing it step by step. So we will construct for this particular we will make a construction of a sequence of simple processes. Now we will choose n to be a very large integer so n is an integer so we will keep on increasing n, n is an integer.

So take an n and have for every step so you divide every interval length is of the length T by n 0, T by n, 2T by n, and so and so forth till T 3T by n that is all that is all. Now as n becomes larger and larger these intervals become smaller and smaller. So we will now define that for every interval how would we define the delta n t because when interval these intervals become smaller and smaller and the length of this interval that this will be larger these lengths would be smaller and smaller and smaller.

There will be more and more intervals of those given lengths. So we will put W 0, it will be same = 0, so it will be 0 if , W T by n it is from here W T by n so W let me just put this is W n-1 T by n. So, you are just taking these values itself. So that is that is what it is sorry that is There. So that is how I have constructed it. Now the next question would be to know whether such a construction can prove this thing sorry this will be W t that would become 0.

So what is the expected value of this? That is an important question. So you can understand if you look at the picture very well I advise you to draw the picture yourself then that is the way you can learn. If you draw the picture is a Brownian motion and here you have kept say 0 and then you have taken the t1 value is here you have kept it here till say t2 and t2 value say is here started like this.

So in that way you will see that as n will become larger and larger these straight lines will become smaller and smaller and smaller become smaller and smaller and smaller and so you will have many of these straight lines as n becomes large and large there will be infinitely many straight lines which would be very hardly vary from the actual value that you will see. So this distance would any way become smaller and smaller.

So if you take their difference for a given sample path and integrate over them then basically you are calculating of the distance between the areas under the curves for a given sample path square of the distance basically not really area but square of the distance then this would turn out to be 0. Of course you can make a very rigorous calculation right of actually putting in the values breaking up 0 to t into these intervals and then actually looking into the difference W0 - W t.

When t is lying between this so the square of the difference you can write them down and then you can separately come and do the job so doing this will take a little bit of time. It looks slightly obvious that you can easily do this but this thing actually getting this thing done of course one might think that how I should be able to change with the expectation inside the integral.

Ya you can do under certain circumstances you can switch expectations and integrals. So if you can switch the expectation integral then the things would be much more simpler then you would know that okay their difference would be 0 their mean would be 0 etc., etc. and sorry the variance would be not the mean that will be the variance so variance would be basically t by n and so and so forth.

So you would add those quantities up right so you would add those quantities up and then you integrate with n becoming larger and larger so the area would actually keep on falling. So that is 1 way of thinking about it but let us not think about it at this moment. Let us assume that we can show this that this will actually work. So this is a very good approximation which it is if you look at it pictorially which it is.

You see in actual finance you really have do not have to bother too much about these sorts of approximation nobody is going to ask you to do such calculations but okay if we find time we will really put up these sorts of things up in the website.

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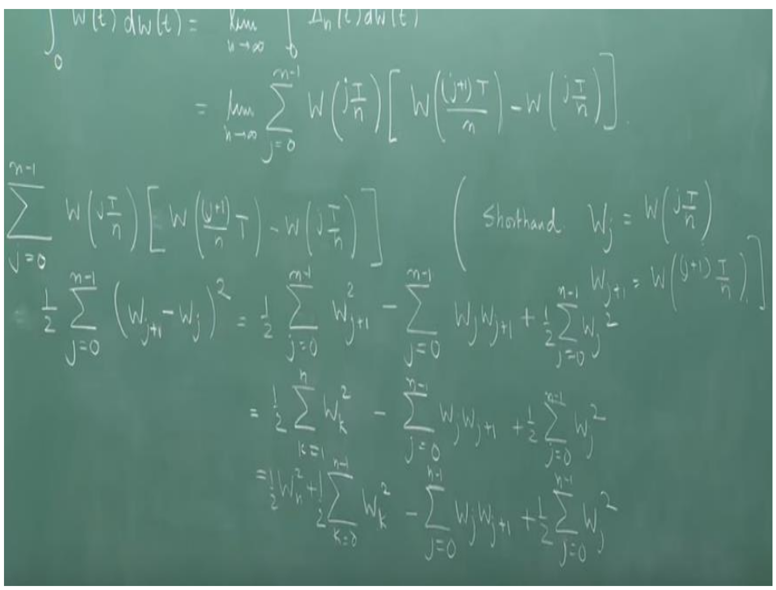

So now what we have is the following that is my definition. So how do I define this delta you already know. So, it depends on so it is from j = n so it is just using the values there. At every step you have to calculate WjT by n when j it is 1 it is j is 0 it is 0, j it is 1 it is W T by n, j is basically you have to do this because these are the function points where these are the function values. You can easily understand these things.

So what have I done. So when j is 0 it is 0 done finished. When j is 1 it is W T by n into W 2t by n - W t by n of course because over that particular interval it will take the value j T by n. Over this interval it will take the value j t j W this will take the value W T by n. So that is exactly the interval that we are considering. So and so forth. So this is the sum.

So essentially now we have to compute this sum and then take the limit. So our job now is to compute this sum. Now because there is so much of clumsy things we will consider some shorthands. We will write W j shorthand let us do shorthand some shorthand notations. In the shorthand notations let us put W j to stand for W j T by n so which means W j+1 would stand for W j +1 T by n.

So with this shorthand notation we will start the computation. Now let me do the calculation. We have to check up this but let me start by doing the following calculation. You will see why I am doing this. I will just go ahead and just tell you a bit more about this. Suddenly I remembered it is a good way to talk about it. You see when n is becoming large see the first part is very simple because when you put any t here it does not matter.

Here W it will be 0 and when n is becoming very very large because W T is continuous and when n is becoming very very large this thing becomes very small this interval. So there is hardly a difference between t and delta and t these values, very small difference and so this square values would actually come down further and that is why their values would basically come down and the expectation would go towards 0.

So I want to calculate this. You will see why we are calculating this. This would allow us to get a fairly good this will allow us to calculate this. So it is just an opening of the brackets nothing else. Once I do open the brackets because W 0 is 0 then so I just very important note that W 0 is always 0. So here because I am putting when I am putting j = 0 I get 1 and I am putting j = n - 1 I am getting n.

So I can as well as write these expression as this particular expression as k = 1 to n W k square so - j = 0 to n-1 W j W j+1 plus half of summation j = 0 to n-1 W j square. So now I will break this up into 2 parts. I will break this up into W half W n square plus summation. Now I will use the fact that W 0 is 0. So what I will have? If I break it up I will have summation k = 1 to n-1 W k square but noting.

The fact that W 0 is 0 I can add W 0 square to it and write here this break it as k = 0 to n-1 W k square. It does not matter because W k is anyway 0. So here is what we have got. But now we will go and transfer our calculation to this part. So we will do our calculation in this part. So I will have half of summation j = 0 to n-1 W j+1 - W j square as I can begin up from this part. So what I will do I will add sorry there is a half here. I will add this half and this half because these js and ks are dummy indices.

So I will have half W n square plus j = 0 so half plus half is added so it is 1 n-1 W j square - summation j = 0 to n-1 W j W j plus 1.

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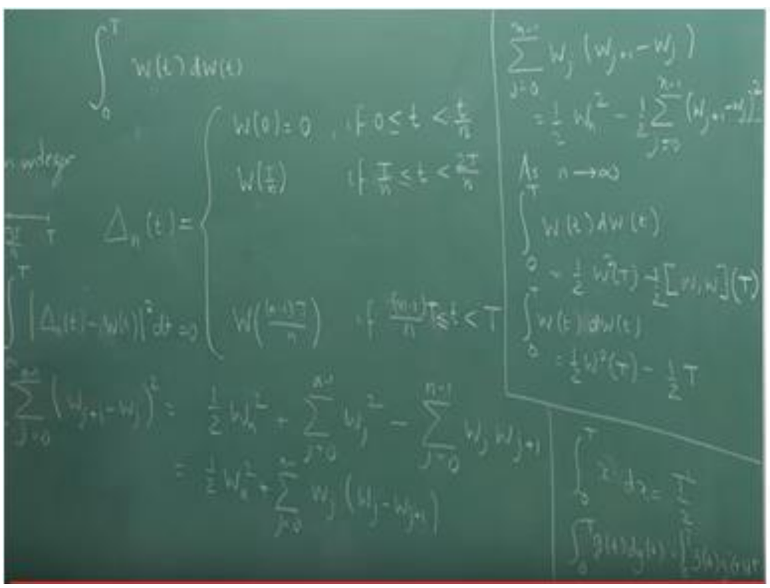

So basically I am getting half W n square plus summation j = 0 to n-1 W j into W j - W j+1. You see I have almost got this expression but with a - sign. So what I will do is the following. But what is this? This is a quadratic variation term. So I will take this part to this side and bring this thing to this side. So my next job is the following.

So I will now write summation j = 0 to n-1 W j W j+1 - W j = half W n square - half summation j = 0 to n-1 W j+1 - W j square. This is actually the form of the quadratic variation of the Brownian motion you see so quadratic variation has come.

So basically, now if I take the limit on both sides what is this? W n square W n square. So what will I get? W n t. So if I put n here W j W n is W t. So here it will be nothing but half W t square. So if I take as n tends to infinity this limit is nothing but the integral. So I will get 0 to t W t d W t = half W t square - this limit assuming that all the limits that I am taking here I am assuming that okay there are in almost surely sense. You can even take them in the sense of limits in the sense of probability convergence in P that will also do.

So I am writing so this equality is a convergence in probability. Essentially this is because we can actually show the quadratic variation course in or can be shown in almost surely sense. So these are actually in almost this equality is always in almost surely sense because this is a stochastic process random variable at time t so this equality holds only everywhere except a set of measured 0.

So this is very very important to understand when a set of a null event if you throw away null event it holds everywhere. So what do you get? So you get 0 sorry this is half here. So what you get 0 to t W t d t sorry W t d W t = half W square t - half of t. You see this half of t is an additional term because if I take a standard integral 0 to t x square dx sorry x dx. This is nothing but t square by 2.

This is nothing but t square by 2. Or if you take any integral as 0 to t g t d g t, d g t is nothing but g dash t g t 0 to t g t g dash t d t because this is the differential of t d t and this will simply give you half g square t. So you see this is something very different and that is the (()) 39:07 of stochastic calculus that stochastic calculus is a quadratic variation actually makes the difference the quadratic variation.

See essentially, we were trying to compute the quadratic variation so these are some sort of a trick. The quadratic variation cannot be forgotten when Brownian motion is in the game. So we stop here and we will talk about Ito’s calculus in the next we will have a Taylor theorem type things that you learn in standard calculus. What is the meaning of Taylor’s theorem in stochastic calculus? So we will learn about those things.