Probability and Stochastics for finance
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Lecture - 12
Ito Integrals-1

So, here we are, on our third week run and today we are approaching or rather entering the fascinating land of the Ito calculus or the Ito integral. Ito is the name of a Japanese scientist who introduced this and with that, he changed many things in physics, his things changed many things in physics and also in probability and Ito integrals and the calculus associated with it, the Ito calculus, is the key to doing a lot of computing or solving lot of models in finance.

So, this is the holy grail of stochastic calculus and we are going to do it step by step. We are not going to be in a hurry, we are not going to push ourselves into too much of rigor. Of course, rigor is a necessary tool of a mathematician, but looking at the diverse audience that I have, you cannot push rigor too much, but I will try to get you some understanding and in this whole the remaining part of the course, the book, I will follow a single book, which is one of my favorite books in mathematical finance. It is called stochastic calculus for finance.

So, this book is written by, Steven Shreve a very famous name in mathematical finance from Carnegie Mellon University. I have taught from this book earlier and this is my favorite book in this subject. I would rather like to show it, this is the book and the interesting part is that, this book now is an Indian edition, both the volume.

This is volume 2 from which I am teaching. So, volume 1 and 2 there are 2 volumes. Volume 1 talks about discrete time. Here we are talking about continuous time and both these books are very a lovely read and those who are serious about mathematical finance should actually procure the copy of these 2 books. This will be helpful in many-many ways and this book takes you from a very basic to pretty advanced level.

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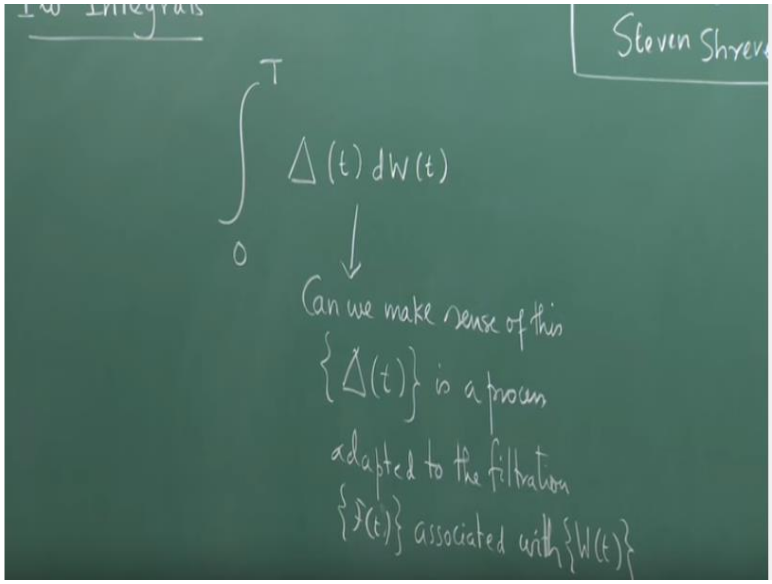

So, now what we are going to do. In this section, we are trying to make sense of the term, sense of this. Of course, you might ask what is dWt. We just said in the last class that, this Brownian motion is not differentiable anywhere.

Yes, but assume that this is like writing the difference. This is the increment of the Brownian motion we and we had been talking about increments of this Brownian motion. So, we want to so the question is, can we make sense of this and what is this delta t?

Delta t is a process adapted to the filtration Ft associated with the Brownian motion. So, just like in standard integration theory, we will start by trying to look into the meaning of this, when this process is nothing but step functions. At every t it is some step function and on a particular interval it will hold a particular value phased value.

So, our job would be now to make you, will see that, finally when we will compute things, you will see that this will give us very different answer than standard calculus. It is the quadratic variation of the Brownian motion that would be the culprit. So, here again, we are going to partition 0 T into intervals. So here we are partitioning where you have this.

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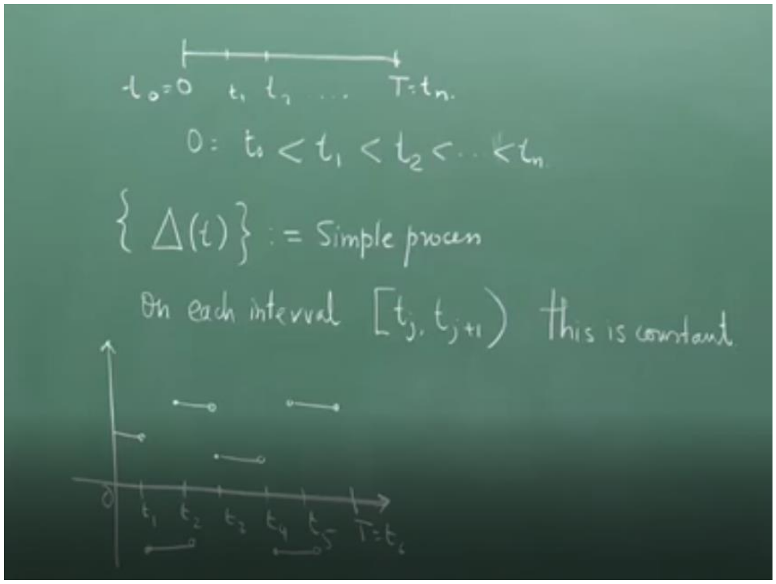

Now, we will talk about a simple process or a simple function basically. We will take this to be a simple process. Simple process means on each interval tj to tj+1 which includes tj and not tj+1, this is constant.

So basically, you are looking at a function of this form. So here is your 0 to T, here is t, here is t2, here is t3 say I will just do it 5, t4, t5 and this is t6. So, what you are expecting that from 0 suppose this, to here, you have some particular value. Then t1, so here you do not have the value, this value is not valid here it starts again from here, t1 to t2 that is why you are giving this round.

This is one particular value and t2 to 3 it could be like this, t3 to t4 it could be like this, t4 to t5 it could be this one, and t5 to t6 it could be like this just (()) 7:24. This is a step function basically and this is an example of a simple process.

So, this is one sample path of that simple process right. So, in this thing, if this is the way you have broken it up the partition, on each partition we will consider that it is constant. Of course, if you take more smaller partitions, the same idea will remain fixed. Okay, there will be 2 intervals where you will have the same value.

Now given this partition on which it is constant, so we will look at only those intervals on which it is constant and that will partition the interval. We will try to calculate it. Now by delta t let us work like a financial guy.

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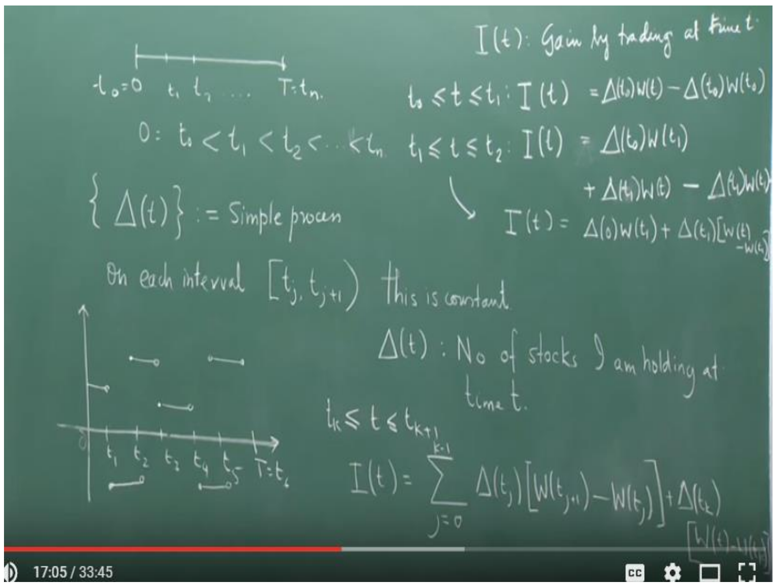

Let us in delta t is the position I am taking in an asset at time t that is delta like a stock so delta t is the number of stocks I am holding at time t of some company for example. So, number of stocks I am holding at time t. Now let us, now assume that Wt the Brownian motion, just like as Bachelier thought, describes the price process of the stock. But you know that, this is really not true because, price process of stocks never take negative values while Brownian motion can.

But for the moment, just for the heck of it, think that we are in Bachelier's days, we are not bothering of those things and we are talking about the fact that W t is the price. So, what I have done so at time t, what is my gain in trading at time t. So, It is the gain is the process is actually a stochastic process, describing the gain by trading at time t.

Now consider the inter t, which is lying between t1 and t0 right. Now I have actually made, I have actually, bought say delta t0 quantity of objects at time t0, and for which I had paid Wt0 as the price. So, Wt0 is the price per unit of course. So, my total money I paid, so let me write It so what is It in this case.

So, total money I paid, to get this delta t0 amount of stocks at t0 is W t0 and total money now I take hold on to this stock at time t at a time to sell number of stocks and at a price Wt, because price is changing at every time. So, by selling those stocks, I make this amount of money. So, what is my gain? Gain is nothing but the difference of this two. Amount of money I spent and then amount of money I actually bought by selling, so I will just subtract these two to know whether I am at profit or loss, n could be negative also.

Now think that what would happen if my t, is between t2 what would happen to this, what would happen to this scenario right**.** So, what would be It when I am in that particular scenario. In that case, of course t0 is in this case 0, so we can write delta 0 also here, does not matter. So, when I am writing this t here, one has to be very careful. I am meaning that, I am ending the trading at this time.

So, what happens when I am telling that t is between t1 and t2 because which means I have held to the stock Wt0 up to the time t1. After that, I will change it, so at time t1, I sell my stock which I bought, this amount of stock at time 0 with the price Wt1.

Then I bought, a stock with this money that I have got, I have bought a stock Wt1 delta t1 at time t1, paying price Wt1 then at t2, at time capital T which is lying between t1 and t2 I sell this this Wt1 holding, a delta t1 holding so delta t1 Wt. So, by selling I totally gained this amount and this is the amount I have spent to actually buy the delta t1 stocks. So, it will be this minus this. So, which I can also write nicely as I t in this particular case I t = delta 0 W t1 + delta t1 W t - W t1.

Again, if you are slightly not comfortable, what I did, I know that my maximum time I can hold the stock, I bought at time 0 is up to time t1, then I have to sell it. That is the period of holding. Then at time t1, I know what is the stock price I may sell it with this price. Now with that price I buy, stocks W t1 stock of the amount delta t,1 with the price W t1. So, this amount now has to be subtracted from this one, because I sold and got some money, from that I spent some money to buy some new stocks.

Again, I sold out stocks at time capital T and I finish my trading, time small t and I finish my trading. I got this amount of money. So, this minus this plus this is the amount of money I have now.

So, this is a very simple thing. Now this can be written in a more general way, that if t, so if you have a t lying between t k and t k+1 then your It, so this is what does it say. Say at every time point, before the time, till time k-1, what did he do? He bought something, he hold something in 1 interval, then sold that thing and bought the new stock by using the money from the same thing that he had gained.

When he sold something whatever money he had got he spent a part of it to buy something. So, he held at this stock, up to time t j+1 and sold this at this amount, but at time tj, he had at, sorry at time t j he had bought the same amount by spending the money delta t j into W t j. And then of course, we will look at the last part here this into this whole thing into Wt minus Wtk.

So, this is the general form of writing It. if you are not comfortable with, the cannot see it I write it again here for your convenience. This is very simple profit and loss thing. I trade make money use that money to buy something, again I sell that and buy some new thing. So, that is the whole thing that is whole idea that is the way the market actually operates.

This It, so this It is often written as I0 to small t. So, this whole thing is written in a shorthand like this. That is how you give meaning to this thing. So, this is what we want to write as dWt. It is a shorthand writing. It is not exactly the differential. There is nothing like a differential of Brownian motion. It is a shorthand writing. Now It then itself is a stochastic process.

So, It is called the Ito integral of the simple process delta t. So, you are taking integral of a stochastic process that is very important and it is not like an integrating random variable. Integrating random variable is just like integrating any function right that is it.

So, here what you create is also a stochastic process. Your integral itself is not a number but your integral itself is also a stochastic process. It is a stochastic process. Son you see there is a huge leap or huge difference in the thought process. Son when you come to this stochastic wall your life completely changes.

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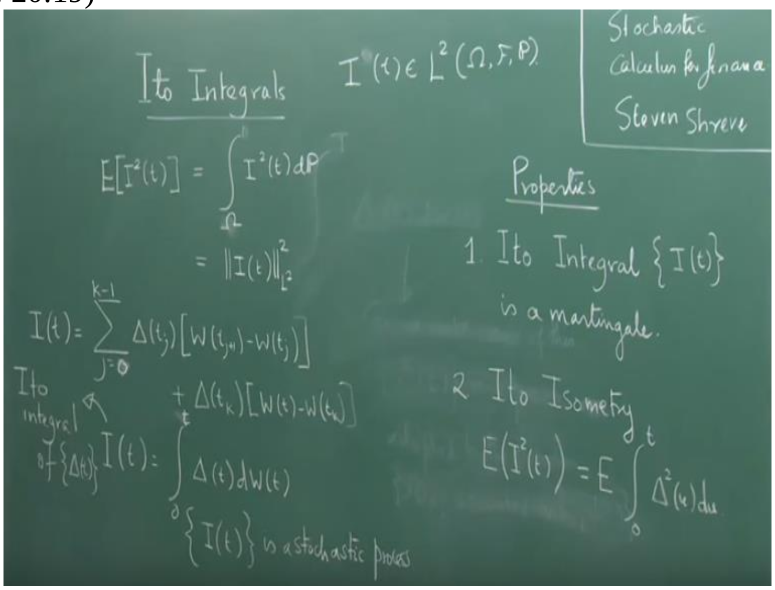

Now what are the key properties of this stochastic process. The key properties are following. The number 1 property is that, the Ito integral properties, if we have time we will prove any one of them, the number 1 property is that, Ito integral is a Martingale with respect to the filtration Ft adapted to the Brownian motion W. The Ito integral is a Martingale.

Ito integral, so when I am taking integral, do not think it is an integration, it is a number. It is not definite integral, Ito integral It means, Ito integral is a stochastic process. Ito integral It, is a Martingale. Martingale are essentially processes which possibly are trying to maintain status quo at every time.

Number 2, proving this is not so simple. It needs little bit of work nothing else, it is just the computation is complex. Another thing is called Ito isometry. We will see, what is the meaning of Ito isometry. Ito isometry says the following. Expectation of I square t, so expectation of the random variable I square t square, is equal to expectation of 0 to t delta square u du.

Now why it is called isometry, would be clear, if you only know something about L2 norms. Say look at this quantity, E I square t. So, if I write here as E I square t this simply means integral right, whatever is the 0 to t, I am sorry, integral over, sample space whatever the sample space is of I square t dP.

So, this because, we are assuming that this is finite. So, once you assume that this is finite you are assuming that I square t actually belongs with a capital L2 of, L2 space of, this is only for those who know this stuff, this function analysis. Those who do not do not bother. Think that Ito divides this formula.

Isometry means they are similar distances. So, distance in the original space between 2 points and distance in the range space between the 2 functional values are similar, then we call it is isometry. So, this actually means if you look at it very carefully, it is, if you take I, no sorry no, not I, It is in L2 f not I square. So, I t this is a square of the L2 norm. So, this is nothing but the square of the L2 norm.

So, this is I 2 is in infinite dimensional space in this when I t itself is so the Ito integral itself is an element of L2. So, it is in the infinite dimensional space is an infinite dimensional object. So, it is so interesting things are coming in and if you look at this.

So, they are telling that this is expectation of this. So, you are telling that the distance, the norm of this is computed, is nothing but the expected value of this integral, because this integral, if this the expected value of this integral means what? This is a normal integral. This is a standard level integration right.

Here, I am not treating delta square as the function of omega basically, keeping the u fixed, but I am treating it as a function of u itself. So, for every sample path you can compute this as t changes fix up the sample path, the scenario omega and you can compute this and then you take expectation over all the sample path, average over all the sample path. That is exactly same as the L2 norm of this. That is the idea.

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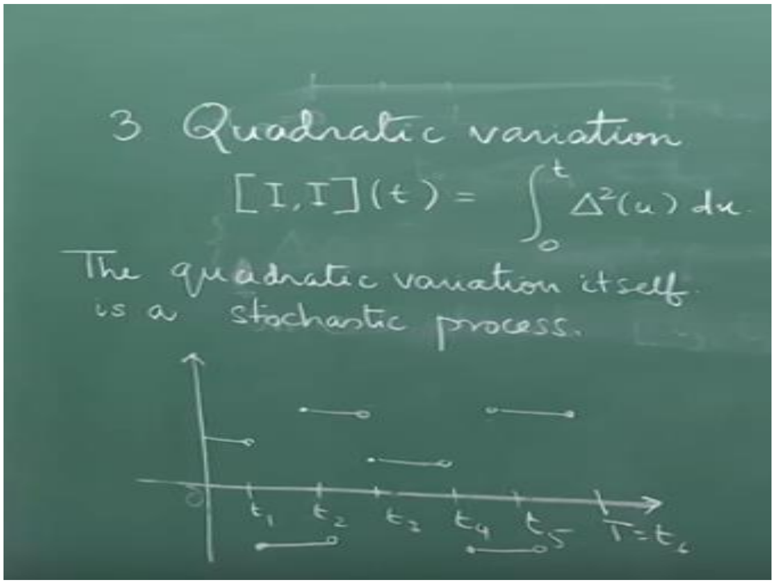

Now comes a very-very important property. So here, all these things are being done for, the Ito integral for the simple process delta t. We have not spoken about Ito integral for a general case and that we will come tomorrow, in the next rather in the next class. The very important thing, that we should now understand, that just like Brownian motion, the Ito integral itself has a quadratic variation and what is that quadratic variation.

So, Ito integral does not have a 0-quadratic variation. Ito integral itself has a quadratic variation. The Ito integral states, the quadratic variation of the Ito interval is being given by this simple thing it is amazing the answer. So, the Ito isometry says the following, that the L2 norm of I the integral, is nothing but the expectation of the quadratic variation. See the quadratic variation itself, whether it is of a Brownian motion or whether of this integral itself is a stochastic process. That is the thing that you have to take into account.

The quadratic variation, but it is very interesting, the quadratic variation has a stochastic process, but for every t it gives you a number, every t it is a fixed number, if for a given sample path, for every omega, it will calculate a number very nicely. So, and for every omega whatever be your omega whatever be your path for a given t, therefore whatever be the omega, this will be the answer.

So, it is path independent essentially. So, the quadratic variation itself is a stochastic process. So, what we have now learnt, is completely for the case, where delta t is that simple function, that is all. Delta t is this so, I know that delta t given the t delta t varies like this. So, given any omega if it is in the time interval, this delta t will have this values that is all. It does not depend on what omega and so delta t is the simplest function.

Delta t sorry, I made a mistake. So, given an omega delta t will take this this this this values in this interval. If you change the omega, it will take some other values in this fixed intervals. So, the intervals are fixed and in those intervals, it is taking constant value but it will depend on the omega. If you change the omega you will have different values and that omega will change the value of I t. So, that making It a stochastic process. Similarly, this quadratic variation itself is a stochastic process.

So, this is a very-very important understanding that we are essentially generating not numbers, we are at every step generating a stochastic process. So, this is something of very important thing that we have to keep in mind. So, when we are going to talk about general integral, this is some adapted process. Delta t is an adapted process, not a simple process like this, just not say function type thing then, we proceed as we do in integration theory.

We write the, we approximate the given function delta t, in as a, we approximate it by a family of simple processes, simple functions, and then you take the integral over those simple functions and then take the limit.

That is the idea, we are going to apply and then we will see for one particular calculation what will happen and there is a so, we had learnt in the last class that a quadratic variation of course, we should ask what is the, what is when we take the limits of those Q pi what was what sort of limit it was.

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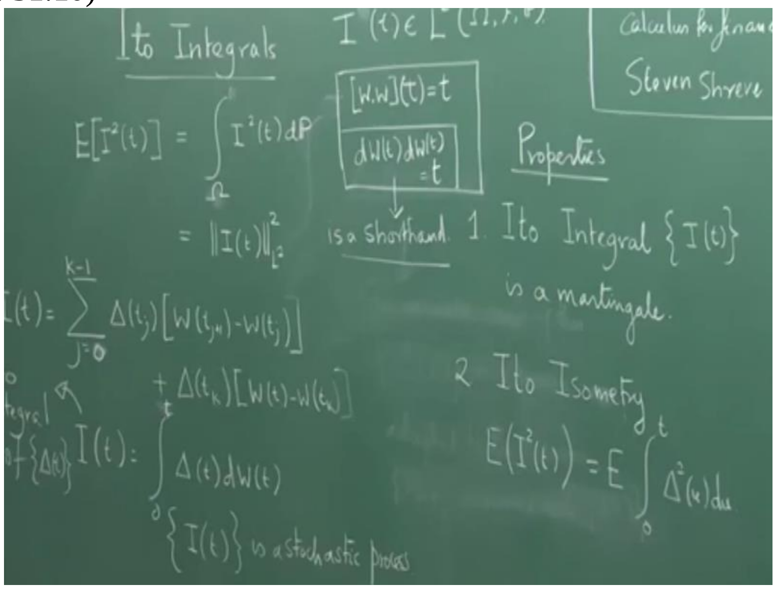

We proved that what the limit we get is a limit in or the convergence that we get that we can always have convergence in probability. But under slight, and also if we just bother about the fact the variance is 0 and the expectation is t then we have something called L2 convergence, which I am not going to bother you with. Or else by slightly modifying the definition, slightly modifying the requirements, of the behavior of the partition, we can show that it can be made to be convergent almost surely. So, you just have to remember this. But remember this is also a process stochastic process.

Sometimes, now because we have started using this d Wt symbol, we will use this shorthand. This actually means quadratic variation. This is this is the meaning of this. We will very, this is nothing but these two difference right the difference square basically. So, we will use this shorthand. Please note that this thing, whole thing, is a shorthand, shorthand to write quadratic variation. But this shorthand would be very helpful when you do finance. They are extremely useful to use this shorthand.

Thank you and tomorrow we are going to talk about, some more interesting facts doing, Ito integral for a general integral, and all these properties that you have learnt here, would be actually translated to that general integral and then we will do an example.