Probability and Stochastics for finance

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Lecture - 11
Brownian Motion-3

Today we are going to do the last part of the Brownian motion, which I am writing as Brownian motion-III. And today we are going to do a very important part, so we will be today talking about quadratic variation of a Brownian motion, what does it mean?

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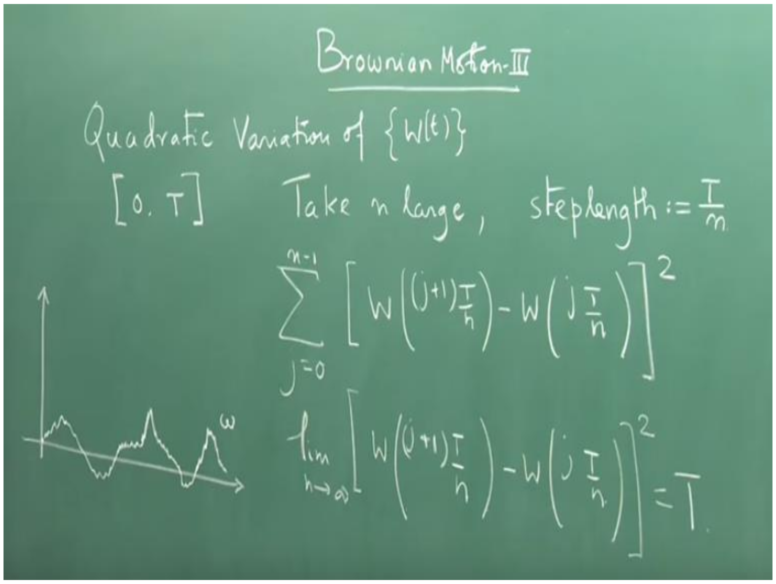

Variation of this stochastic process Wt. W actually stands for Wiener, Norbert Wiener who actually formally studied the process in detail and made lot of applications.

So, a natural thing to do is, that if you have given interval 0 to T, there is no natural way to break the steps, like for the symmetric case symmetric random walk here a step every step 0, 1, 2, 3, 4. So, here you do not have any selected step length, so what you do is take n very large and consider step lengths of the size T by n and form the following sum, it is not so apparent what would be the limit, if I take the limit of this quantity. So, I take limit of this quantity as n tends to infinity, what is the idea? how to compute it? It is not so apparent that the limit would be T.

So, before really getting an idea, what it means, let us talk about the first variation and second variation of a continuously differentiable function.

If you look at the sample path of a Brownian motion, then for a fixed omega, so once you fix an omega, it starts from 0, this is too much zigzagging basically, I am unable to draw, I am putting straight lines at some place. You magnify them there will be zigzagging. For every small zigzagging, there are many infinitely, many zigzagging's.

So, if you look at this it essentially tells you that the curve is continuous, as a function of T given a sample path omega, so this for a sample path say omega, the curve is continuous in T, but is not differentiable at any point that you want. And so, second variation will take the value T. It will accumulate, some second variation, while will show that, if you have a continuously differential function second variation will always be 0.

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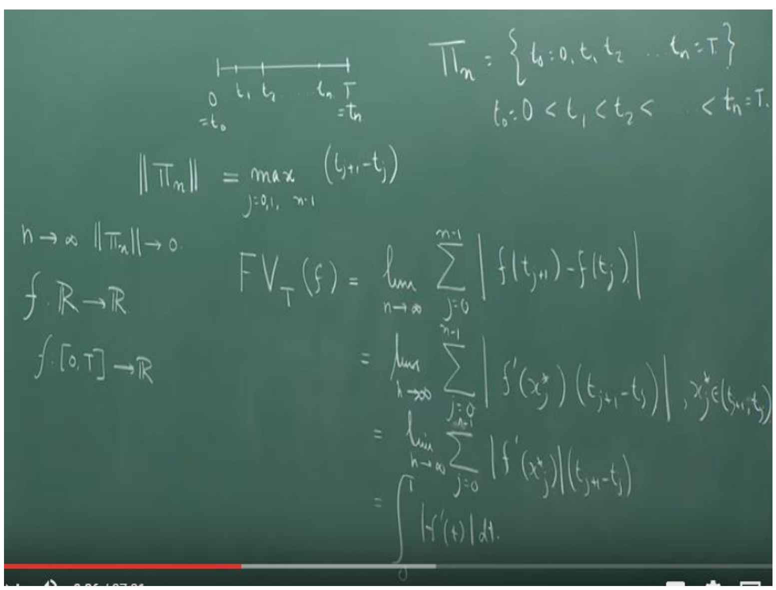

But let us just talk about what is first variation. So, if you take the interval 0 to T, the key point is to, break it up into partitioning points say n partitions. So, the nth partition will be given as points t0 equal to 0, t1, t2, tn equal to t, where you of course have n equal to T.

So, this is the basic partitioning of an interval, that you do to compute the integral. So, if you write the first variation. So, you define some quantity of this form, and max of, so for at the nth partition, this is what is called the maximum size of an interval, because these intervals need not be of equal size. One thing is very clear that, as you keep on increasing the number of points, this one will go down to 0, this will come down, whereas this is the maximum of all possible gaps but if you put in more points this value will come down right.

So, once you understand this, you can immediately conclude for the time being, that if n tends to infinity, this is the first observation. Once you make this first observation, you see how simple it is to calculate the first variation. The first variation of a function over the time interval 0 to T of the function f, which is continuously differentiable. So, you will take a function F say from R to R does not matter, you can take R into R, we are just taking R to R because, this W as a function of t is a function from R to R for each sample path.

So, let us look at it, R to R or 0 t to R does not matter. If you are more comfortable, let us just put this is what. So, I am just bothered about this path. I am not bothered about the function elsewhere because then I can simply write this as, limit n tends to infinity, which is same as telling that non-pi n goes to 0. This one now how do I look at it, because once you have some information that this has a derivative and you want to have some information about the function always use the mean value theorem that is the biggest trick and a simplest trick in analysis.

So, what you are doing. When you are taking the difference of this so by mean value theorem there would be some xj star okay into tj+1 -tj, where xj star is an element in the open interval tj+1, tj, not the closed interval. So, this is nothing but f dash x-star j, but if you look at it this is nothing but, a remand sum for the function mod fj mod f sorry. So, it is a remand sum for the function mod f and this is nothing but the integral from 0 to t, sorry not infinity, n-1, so it is nothing but the integral by very definition of integral from 0 to t of mod f dash t dt.

So, this is what is the meaning of first variation which is pretty natural. So, let us come to the definition of the second variation. What is the meaning of the second variation of a function? Of course, you will do a little bit of jugglery I am just removing this part. So let us look at the second variation or the quadratic variation.

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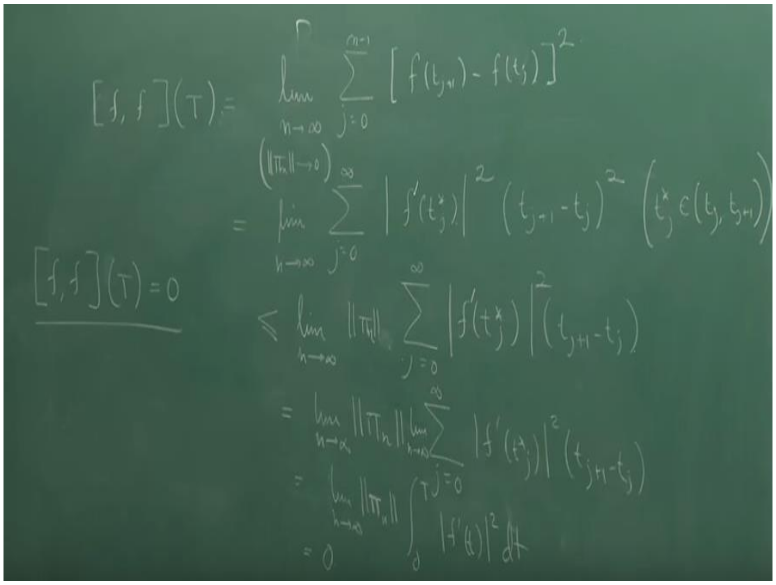

So, quadratic variation is usually symbolized like this and this is written as, limit n tends to infinity. Limit n tends to infinity is same as writing norm of pi n tending to 0 okay the same thing, either we write that or we write that. So, as n goes to infinity, so j is equal to 0 to n-1. Instead of having the modulus, the absolute value of the difference.

So, what does the first variation do. It counts the total number of times the function goes up and down up and down up and down. That is what it is counting, it is counting the variation. From here say if this is positive and this function value is say lesser than this it goes down and then possibly it can go up so it is counting the up and down total up and down the crossings.

So, here I will write the same thing but instead of looking at the variation in terms of the absolute value, I am doing in terms of the square, something like the mean square error type of thing that you must have heard if you have learnt some statistics. So, this is summation. Now I can here write the same thing, but I have written x-star j where x-star j is here anyway, I will not write x star j, I will write say f of t star j whole square into tj+1 -tj whole square okay.

Now if I write this tj+1 -tj into tj+1 -tj. This breaks the square into two parts. The product of the two similar parts. Then one of them is obviously less than equal to this pi n. This pi n is the maximum of the interval size. So I can write this thing of course as, less than equal to limit n tends to infinity, norm of pi n that is, into summation j is equal to 1 to infinity

So, here again I have applied the mean value theorem if you have not observed that I have done the same thing only instead of x-star j I have put t star j x t star j is an element between tj sorry, I made a mistake here, it should be tj and tj+1 tj+1 is bigger than tj, please correct that tj to tj+1.

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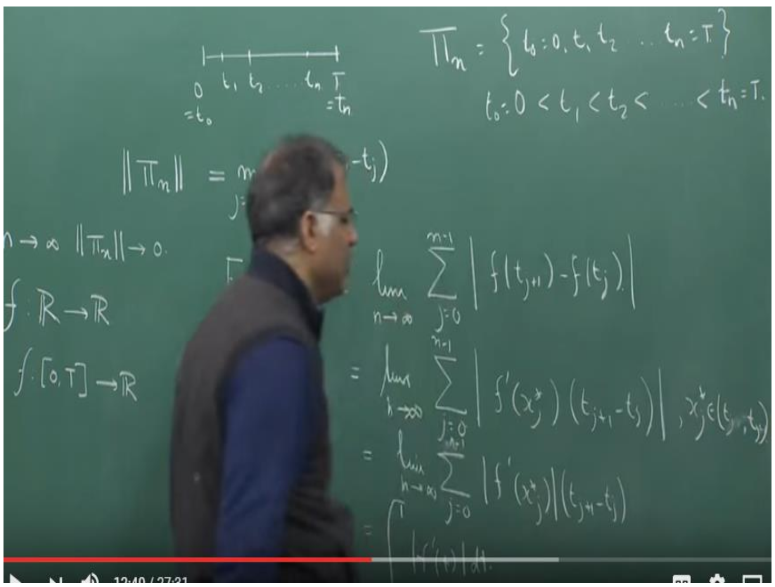

So, it is lying in the open interval tj and tj+1. So, basic mean value theorem (()) 13:02 mean value theorem basically. Now, you will write f of f, sorry f dash, so here we observe this f dash. So, we will have f dash of star j t star j into t j+1 -tj. Now limit of this exist, limit of pi n. Limit of non-pi n exist this is 0, and limit of this also exists, because these are nothing but the integral of the mod square. So, what you will have.

Of course, because I have taken f to be a continuously differential function its square is also continuously differentially function and hence it is integrable, basically in the Newtonian sense and hence in the (()) 13:49 sense.

So, this means, this is nothing but, limit of both of them exists. Limit n tends to infinity so this is exactly equal to non-pi n, so here also I have to write limit n tends to infinity, f dash t star j whole square tj+1 -tj. So, this is nothing but limit of n tends to infinity non-pi n and non-big pi n and this one is nothing but integral 0 to t, f dash t whole square dt.

So, this is a finite number because this is a integrable function. So, this is nothing but, that has to be one has to make sure that this is finite. So, because of our assumption this is finite this limit is going to 0 so this whole limit goes to 0. So, what have you proved. We have proved the following that for a continuously differential function f defined from on the interval 0 to T ,it does not accumulate any quadratic variation.

The square of the ups and downs the total accumulation of the ups and downs up but the square of the values is added up and that variation is 0. That is very very interesting because that does not happen with this one but it does not happen with this but you will see immediately that the behavior of the, you can immediately understand also the behavior of the Brownian motion is so different because it accumulates quadratic variation while a smooth nice function does not accumulate any quadratic variation.

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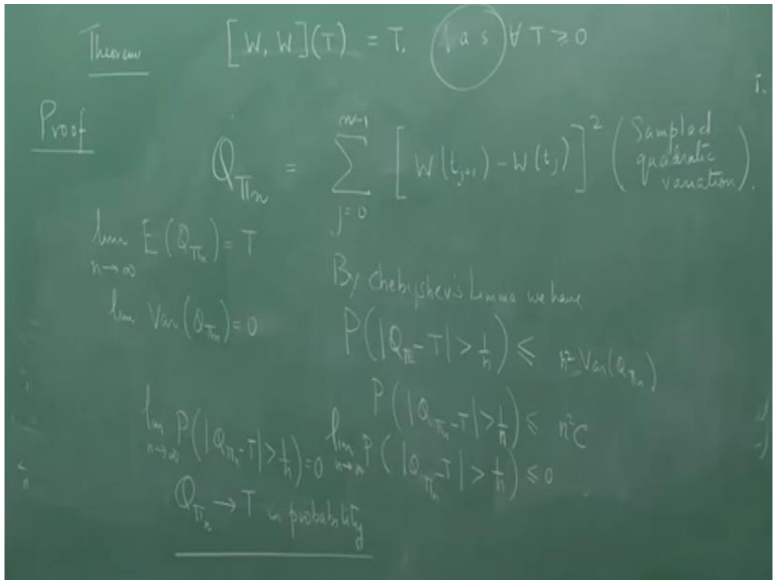

So how do I go about proving that the Brownian motion actually accommodates some quadratic variation. So, we will just prove this fact about Brownian motion. So, the theorem W W T is equal to T almost surely for t greater than equal to 0. Now it is very important to note that, this quadratic variation here is actually a random variable, because W itself is a random variable, is a function but is a random variable so when you make these sorts of variations you are actually getting random variables.

So, your convergence is in the almost surely sense and that is we are going to show and you will see a fantastic application of Borel-Cantelli lemma and a fantastic application of the Chebyshev’s lemma. That is what makes mathematics beautiful actually.

So, when you deal with random stuff always be sure that, this guy would be always around you, almost surely this term, which means same as almost everywhere, this term would be always around you. So, let us try to prove this fact. This is known, this is very-very much a useful thing in finance. We also at the end write down some shorthand, which will be used in finance when we do the Ito calculus.

So again, you do the same sort of partition. Partition in the interval, portioning is something I am not going to write, because I have already written how I am partitioning the interval, what is your big pi n, what is the meaning of non, non-pi n is going to 0 as it goes to infinity.

So, I am going to define a variation, total quadratic variation on a given partition the nth partition, which as before, would be, when j is equal to 0 to n W T j+1 minus W T j. This is sometime called sample quadratic variation.

So, when you are taking about this sample quadratic variation, you have to observe that, this Q pi n is actually a random variable, because this is random variable, I can put a particular scenario omega and so you will have a qn omega, that will give you a particular value of this, for the same partition. Now if you change the scenario omega to another omega 1, then for the same partition it will give you a different value.

Now, what we are going to prove first is an amazing thing. Let me first tell you, what we are going to prove. We will first going to prove that, limit n tends to infinity, expected value of Q pi n that value is T. This thing will allow us to prove that, and we will prove that, limit n tends to infinity the variance of Q pi n is 0.

Suppose, I have proved this, basically what I am telling, that if you average over all the possible parts omega, then you essentially get the value of the quadratic variation, that is the meaning. You take this and average over all the possible parts then you get the value of the quadratic variation, which I claim to be T.

Now how do I actually say that, this Q pi n goes to T, almost surely and here you will see the use of Chebyshev's lemma. We will prove this later on, if we have time, but let me, once I know this how will I use this, Chebyshev's lemma and the Borel-Cantelli lemma. Actually, I will use this idea.

So, now my Chebyshev's lemma what will we have, if we have this information then by Chebyshev's lemma we have, this minus T sorry, Q pi n I should write, minus t bigger than n, this one is strictly less than 1 by n square variance of Q pi n. So, variance of Q pi n that been bounded and that that goes to 0 and hence it is bounded. I can still write this to be, sorry, it has to be 1 by n. So, this will become n square. That will be n square into C.

So, what does this tell me. It tells me that, limit n tends to infinity probability of mod Q n, sorry Q pi n minus T strictly bigger than 1 by n is less than equal to 0, but probability is never a negative quantity, so limit n tends to infinity bigger than 1 by n.

So, it tells me that as n becomes very large, then the set of all omegas for, which this is true is, as measured 0, which says that Q pi n converges to T in probability. So, if it converges to T in probability, does it converges to T almost surely. Let us look back at the literature. So, if you go back to our very old lectures on the Borel-Cantelli lemma section, then you can see that, we have spoken about this, so if xk converges to x almost surely, we have proved that it is Q pi n goes to T in probability okay.

Now does it mean that, Q pi n will go to T almost surely. This is not a clear thing that it will go to Q n, will go to T almost surely, but that is what essentially you have to prove. So, we have come to this conclusion immediately by applying Chebyshev's lemma and this information. So, what we have that Q pi n goes to T.

Now we can use some other techniques some more involved techniques to actually show, that Q pi n also goes to T almost surely, which we will put up in the notes, in your, what is that called in our forum because that will take too much of time to describe. So, just by applying the Chebyshev's lemma by noting this notion we have proved that, it goes to Q n converges to T in probability. So, what we have to show that Q n converges to T almost surely.

So, this is something which is quite involved which we will not do in the lecture, but we will describe it in the class, so hence we have come to the following conclusion. That this is equal to t. You have obviously showed that this is equal to T in probability, but this is equal to T also almost surely. That thing needs to be proved. That needs a little rewriting of this Q pi n and needs a little bit of, you know tweeting the whole thing so that we can apply Chebyshev's lemma and Borel-Cantelli lemma both to come to this conclusion.

So, we will take a little bit of, it will be a little more complex thing, so we will not do it now. So, what we have proved here that if I can this is easy to prove, this will take some time to prove which we will not again prove it. So, we can assume that this if you agree to these two things this is of course can be proved very easily you can prove it yourself and this is take time then by applying Chebyshev's lemma.

You can at least prove that Q pi n converges to T in probability but proving that it also converges to T almost surely needs a machinery needs some more machinery which we will not get into the class but we will put it on the forum.
Thank you very much.