Probability and Stochastics for finance
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Lecture - 01
Basic Probability

Welcome to this course on probability and stochastics for finance. I will not start by just blabbering some technical things at you. I want to tell you that, everybody nowadays seems to be interested in finance and I do not find the reason, why everybody should be interested in finance. Obviously, people are always interested in making some easy bucks, but let me tell you, it is not so easy, to make easy buck in a financial market. Of course, there are lot of stories around how people like, Warren Buffet has done it or people like x and y and z has done it.

But those are all exceptions and exceptions do not prove the rule, or rather exceptions are proving the rule that it is not so easy to make money in the stock market. Because the financial market is filled with uncertainties and unless you really known the language by which uncertainty is understood by human beings. There is no way, that you can have a better understanding of financial market or whether you actually get expose yourself to the risk of investing in a financial market thus, rather than investing in something like a fixed deposit in a bank.

So, before we do so, before we get into, actual issues of finance, we need to know the language by which uncertainty is quantified, specifically those tools from the theory of probability and stochastics which are actually needed to study finance. Now, without understanding probability, it is almost no chance for anyone to truly understand the intricacies of the financial market, the financial products and how to price them.

So, before we, this is a two-part course, the first part is on probability and stochastics, second part is on the issues of pricing of financial commodities. Now let me, start by talking about probability. What is probability?

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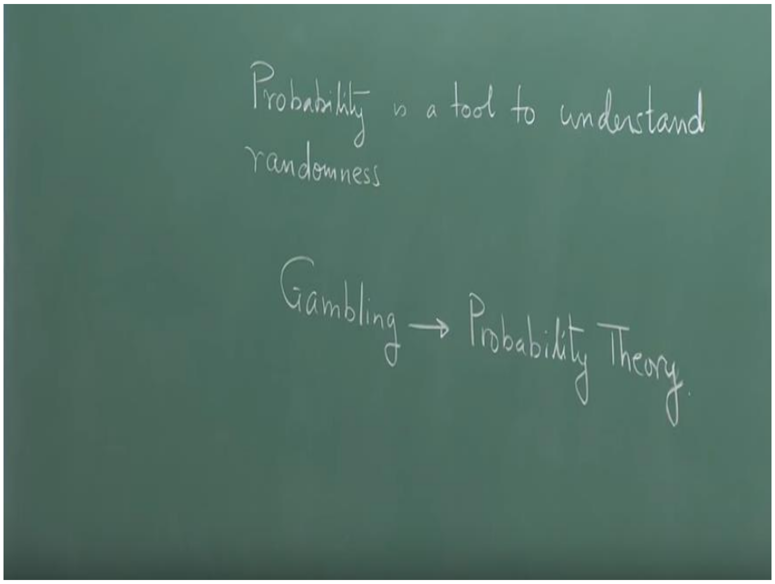

So, probability is a tool to understand randomness. So, I am writing in the board, rather than on the screen, so that as you listen to the lecture it will be advisable for you to makes the notes yourself. So, what is randomness? It is a very difficult thing. It is so difficult to really understand what randomness is. Randomness in a much more crude way, we can say that, something is random if we have no prior knowledge of what outcome would certain activity give.

For example, if you toss a coin, you do not know whether it will come out heads or tails, but you are sure that only that there are only two possibilities if you toss a coin, either it comes on head or comes out to be tail. Now the real mathematical ideas of probability came out by looking at the problems of gambling, so it is gambling to probability theory. So, probability theory did came out of a wise, rather than some virtue.

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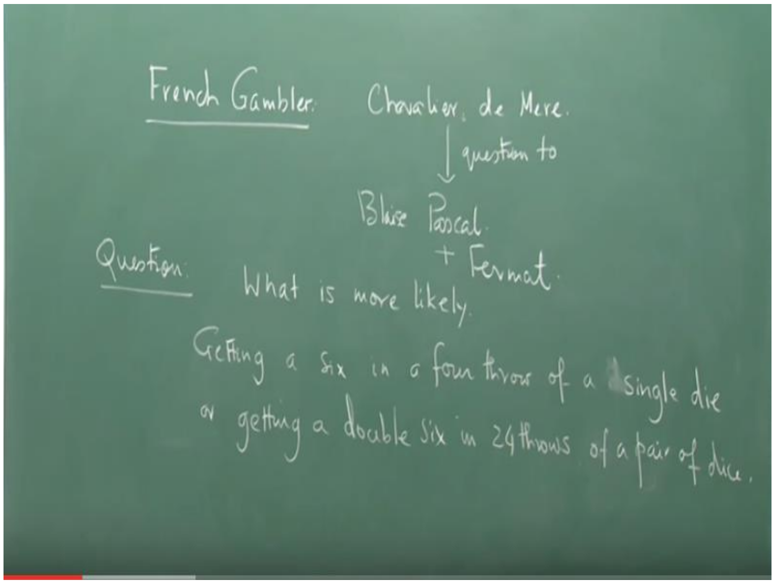

There was a very famous French gambler by the name of Chevalier de Mere. He had a mathematician friend called Blaise Pascal. So, Chevalier posed the question, following question to Blaise Pascal. Most of you have heard of about Pascal's triangle, when you studied binomial theorem. So, he posed the following question to Blaise Pascal. So, what was his question?

So, they were gambling. So, in those days, gambling were done through throwing out dice rather than nowadays the sophisticated (( )) 5.15. So, the question was the following. The question was that, what is more likely, getting 6 in a 4 throw of a single die. So, if you just throw 1 die 4 times, what is the probability of getting 6 or getting a double 6 in 24 throws; of a pair of dice. Okay, so you throw, take 2 dice and throw, we will do the experiments in the next class.

So, which is more likely. So, Blaise Pascal who had denounced mathematics has a bad form of perversion at that time. He had then got back to this question and in between him and Pierre de Fermat, who known for his famous Fermat's Last Theorem, they figured out what finally gave rise to the modern theory of probability. So, it is Blaise Pascal plus Pierre de Fermat. Now we will answer this question. In the next class, we will do the experiments or I will have my TA doing the experiment.

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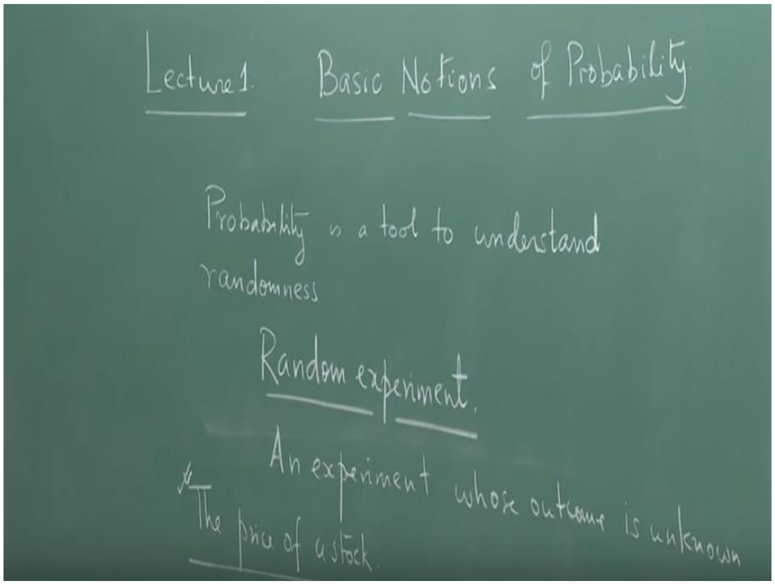

So, in order to more technically study probability, we should first study, what is called a random experiment. So, what is a random experiment? So, random experiment is a, experiment whose outcome is not known to us. Just like, throwing a die or pair of die or whether I will be involved in a car accident in the next 1 hour. In fact, I am going to drive in the next 1 hour, so it is really a fact, it is a question which, to which I also do not have an answer. I cannot say, it is half, I maybe or may not be, it depending on my driving experience, it is the way I am driving, my probability of my getting involved in accident will change.

This way of changing probability based on more information is what is called the Bayesian ideas, is now very important in mainline statistics, but we will not get into that right now. So, Random experiment, an experiment whose outcome is unknown. I am not writing down the tossing of coin or throwing of a die, I am telling it just repeatedly.

Another important thing is that, if you are in a stock market. You known the stock price today, you see it on the screen, you do not know the stock price after two hours or at the end of the day. So, the price of stocks, is random quantity. So, these are examples. So, price of a stock, so this is an example of a random experiment.

Suppose I am just looking at price of a stock every day, so I will not get the same price and so and I do not know what is the next day’s price. Possibly there is a price range within which it will vary, but I do not know what is the next day's price. So, I cannot predict before and exactly this will be the next day's price. So, the price of a stock in a stock market is actually, is a form of random experiment, if I want to look at it like that. So, this is something which is random quantity and hence has to be guided by the laws of probability to which we will soon start talking about.

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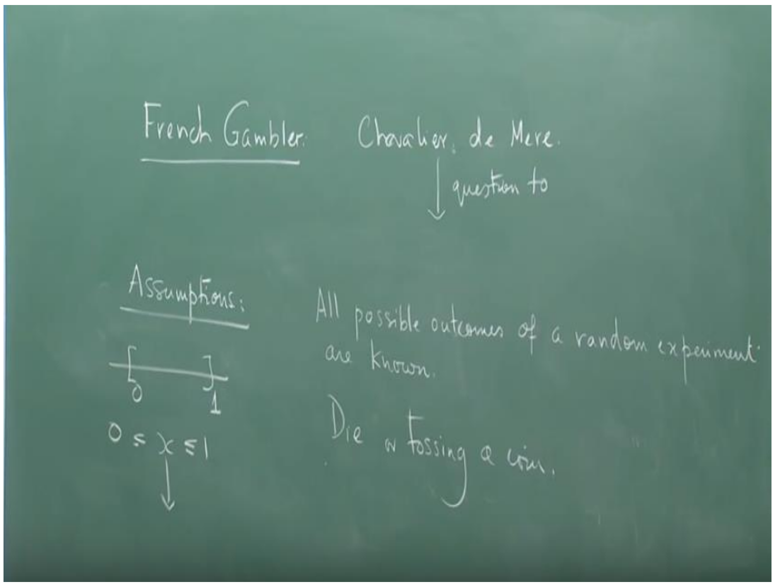

So, in a random experiment, what is known is that, I would at least have an idea of all the possible outcomes. All possible outcomes of a random experiment are known, that is something which is an assumption we always assume. Without making some basic assumptions which looks natural, science cannot proceed, so the assumptions is the following, that all possible outcomes, all possible outcomes of a random experiment are known.

For example, if I throw a die, I know there are only 6 possible outcomes on the face, there will be either dot 1, dot 2, 3 dots, 4 dots, 5 dots, or 6 dots; there cannot be anything else. So, the number of possible outcomes could be finite or could be infinite. If I am telling that, okay I am choosing 1 point randomly from the line 0 on 1, I am choosing 1 real number out of the interval 0 1, then my all possible outcomes is any number between 0 and 1.

So, any x between 0 and 1 is a possible outcome. Again, I have to draw any number out so you have infinite possibilities, but here for throwing of a die or tossing coin, I have finite possibilities once it is done.

Let us go to very basic definition of probability, that you even learn in your high school. So, let us see, what are the plus point of this definition and what are the minus point of this definition. And why we need to shed this definition and go to an (()) 12:58 formalization of probability by understanding that we have some idea of what probability means in some.

It is something like a proportion, of occurrence of certain things, among a universe or certain things, but we do not really have a full and give a true definition of what probability is, so we basically make some rules of the game and we play the game by those rules. (()) 13:23 or trying to define certain rules, has been the standard in mathematics, since Euclid wrote his 13 Books of Elements.

You studied Euclidean's geometry at school and possibly, do not know about all these things, but it is very important that, as you learn more mathematics, you have to start appreciating that, many of the very basic things cannot be defined, just like a set cannot be defined, probability cannot be defined. It has to have a basic (()) 13:49 framework and once you accept that (()) 13:51 framework lot of other things would follow in a natural way.

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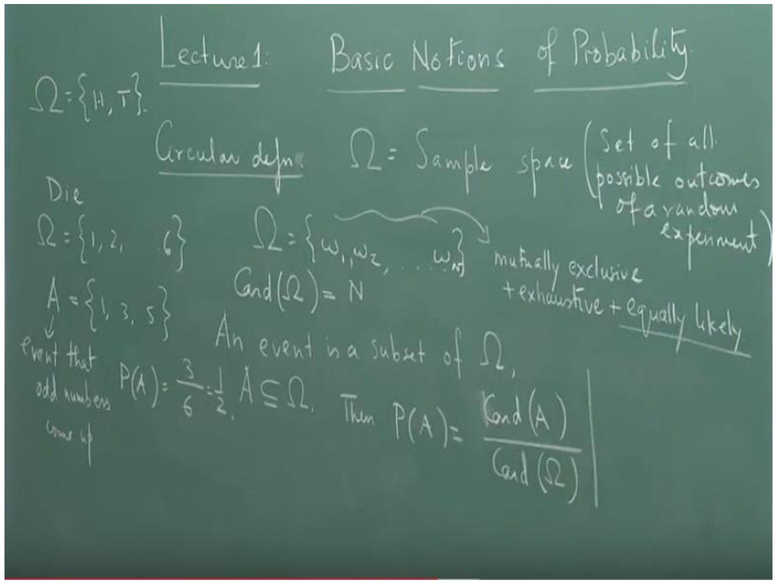

So, the circular definition, that is taught to all kids in the high school, what does it say? So, the set of all possible outcomes of a random experiment is collected in a set, which is called the sample space. So, in order to start any discussion about probability, we have to talk about the sample space, is a set of all outcomes, all possible outcomes.

Now, this set we can write something like, okay we are not, okay once I write like this, I am basically assuming, that I possibly think that, there are countable number of elements or infinite or finite number of elements, but okay let us just write it, for the heck of it, for the moment. So, it is assumed that, so these are outcomes of a random experiment which we have collected.

So, it consists of outcomes which are mutually exclusive, exhaustive, and equally likely that is the definition. So, these omega 1, omega 2, omega 3, omega 4, which are the outcomes of random experiment, they are mutually exclusive; means if omega 1 takes place omega 2 does not. If omega 3 takes place omega 4 does not and so and so forth.

Exhaustive means, these set of outcomes, are the all possible outcomes that you can have in the random experiment, like head and tail. If you take the tossing of a coin, then your sample space would be written like this head or tail. Now, suppose I just have finite number of them and then I talk about the cardinality of this which is N, N objects. And even it is defined, to be a collection of some sort of a subset of these outcomes and even if it is a subset of these outcome, for example I say what is the occurrence of odd numbers when I throw a die, so it can be 1, it can be 3, or it can be 5; so, all these 1, 3, 5, they come collected in a set.

So, for this moment, an event, because now I just considered finite number of them, an event is a subset of, consider A to be a subset of omega, then the probability of A, asks you, how much proportion of A is in this big omega, so you write, this is nothing but the cardinality means the number of elements in A. Of course, this is finite.

One immediate thing you realize, that this quantity is greater than equal to 0 but lesser than equal to 1, because A being a subset and this is a finite set A would have lesser elements than omega has. This is a very useful definition. This gives you a very intuitive idea of, what probability is and it allows you to actually compute probability in most cases, but there are certain flaws in this definition.

The first flaw, is that we assume or we mention the term, which I will now write again, that all these events, these outcomes, are mutually exclusive plus exhaustive, but they are the only ones, plus equally likely; equally, likely simply means that, you are assuming that all of them can occur with, so you have actually assumed basically, a definition the idea of probability within the definition of probability. So, you cannot consider this as a fool proof definition of probability.

The second point is that, it this is alright, when you are talking about a finite sample space, but it is not alright, when you talk about infinite sample space. Now suppose I throw a pair of die, and say, what is the, throw just 1 die and I say what is the probability that I find an odd number on it.

So, I know that for a die, the sample space is 1, 2, 3, 4, 5, 6 and A is the event that odd numbers turn up and if this is my event then I have 1, 3, and 5. So, according to this definition, the probability of this event A is 3 by 6, which is half, which also looks natural because they are in equal proportions the even and odd numbers here.

Now, so this definition cannot be taken as a standard definition because it has a circularity, because it is assuming the notion of probability, within its description and also it cannot handle the case, when I am talking about infinite sample spaces.

So, we have to go to an (()) 20:50 formalism, which was given by Kolmogorov in the 1940s. Kolmogorov the greatest Russian mathematicians, of all times, or rather one of the greatest mathematicians ever in this world. So, let us write down, so we are now writing the Kolmogorov's Axioms for probability.

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Tomorrow, or rather in the next class, we will first try to give an example, as to why this definition of probability does not work, which is, which will be called the Bertrand paradox. In that example, we will not give it now, we will tomorrow, in the next class, we will give you, we will do some examples.

What we will show is that, if we consider this definition of probability and in the Bertrand paradox, the same question will have two different probabilities, so which will simply say that this definition is not so useful. So, we will not get into doing Bertrand paradox right now.

But assuming, the difficulties that lie in here, first I will list down the Kolmogorov adjuncts of probability, that I have. To understand that, I understand probability is some sort of a proportion and then I need to go on about it. For example, if I take a coin, you can do the experiment at home and toss it 100 times and note down, how many times head and tail have come, then their occurrence is almost same; the number of time head comes and number of time tails comes is almost same.

The important issue, is that. When I am taking into account a sample space, which can have infinite outcomes, then the whole notion of events, the idea of events change, that every subset of such a, infinite set of outcomes, need not become an event.

This is a deep result from measured theory, because probability is viewed as some sort of a measure, so like measuring a length of something, but I am not going to stress on that. But certain things you accept from me just on faith for the time being, you can read up yourself but just for the heck of the course just for the faith you just accept that, if you have infinite number of sets, all possible subsets are not cannot be considered as events. But if you have finite number of sets like you have it here all possible subsets can be considered as events.

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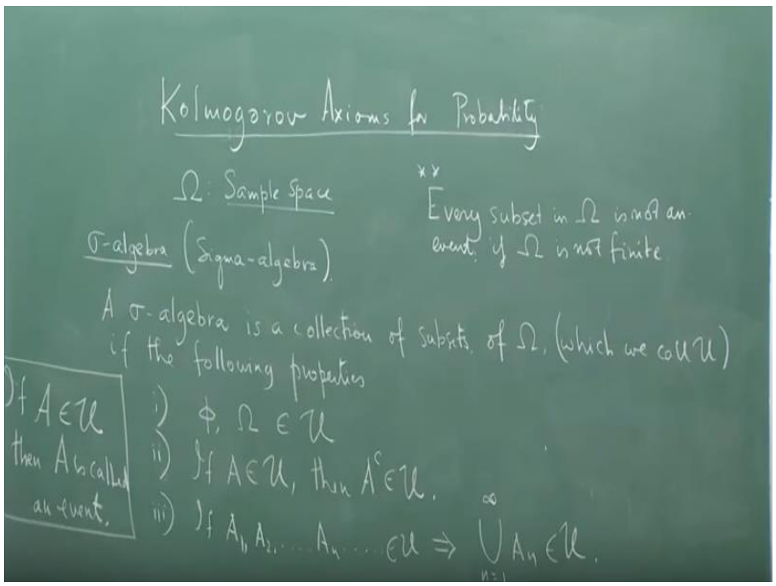

So, any discussion we will start with a sample space and to be more mathematically erudite, we will start with the definition of a Sigma-algebra, this is a sign called sigma sign in Greek. If you want to write in English, probably you can also call it sigma field and I being truly an outsider to the field of probability. I would like to continue, what my math colleagues call it, as sigma algebra.

So, sigma algebra is a collection of subsets of omega, which we call U. If the following properties hold. Actually, those subsets, which follow these properties, which we will list down can be called as events. The empty set and the whole sample space, is an element of that collection of U. U is a set consisting of sets; it is a collection of subsets of the set omega.

If A is element of U, then A compliment is also an element of U, means if an event A, if A is an event then its nonoccurrence is also an event. This is very natural way of looking at events. And the third which might look slightly counter intuitive, but that also is very important that, if you have a countably infinite sequence of events, then it implies and if all of them are element of U, so, if all of these are element of U, then it implies that if it is the union of this, set union, the infinite union, that would also belong to U.

But every such, every subset of this need not follow this property. If A, a subset, is a member of the Sigma-algebra U, then A is called an event, then A is called an event. So, this is something you have to keep in mind as you work on, that every element in U, so this is something very important, every subset in omega is not an event, if this is not finite. If it is finite, then it is an event then every subset is an event. If it is not finite then, finite means, it does not have finite number of elements, then you cannot say that this is an event.

Now, let us write down the Kolmogorov's laws, which are also very important. So, this is, so I am now defining a function which maps an event to the set the number between 0 and 1, because from this very crude definition.

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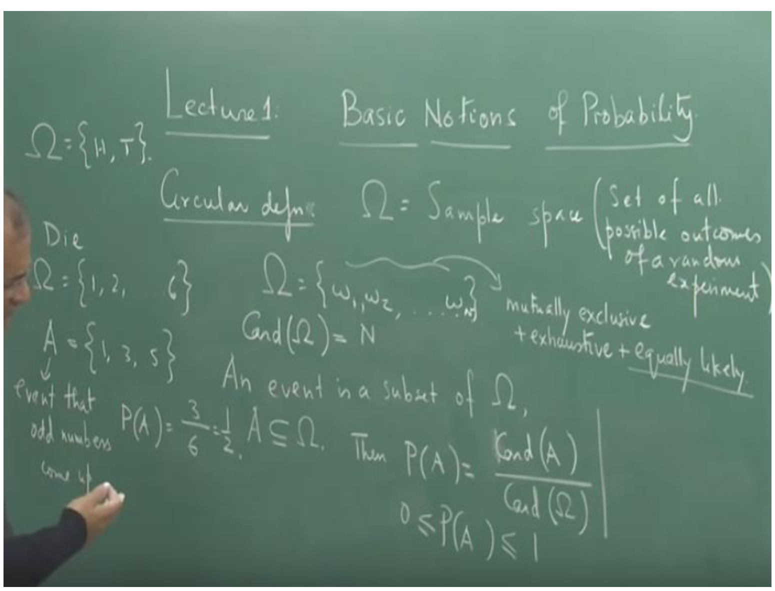

You can immediately see that, probability of any event A or at least finite event, when A comes out of a finite sample space has to lie between 1 and 0. So, P, the probability can be viewed itself as a function which is taking a set and putting that set, mapping that set, to a number between 0 and 1. So, that basic idea is already ingrained from the circular or not so good definition of probability.

So, once taking the clue from this, or rather a rough or a crude definition. We can now formulate, what are known in the literature or known to probability says the Kolmogorov axiom’s. Okay, we should have put it that side but does not matter, I will just write down the axioms.

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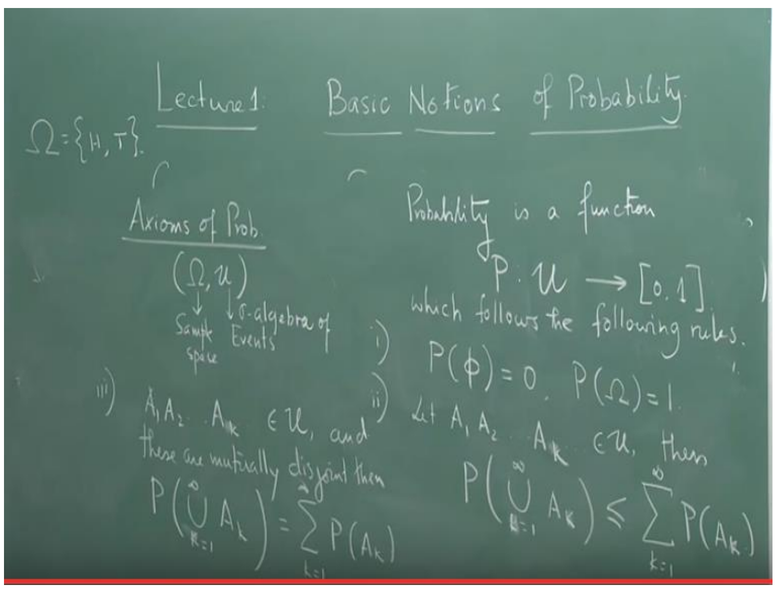

Axioms are probability. Sometimes I will write por or prob, prob for short because I am sure most of you are familiar with the word probs and stats. So, with this young generation I think in the SMS generation, we can always use short forms.

So, once you have a probability, whenever try to talk about probability. So, you have a sample space and a Sigma-algebra of events associated with it. So, to begin with, you should have a sample space and you should have a Sigma-algebra of events, so this is given to you. This has to be known, without this nothing can be done. So, probability is a function P, which is, whose domain is U the Sigma-algebra. So, it takes an event, from the Sigma-algebra and maps it to some number between 0 and 1 which follows the following norms.

So, what are the following rules? Number 1, sorry, I think first we have to because, phi and this, phi or the null set is also called the null event impossible event, like I will live forever is an impossible event and I will die one day, is a sure event. So, this is this also comes out of from the crude definition this idea, but since we cannot use the crude definition for collection of infinite sets, infinite sample space, so we give them as rules.

Let, these are all elements of U, then probability of the union, this is we are just putting additional union, we did not put it, we can actually prove it, prove this value but okay forget it. Sorry, I should write k here, this is the second law.

And the third law, which is the more interesting one, that if you have Ak, I should write here, it is much more meaningful, is a element of U and so you are having a countable collection of sets, note these are not infinite collection of subsets or not arbitrary collection, but a countable collection and Ak element of U and these are mutually disjoint, then probability of the union, union of these countable collection is nothing but the sum of the probability.

So, this probability measure P, we call it a probability measure, this probability measure P, this sample space and U, all form what is called is should write it here the probability space.

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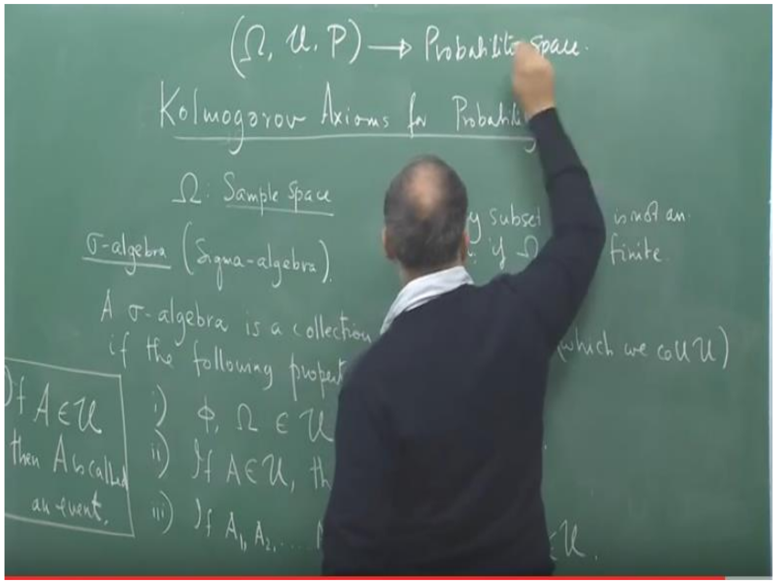

So, if you can have a sample space, a Sigma-algebra of events and a probability measure on the Sigma-algebra of events then this forms what is probability space. So, before we start talking about and we want to discuss or compute the probability of certain events, we have to be sure, what sort of probability space we are considering for that random experiment. Those are the things will come in the next lecture.

So, what does it tell you? It gives you certain, it takes ideas from that crude definition and puts them in form of laws, so that even if you are talking about infinite sample space, any measure or any function which takes events to this interval 0 and 1 and satisfies all these rules we will call it a probability measure on that.

We will soon come to its utility, very soon, that this sort of definitions are useful, and we will talk about Bertrand paradox in the next class and also next class we will do two problems. This Bertrand paradox will solve the Chevalier de Mere's question to Pascal. We will talk about the Buffon's needle problem, which actually I had been used to estimate pie by D' Alembert and we will also then talk about how to construct sample spaces.

When you are doing, no it is not sample space, how do you construct the probability spaces. If you know a sample space and if you can construct the algebra of events, Sigma-algebra of event, how do you, define this measure so that a probability space can be constructed.

See the term Sigma-algebra, why it is called Sigma-algebra because if I just had finite number of them and this had been occurring only for a finite number of events, then we would have called it just an algebra, it is called a Sigma-algebra because we are talking about a countable collection. That is why the term infinite sum, actually sounds sense that is why the term sigma is used so with this very basic idea. I would stop today and then we will go to the next class where we will do this interesting sort of problem solving.

Thank you very much.