Basic Calculus for Engineers, Scientists and Economists

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Lecture - 09
Derivative-2

Today, we are going to talk about Derivative-2 means second part of the derivative. So, first we are going to talk about the rules about computing with derivatives. Then we are going to talk about the famous chain rule, which is among them.

And then we show a very useful application of that rather we will try to correct one misnomer, which usually is done by calculus student. And then we are going to talk about the second derivative and higher derivatives.

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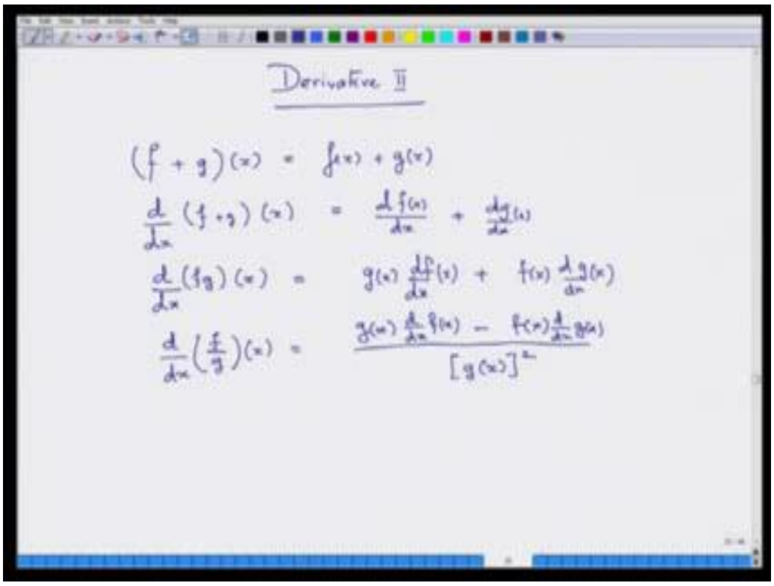

If we have a function f a near function g, so by a function f plus g, the added function if we evaluate at x it is nothing but f(x) plus g(x). The question is what is the meaning of the d dx of f plus g at a given point x. Even simply proved even from the first principle the limit the definition that this is nothing but d dx of the f(x) plus d dx of g(x) or dg dx you can write, whichever way you thing it is comfortable to you. Now, what about subtraction of course, you replace plus with minus, about curious thing what is the meaning of this.

Here again if you use the definition I think you should everyone who has forgotten is high school calculus should make a little trial of this. It is always good to do some exercise by yourself. Then this would be nothing but g of x into d dx of f at x plus f(x) plus d dx of g at x that would be the answer.

This is a well-known fact and of course, we need to talk about this, of course, g(x) here cannot be equal to 0 any time. If this is the case again, the first principle will show that this is what you have, you will have one should be very careful in this g(x) into the derivative of f minus f of x into the derivative of g, this will become helpful in many times. The key rule that I want to mention here is the chain rule.

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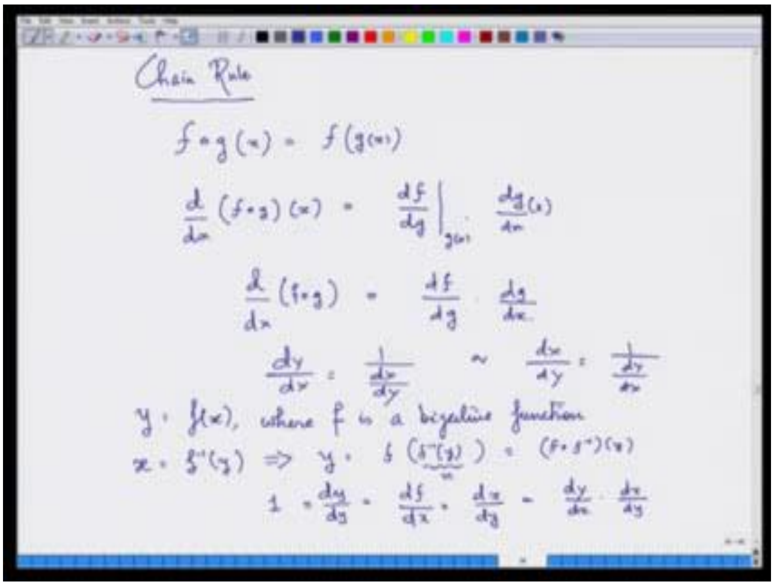

Now, suppose you have function f you write function f compose g evaluate at x, I already spoken do you award this composition of functions this can be viewed as f, g first computing g(x), g first operator on x then f is operator on g(x). Now the question is what is the meaning there now, d dx of f compose g(x). This is first, f here first looks like a function, whose independent variable is the function g the value of g(x) the function g itself is the independent variable here.

Basically, then you have to derive take derive do it in two stage, take the derivative f with respect to g. I am writing x means to say that I am evaluating it at point x, if you do not want to write the x that that is also is fine. So, you can write just d dx of f compose g. So, g d dx of g now of course, the d f dg evaluated at the point g(x), right that is more important, into dg d of x evaluated at the point x. So, even write this in a more operand shorthand that we use.

Now, I have often seen calculus, students do the following mistake. So, often they, often write dy dx as 1 by dx dy or dx dy as 1 by dy dx, when I have ask some as for my experience goes, when I ask certain students they simply say that is ok, it is so obvious, it should be like that. But is not so obvious, it is not you are not inverting fractions, you are not inverting numbers. It has one has to remember that here; these are operators you cannot invert them as numbers.

So, how do you get this thing this comes from the chain rule. Now consider a function y is equal to f (x), where f is a bijective function means it has an inverse. So, which means x can be written as f inverse of y, which would imply that I can write y as f of f inverse of y that is. There is 1 y into 1 x. I bring the y back to x and then you can if I apply the f it will take me to y. That is exactly what is happening. It have the function has we absolutely bijective 1 to 1.

If these happens, now what is f inverse y it is actually x. So, what is now, dy of dy, now you know dy of dy is 1. This can be obtained by the first principle that is just a compute two definition the limit, which is obvious at least if it is very simple you can just check it out. If I tell you this, then this will in basically telling letting you basically would we like to teach a b c d. Now, what I need to do is (Refer Time: 07:31) how do I apply the chain rule in this case. This is basically you can write this as f composed f inverse of y. So, how do i apply the chain rule? So, f is first viewed as a function of f inverse y, but f inverse y it itself x. So, basically this is this can be viewed as x.

So, what you are doing your first taking the del f of del if inverse, which is del f of del x into the chain rule del x. This is f inverse recognize function y. So, f is actually your x basically is your y. I can write instead del f del x is same as del y del x, because y is equal to f (x) this is. So, what is this product of this to derivatives, these are of course, numbers dy dx itself is a number, dy is not a number, dx is not a number. This product is equal to 1 and then of course, you can write that dy dx is 1 by dx dy 1 has to remember this is only possible in this you just know in y is a invertible function if is y is not invertible function you cannot write such stories.

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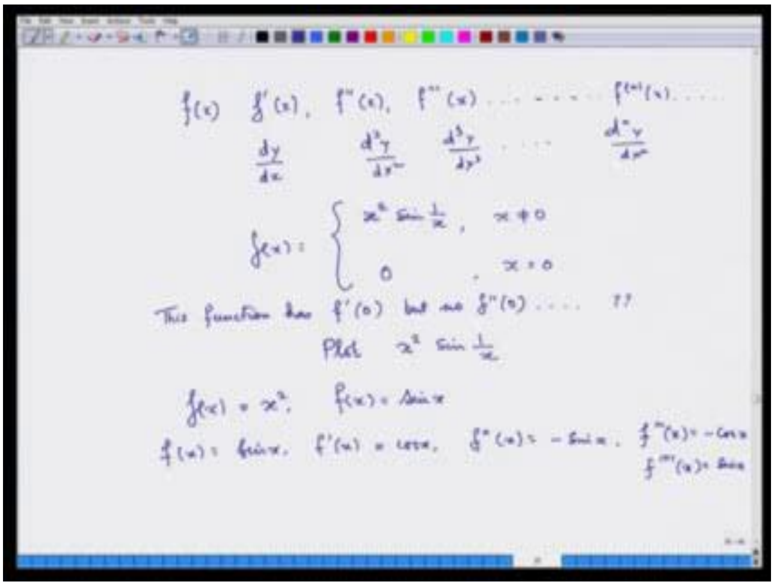

If you are can take the derivative of f, if it is differentiable, then you can take the derivative of f 1 more third first time, second time, third time and so on. For example, if you are f of x then you will take the first derivative and suppose you can take the second derivative, you can take the third derivative and do like this. Now this is of course, written as dy dx or del f del x does not matter, whatever you want to write I am just writing as dy dx. This is written as d 2 y dx 2 second order derivative, third order derivative is written as d 3 y, y is f actually dx 3 and.

So, order, I can have the any thought derivative d n y dx n, even compute this by the same formula called (Refer Time: 10:00) formula, which we are now going to be discuss here. It might be, for example, it might be possible that there is a function, which is differentiable at 1 differentiable first, what not differential for the second time.

For example, if I write a function like this, which is, x is square sin one by x where x is not equal to 0 and is equal to 0. When x is equal to 0, then this I would like you to try out at home this is very important. This function has f dash 0, but no f dash double 0 check, whether what I have told is true is very important that those, whose have access to internet computer, etcetera or MATLAB or Mathematica they should try plot. This can we consider as one home work for those who are just enrolled in the course may not take exam, but they can just try out for fun.

Now, let us look into this whole issue once again. So, you can have a function can have derivatives nth time and all for example, f(x) equal to x square, f(x) is equal to sin x. For example, let us take f(x) equal to sin x, f dash x is cos x, f double dash x is minus sin x. So, f treble dash x is again minus cos x and f four times x is sin x. So, you see it has a pattern which emerges and you can actually I will like you to write down the general formula in this case. So, f(x) that is a little bit of trial and error in mathematics. So, f(x) is equal to sin x, f dash x is equal to cos x, f double dash x equal to minus sin x, x triple dash x equal to minus cos x, if four dash x equal to sin x. Now one this is done.

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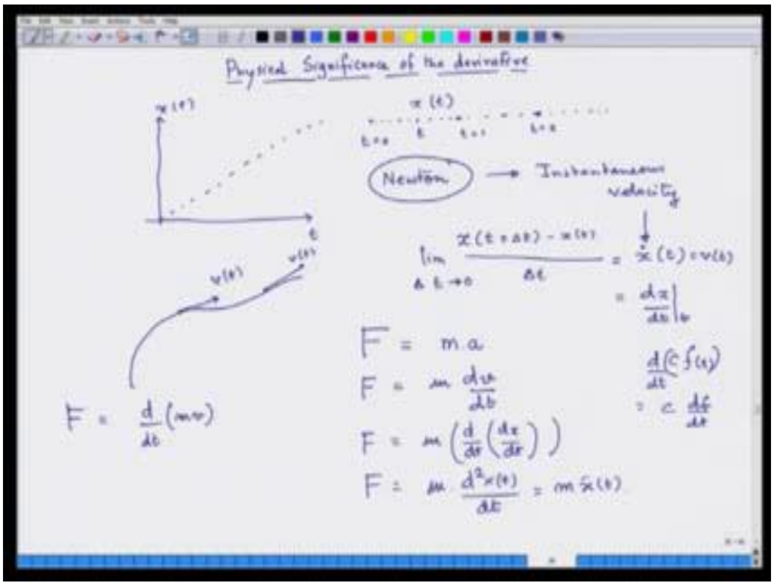

Let us talk about physical significance of the derivative. A physical significance of the derivative a physical significance of the derivative now suppose you have a particle which is moving may be in a straight line does not matter or may be at every given time you note this position. So, may be the particle is actually moving starting from a time t equal to 0 of free particle is moving around a straight line at every time t equal to 0, 1 t equal to 2, etcetera 1 or 2, 1 side, 2 side does not matter. At some time t we are noting its position. So, at any time t is position is been given by x of t. So, you can have t an x(t). I can start with 0 and it can just go up like this or it could be some curve does not matter.

Now, what is the meaning of velocity? At a given point that was one of the very important questions that was first faced by Newton while developing mechanics. It was Newton who brought in the idea of instantaneous velocity. I can tell you that, whenever I thing of this term instantaneous velocity means velocity of particle.

For example, a particle is moving, a particle is moving across, a particle is moving and then this is a particle; which is moving let the pen is moving and then at any point a suddenly at to one particular intent you want to know the velocity. Of course, you know velocity is a vector, which is it has a quantity on a direction. Is moving on certain direction, suddenly you want to know its velocity.

For example, a particle is moving along curve at every point its velocity is along with tangent to that curve. It is not so obvious and not to include if that you are talking about the velocity at right at a given in instant of time. So, what do have to measure, that how do measure velocity value method speed. So, our car has traveled a distance from a to b. So, what is the distance between a to b and the time it has taken to go from a to b we take the ratio of the distance covered by the time and we say that was the basically the average speed of the car.

Now, here you are talking about just small movement away, you are taking about velocity exactly at a point. So, how will you do it? Newton first thought of incremental changes. Let us look at the velocity on the distance profile from t and from t I had very small time delta t. In little time how much it has moved, thing of delta t as positive at this moment you can think of it is negative means you are looking it from the backward thing. See the interesting part of classical mechanics is that if you know the state of a particle at a given time, you can tell about its evaluation in the future. You can tell what has happened to it in the past and that is what state it where in the past and that is why it is called deterministic mechanics.

So, once you know position of particle and it know its initial condition you can tell like what will happen to it in the future, how the whole trajectory of the particle evolve and whole classical mechanics is based on this. The classical physics is based on this on a way of scientific thing in is based on this principle. It from t it has moved to a point just a little bit time. Instead of looking at one instant you looking at just what has happened you get just you get just gone a little bit ahead. It is a very very intuitively difficult concept and then he says let me now look at a speed what average speed it has taken.

So, because it has gone in gone a very little distance and at a very little time; time elastic is so small that that direction possible as not changed as a result which it see even of vector, but we are computing it is you know magnitude. Now, you are computing taking the velocity. This is what the change is; this is what we will feel that these averages speed basically. This quantity x(t) plus delta t minus x(t) by delta t is average speed.

Now, what Newton says is that I want to look at it only at a given point, so at the point t. So, what I do is I keep on decreasing my time interval delta t. So, hence they are first came in the concept of limit, where he is talking about what happens to this ratio, when it when this limit, what would happens to this ratio as I made delta t smaller and smaller and smaller.

Remember if I make delta d smaller and smaller these distance also become smaller. So, do not think that it has 0 by 0 things I can forget about it. Now the interesting thing is that if this is a number, then that is called the instantaneous velocity is given in this say at that a point t. This is called the instant or rather I should write it just give the same arrow, this is called the instantaneous velocity at the point t.

At the time t and this is the very very fundamental thing it will very difficult to graphs this feature at the beginning. But then like many things in mathematics and any things in the physical science you start getting used to it was it turns out to be very useful idea in describing motion of particles. These also written as v(t) and this v(t) also written as dx d t evaluator at time t if you want to set up at this.

Now, another use of high derivatives is in the expression of the famous second law of Newton, which says force is mass into acceleration. So, acceleration is the change of velocity. So, force is equal to mass into acceleration that is d v of d t change in the velocities. So, derivative is the rate of change.

In fact, when Lord Kelvin J J Thomson was teaching derivative if the calculus is students of physics in Cambridge, they got very confuse with the very standard way of talking about derivatives, then he told them do not worry derivative is nothing, but the velocity. So, when you are whenever you are looking at a rate of change of quantity with respect to time you are talking out derivative with respect to time. So, acceleration is how the velocity changes with respect to time you press break accelerator of car you will see that the car speeds up. The velocity changes, so you would release the acceleration it release, the accelerator the velocity comes down.

For example, when you pressing acceleration of car, a car moves forward, its velocity increases, or relieve the accelerator its velocity decreases, either way the velocity changing. And this any with respect to time is captured by the derivative. And what is the meaning of this acceleration it means, what is the change in velocity if the time changes by one unit that is exactly what is the meaning of rate of change, which is very clear very natural understand. But this whole idea came with this bringing of this idea of the meaning of velocity at a point. It is so surprisingly a complicated idea it is not such a simple idea I would like you to have a look in the internet on this whole issue it is not so simple, but it guides all our physical sciences.

Now, because v is again x dot t you can write this as m times d d t of dx d t. So, basically this is the second derivative. The force is mass into the second derivative of the distance with respect to time. So, sometimes it is also written as m of x dot dot t. Note, I just want note just for curiosity that even write this whole thing as d d t of, because we are if there is a constant, if there is a if you multiply a function c by a constant f.

And then you take is derivative with respect to t or x whatever you want say it is t here. Then the constant actually comes out, it becomes. So, here you can pull in this mass the constant by the way mass is also not a trivial concept. This m v is call the momentum force is actually propositional to the rate of change of momentum and that was exactly what was it in my Newton in his principle it was not told that force is mass into acceleration. This is just for a curiosity; and with this, I would like to end the talk.

And tomorrow, we are going to talk about a very important thing without which nothing happens in the world as for Euler. Everything in this universe take place will never takes place nothing in this universe ever take place without some quantity been maximized or minimized, that was the thinking of Leonhard Euler, the great one of the greatest mathematician of all times. And we will talk about maximization and minimization of functions; trying to find the maximum value of function and minimum value of function and thing, which I had been in (Refer Time: 24:00) for last more than 20 years.

Thank you.