Basic Calculus for Engineers, Scientists and Economists
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Lecture - 07
Limits and Continuity-3

I hope you have enjoyed the first week lectures. If there are any problems you can always write back to the forums.

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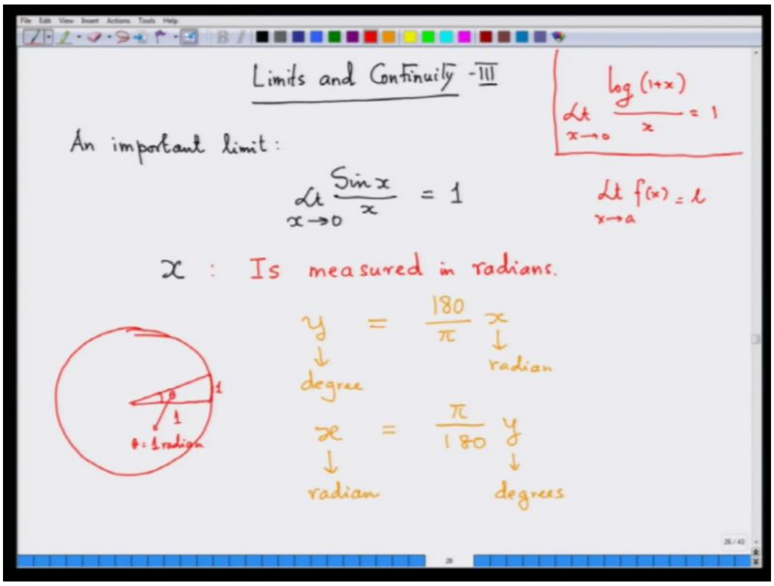

My teaching as tends will definitely help you out. It is the second week now and we are going to end our discussion on limit and continuity. Today we are going to talk about a very important limit, limit sin x by x as x tends to 0 is 1. There is another important limit which we are not going to talk much about, but we will show that it can be used to compute the derivative of logarithm of x when we do it from the first principle that limit is, it looks very strange because you see at 0 neither this function or this function is defined.

So, when we are talking about limit fx x tends to a is equal to l we always had a deleted neighborhood because we never considered a in the neighborhood surrounding a because at all points in the neighborhood other than a we expected the function value to be computed. That part has to be in the domain away, but a need not be in the domain of the function f. Here you see for whatever x that you take in both sides of 0 sin x by x is a perfectly valid function. It is a numerical value, but at 0 it is meaningless because it will come 0 by 0 then, but still has a finite limit. So, this is effect, this is a, which is not in the domain of this function because 0 is a, a is 0 here and l is 1 here.

We are going to give a very small geometric proof for this fact there is and another is also another easy proof of this fact which you will learn when we will talk about L'Hospital's Rule of finding limits, when we are going to study derivatives in the next classes. See it is important to understand that whenever into trigonometry we are talking about sin x, cos x, tan x whatever is x is measured in radian. What do we mean by radian? It is the angle, here is a circle. So, say circle of radius 1 centimeter to if you swipe an arc of length 1 centimeter then the radian that is swipe then the angle that is swept this angle theta, if this is also 1 and this angle theta is called 1 radian and 180 degree is pi radians and that is the interesting thing.

So, 180 degree is pi radians, one radian is pi by 180 degrees. So, x radian is 180 by pi into rad x degrees. If you want to compute the radian out of the degrees, for example, when you put x is equal to pi, where pi radian that becomes 180 degrees when you are x when you are putting x equal to one radian you are putting 180 by pi degrees. This is very, very interesting that you know you can play a lot with this thing pi. So when you want to compute x here. But when you are doing actual computation we are essentially concentrate with degrees, when we take a composite cool we do not talk about radiance we talk about degrees.

So, when we are talking about degree of the angle then if I give a degree I should be able to calculate the radiance because in mathematics when we are calculating on trigonometry when you are calculating this sort of sin and cos functions then you are essentially talking about radiance. Now, the question is how can I make a guess? It does not see then I will right away write down a proof, I can right away write down this result if I know the L'Hospital's rules. If I do not know the L'Hospital's rules writing this result may not be so easy, it needs a slight of geometric intuition.

So, when we do not have our geometric intuition right away, when we do not have any analytical tool right away to compute this, we have to depend on experimentation.

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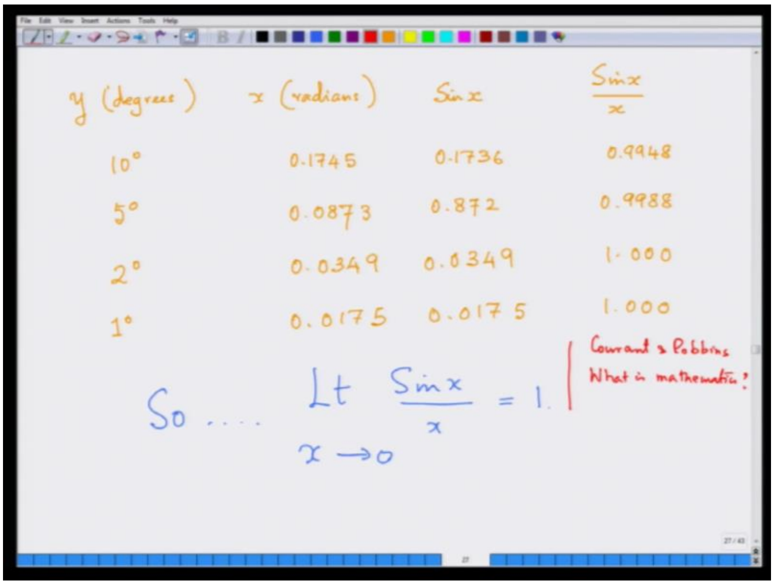

Experimentation via computation is now one of the central things of mathematics. Mathematics was largely experimental tool, experimental subject long back in antiquity people did computation and then try to understand whether certain things are true, but in many case computation cannot be done for infinite things, computation can be only done for finite number of things, but the importance of recurring mathematics is that if a thing can be true for a finite number of numbers, but a thing may be just not true after that.

So, that is why you need to talk about, prove that is why talk about trigger, for example, many of you heard about the (Refer Time: 05:52) theorem which says that if you have any equation of the form x to the power n plus y to the power n is equal to z to the power n, if n is strictly bigger than 2 that is 3 and above then you do not have any integral solutions to it.

Of course, there was huge trail for proving this problem, people prove that if n is 3, 4, 5 or n is up to 10000 I can prove this, but that does not make it a result because up to 10000 I know that this is not true, but beyond 10000 there could be n for which it is true to prove it finally, what Andrew, a very, very sophisticated developer in mathematics. But experimentation is very important as we are coming back to experiment nowadays, and there is a separate branch of mathematics called experimental mathematics because experiment is always giving you an insight, it tells you the possibly this is the way to look for the answer.

Here what I do because x is going towards 0, I tend to decrease the degree of course, if x is now, if y 0 degree then we have 0 radian. So, when I am decreasing the degree I am decreasing the radian also. So, for 10 degree of this sin x is this sin x y is this for 5 degree. This is your radian 0.873 then you compute sin x, then you compute sin x y x, again when you compute 2 degree because you are approximated the calculations it looks like this. So, this little calculation is due to Courant and Robbins where is found in other places from a lovely book which anybody interested in mathematics should read and see his book called, what is mathematics.

Of course you can. So, 2 degrees is far away from 0, 1 degree is very far away from 0. So, how we are getting 1 now if I go down further, I would leave it to you to compute it down further because now you know to compute it you know what to do just put in a degree get the x compute sin x, computing sin x, now it is if when you can do it with the calculator just do it with the calculator then see what you get sin x what is the value of sin x by x. So, i believe in you get 1.

Basically then you can have some confidence to say that let that limit of sin x by x is actually equal to 1, but that needs the proof this is all guess work of course, this give you a lot confidence because your computation in front computation is there is to gain inside.

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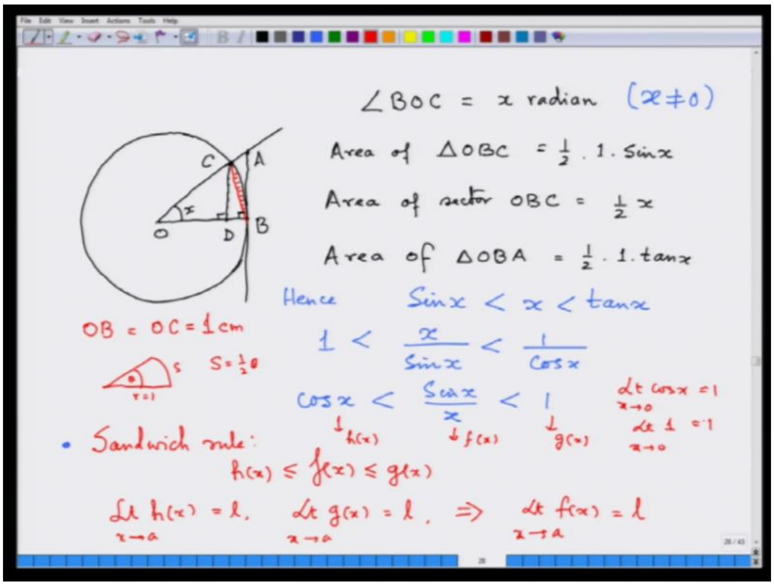

Now, let us talk about a proof. So, what we do is that we take the circle where o is the center and it is the unit circle the radius OB is equal to OC is equal 1, 1 cm it does not matter if you know uncomfortable not writing unit write 1 centimeter. So, this called a unit circle whose radius is 1 unit one something could be meter centimeter let us just take 1 centimeter.

Now, what you do let x taking b as the base OB as the base to swipe out at the angle of x radians of course, which is non zero and let it cut the when we swipe it off it stops at the point C. So, BOC this angle is of x radian, where x is not equal to 0. Now, what you do from C drop a perpendicular on OB where it touches OB at D and now draw a perpendicular sorry a tangent from through B to the given circle. Now, you know that the radius OB would always be perpendicular to a tangent. This is basics school geometry now extend the line OC. So, that the tangent through D meets OC at the point A.

Now, I will calculate the angle of OBC which is this angle OBC then I will calculate the sector O circular sector OBC. So, the circular sector has this additional area this one the circular sector has an additional area which is this one. So, the circular sector OBC is a triangle OBC plus this additional area and this red marked area and then we will calculate the area of triangle of OAB then of course, you can know that the triangle OAB includes both the circular sector and the triangle OBC. So, and that triangle OBC is area must be smaller than A area of the circular sector and which is smaller than the area of triangle OAB.

Now, once that is done I need to calculate area of triangle OBC I am not going to do this simple school geometry for you, area of a sector people might worry, how do I find the area of a sector of a circle. So, if this is my s, this is my theta, how do I calculate the area of a sector s is actually equal to half square theta right, for example, if I have taken a semicircle then s would be nothing, but half of pi r square, if r is the radius, here the radius is 1.

It will be its half of the angle subtended at the center if radius is 1. So, its half theta which is half of x, half of theta, if where the radius is equal to 1 and then also I am not going to find the triangle OBA for you, this is very basic geometry when you listen to this talk you stop the talk and then you can tried to calculate them and come back.

What it does it mean, the area of triangle OBC is strictly less than the area of circular sector OBC and that is again strictly less than area of triangle OBA. So, half of sin x is less strictly less than half of x which strictly less than half of tan x. So, sin x is strictly less than x which is strictly less than tan x. So, if x is not equal to 0 and x is very near 0 then sin x is not 0 and we divide by sin x of course, this division is assuming that I have taken x to a very small and sin x is of course, not 0 in that case.

So, once this division is done again then revered back to have cos x strictly less than sin x by x strictly less than 1. Now, all of us know that cos of x is a continuous function at 0 because limit of cos x as x tends to 0 as 0 is one that you know. So, if you just look at the graph of cos x you know what I know. Now, how do I get the fact that the limit of sin x by x as x tends to 0 would also be 1. For this we introduce or not introduce rather we state here what is called the sandwich rule, the sandwich rule states that if you have a function f lying between two functions h and g and when x tends to a, both h and g tends to the same limit l then f will also tend to the same limit l, when x is tending to a.

So, here if I take limit of cos x as x tends to 0 that will go to 1 and limit of 1 when x tends to 0 because this is independent of x will just go to 1 it will just 1. So, since f is here f is playing sin x by x is your effects g x is just g x is equal to 1 constant function and h x is your cos x. Now, in this in our case limit of cos x as x tends to 0 is one limit of 1 as x tends to 0 is 1.

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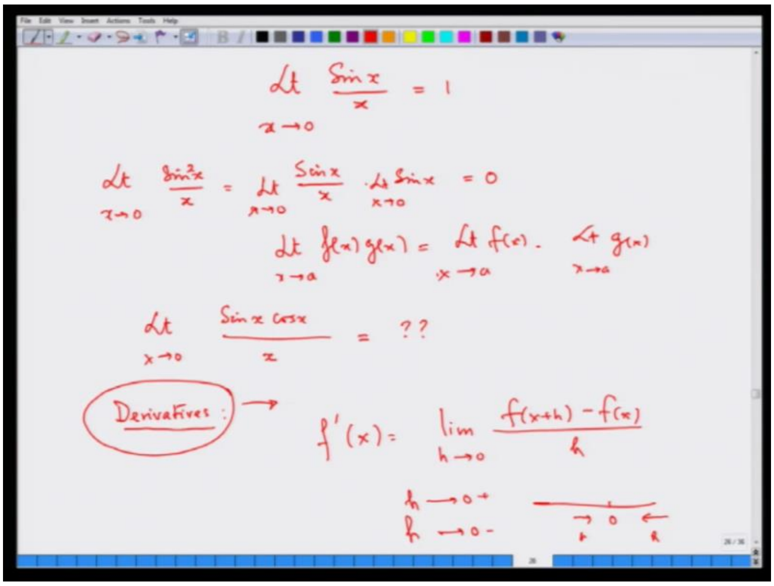

So, we conclude that limit of sin x by x as x tends to 0 is 1, how what where is limit is useful, for example, if I ask you to compute the limit of, what will be the answer, it look very surprising unit 0 by 0, but now we can write this as sin x into sin x. Now, we know that limit sin x by x is finite quantity which is 1 as x tends to and limit sin x is x tends to 0 is 0. So, if you know that the multiplication rule that if f and g are two functions and both of them tend to some limit then they will product the limit of the product is the product of the limits.

These are high school math. So, I am not getting you putting too much stress on actually stating them, but you need to. Now, we will have that this is nothing, but unit of fx x tends to a into limit of gx x tends a.

Now, sin x by x, limit of sin x by x as x tends to 0 is 1 and limit of sin x as x tends to 0 is 0. So, this answer is 0. So, you can have lot of problems of this form an it really does not matter, for example, if you have say limit of all this of trigonometric limits of x tends to 0 sin x into cosine x by x. So, what would be the limit of course, I leave these things to you. So, what we have done is giving you very, very broad idea in these three lectures about very important concepts about limit and continuity. With this idea we will be able to move forward from the very basic working with functions to the first steps in true calculus, where we are going to talk about derivative.

To give a very brief introduction of what is going to come next we are going start talking about derivatives in the next lecture what let me give a just a very brief idea what. So, this idea of derivatives was first introduced by live of course, by the great Sir. Isaac Newton, who bought it into study mechanics, derivative represented the velocity of a particle instantaneous velocity of a particle in motion maybe along a straight line does not matter. There Newton wanted to look at the following that if I look at the velocity of the two objects, the two very close points or rather I look at the change of displacement in a very short time.

Then this ratio is a velocity of the object within that very short span of time. So, when the time becomes very, very small where does this ratio tend to earlier this obviously, thought that such things are impossible motion is almost impossible, but that is not really true and what Newton would showed that yes indeed that contain to a non 0 quantity and that whatever how is when the time become small this distance will also club to 0, but they will actually get something meaningless 0 by 0. So, this is not really true.

Then that idea of rate of change of a distance which we start as calling velocity or instantaneous velocity became one of the main soul fact in mechanics velocity and its again derivation the acceleration which actually lead to Newton's second law force is mass into acceleration became the corner stone of mechanics on which all of modern physics is built.

So, it is very, very important to study derivatives and we are going to study this thing. For a mathematician it is important to know that derivative is defined in this following fashion I just I am giving you this definition, so that you came with the little bit of refreshment of your refreshing your class 12 notes or class 12 I thinks the books whatever you want to know this definition here h of course, means h is going to 0 from the right and h is going to 0 from the left. So, both wise and this notion of a derivative of a function are also central to all mathematics.

This is a very, very important notion that we are going to start studying in the next class. So, please understand and I want to make it very clear that the famous mathematician who rather invented the amazing geometrical notion of factors they about had once said that you would understand calculus much better, if you know at the very outset at every continuous function did not have a derivative.

With this I would like to stop today's talk and I hope that you go through this derivation, the geometrical derivation of the limit of sin x by x as x tends to 0, it is a good thing that a lot of things can be done by geometry, it also shows geometric reason.

Thank you very much.