Basic Calculus for Engineers, Scientists and Economists
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Lecture - 05
Limits and Continuity-1

Today, we are going to speak about Limits and Continuity of functions. This is our limits and continuity of functions.

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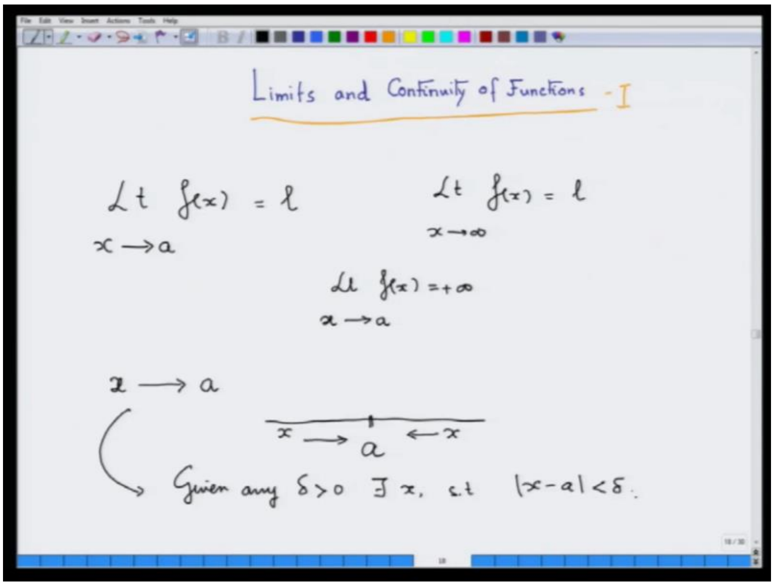

This is our first story. So, what do I mean by this symbol you will often find in calculus books. So, limit of f x as x tends to a is l and limit of f x as x tends to infinity is l. Also we are going to explain to you what you mean by limit of f x, x tends to a is plus infinity or minus infinity let us do it for plus infinity.

So, we are going to explain to you very carefully with examples what do we mean by this. Now, first let me understand what I mean by the term x going to a. It essentially means x is coming to a, but still not coming; coming, coming but still not coming as I told you in one of the very first lectures. Teacher explained to me the notion of limit in this form. This is my 'a', x coming to a means x can approach from either this side or x can approach from this side. So, x tends to a means the meaning of this is following given any delta greater than 0 there exist x such that mod of x minus a, is strictly less than delta.

However, small delta may be I will always find an x such that its distance from a, is less than delta. So, I can bring or more just telling in English language I can make x come arbitrarily near a, x tends to a means x can be brought arbitrarily near a. But it might not be equal to a, anytime. That is the meaning of x approaching a. So, what is the meaning of f x?

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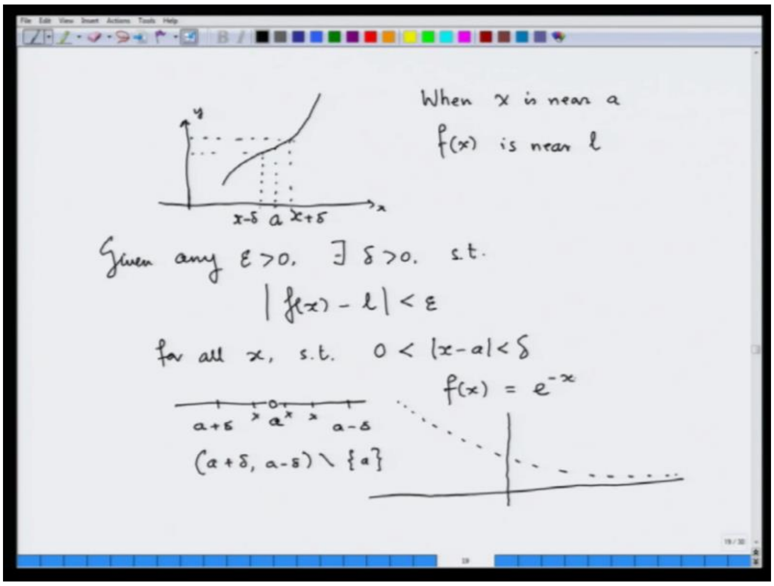

Essentially, so look at the nice function like this and this is your 'a'. So, x approaches a, so we take a very small neighborhood around a. So, say if the delta neighborhoods, then we want to see in this particular case if I take any x here, here or here f x is banned. So, f x is whatever be the value of sorry; this is whatever be, this is my x axis and this is my y axis I am not repeating here, these are very common things. So, you take a nice function, it is a very nice looking function, smooth and essentially a continuous curve. You see that whatever x I take here we always have f x lying in this particular band.

Essentially it means that whenever x is near a, when x is near a; in this case the limit here is f of a, but it need not always be. So, when x is near a, f x is near l this is essentially the meaning of this term - limit f x, x tends to a is l. Now mathematically how do I put it in a more proper format? That is again done through epsilon and delta language which you have to grasp a bit for your own understanding. Then I say, can you tell me that when x is near a means if I say I want f x to be within epsilon distance from l, can you tell me if there is any x which is within some very near a, for which this will be true. That is the idea. That, if you give me how much I should make f x close to l, I should be able to tell how much x should be close to a such that for all those x's f x must be close to l within your given prescription.

So, given any epsilon greater than 0 there must exist delta greater than 0, such that f x minus l is strictly less than epsilon for all x such that the distance of x from a is less than l, but x is never equal to a. That is here, we essentially look at a punctured neighborhood because l may not really been the domain of f. So, basically you look at a punctured neighborhood

Here you are looking at what is called as punctured delta neighborhood. Here this neighborhood is written as a plus delta, a minus delta. And from this set I have taken away a itself, it is called a punctured neighborhood away. Because I do not want x to be exactly equal to a. Sorry, it should be delta. So, I am deriving x minus a, should be strictly less than delta that is the distance between x. So, x has to be here or here or here, does not matter. But x cannot be equal to a, because I do not know whether a is in the domain of f. That, as an example of such a fact let me consider the function f of x is exponent to the negative power, e to the minus x. If you draw the graph of such a function you will observe the following.

That here the function value is gradually when I am increasing x. That is in this case where x is going to infinity, when I am increasing x this value is coming towards - for example, is bringing me close and close to 0, but there is no x for which e to the power x becomes 0. Another example in this particular case for example, let me construct here.

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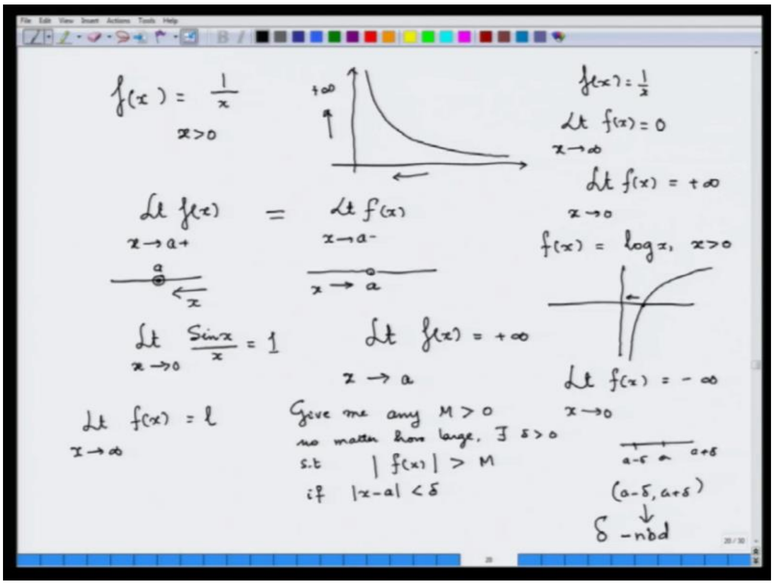

For example, if you take f x is equal to 1 by x. And we look at the graph of this, if you take x is tending to 0 the function value here is going towards plus infinity. As you are bringing x near 0, I am just defining it for x strictly bigger than 0, the function value is coming near infinity. But let me explain to you something very important. When you are talking about limit there is also a concept of left limit and right limit, that is you first have to look at where the function values are going when x is coming to a from right side. So, you do not take a - a is out. So, when x is coming from right side. This is called the right limit and then you have to also look at the left limit when x is coming from the left side.

This a - which you have taken out, looking at x coming from the left side; what is the relation between these two? If you have equality among them then you set the limit exist, if you do not equality among them then you say the limit does not exist. For example, you will soon learn about a limit - an important limit of this form, limit sin x by x as x tends to 0 is 1. You will soon learn about this limit. These are very important limit.

But you know that this function is essentially has no definition at x goes to 0. So, you cannot have, there is no definition of f 0 in this case. So, 0 is not in the domain of this function, but still. That is why you cannot say that this one is equal to f of 0 or sin 0 y 0 you cannot say that, because sin 0 y 0 has no meaning. This is an example where the function value can approach a limit when it is approaching a given point 0 in this case, but that point a might not be in the domain of the function. This is something that you have to keep in mind.

Let us again look at limit of f x, as x tends to a is plus infinity. What does it mean? And also let us look at limit of f x as x tends to infinity is l. Here it means as x becomes bigger and bigger and bigger, function value approaches l for example, here if you look at this function f x is equal to 1 by x and you say limit of f x as x tends to infinity, what does this mean? It means that as I am increasing the value of x where does the function value go - here it goes down and down and down and down and goes to 0. This is some that you have to keep in mind, you have to remember.

Moreover, let us look at something more interesting. For example, if you look at; you have an understanding of this, but what is the meaning of this? It says even when x approaches a, for example in this particular case, if you say limit of f x when x is equal to 1 by x and when x approaches 0 then, what is the value of f x? That f x value is increasing and increasing and increasing, that is plus infinity. For example, if you take f x is equal to the logarithm of x, you know logarithm of x the domain is only strictly greater than 0. So, again if I approach 0 I am only approaching it from the right hand side, because there is no way either. Approaching it from the left hand side is meaningless because there is no definition of the function on the left hand side. So, this is log one is 0.

So, you see what is happening to log x. In this case limit of f x, x is coming to 0 from this side as x is approaching here, this is becoming minus infinity it goes down and down and down and down. The various ways a function can behave depending on the various ways as x goes to a, for example, if you want me to write this how do I write this in some sort of an epsilon delta type thing, of course, here you do not have epsilon delta business because function f x goes up.

What it says? That, give me any M greater than 0, no matter how large. There exist delta greater than 0, such that mod of f x or f x assuming say as in positive function does not matter. Mod of f x is bigger than M if mod x minus a, is less than delta. Whenever x is within the delta neighborhood of a, it is called the delta neighborhood. This is always called the delta neighborhood a, a plus delta, a minus delta. This is an open interval. This is open interval. This is called the delta neighborhood, delta neighborhood. I am doing a shortcut nbd. So, whenever x is in this, mod of f x would be very large.

Of course, you can say that if the whole thing is negative how I can write. Of course, it will not be negative. If this f x is tending to infinity, at certain time the function value has to become positive. So, you may say that I can just forget the negative values and just concentrate on the positive values and see where it is going when x is approaching a. If this is the case then f x must take a positive value after sometime, it cannot just stay (Refer Time: 14:19) negative values.

If it is having negative minus, it goes to minus infinity it has to take a negative value after sometime. For example, the case of log x if the function has started taking negative values just after log 1 which is 0. For after x equal to 1, it has started taking negative values. So till, suppose x is approaching zero, so I am coming along this side the function values are all, you know the function values are all positive and then once it crosses 1 and goes mod near 0, the function values have to become negative. These are something you have to keep in mind.

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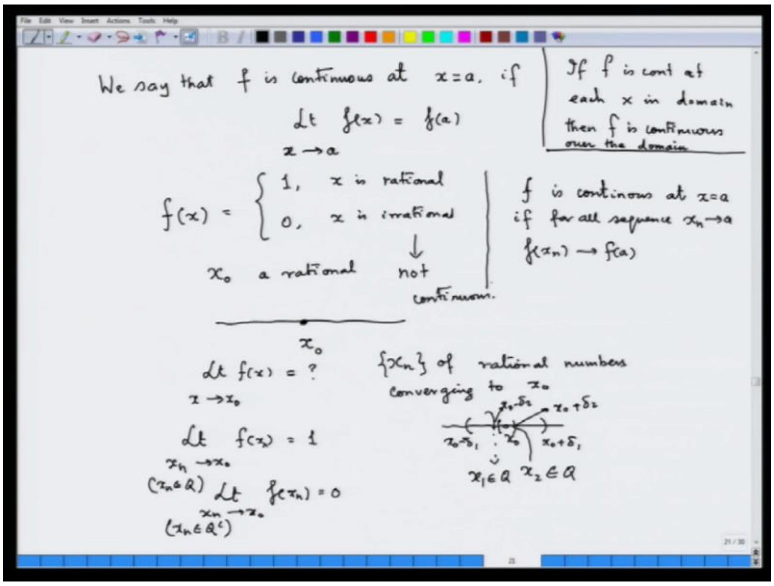

We say that f is continuous at x equal to a, if limit of f x as x tends to a is equal to f of a. For example, what is the discontinuous function? Let me define a function, f x is equal to 1 when x is rational, is equal to 0 when x is irrational. Here I am talking about a function to be continuous at a point a. If this function is continuous at every x equal to a, we say that the function is continuous throughout its domain. Here is just an extension of this idea; if f is continuous at each x in the domain then f is continuous over the domain. It is alright, x is rational - x is irrational.

Now, what is happening let us see, let us look at this function. So, we are going to test its continuity, f x is 1 when x is rational, 0 when x is irrational. And let me take x naught, rational number. So, I will check whether at any given rational point it is continuous or not, so look at x naught. Now when I say, so I have to now look at limit of f x as x tends to x naught, what does it mean? How do I look at this thing? Here you see a use of the sequence to answer such a question you will need the help of the sequence.

This definition of continuity can be written in terms of a sequence. It is as follows, we say f is continuous at a, x equal to a, if for all sequence x n converging to a, f of x n must converge to f of a, then it is continuous. If it is not happening, this is not happening then it is not continuous. This idea would be used to understand this problem.

For example - I am asking what is this question. Now consider a sequence x n of; so you take a sequence x n of rational numbers converging to x naught. So, how will you do it? For example, take any sequence - take a sequence like this, take a neighborhood around x naught, x naught plus delta 1, x naught minus delta 1. So, choose any rational number here say x 1. Then make a much smaller neighborhood which that does not contain x 1, right.

Here in this, you say that neighborhood can be say x naught minus delta 2 and this is x naught plus delta 2. So, you choose another x 2 and even keep on decreasing the size of the neighborhood. Basically, now you have consider a sequence and you are decreasing your length of the neighborhood and each neighborhood you can still find rational numbers coming towards x naught, so this x 1 is a rational number. Now here you choose some x 2, x 2 that is also a rational number, so x 1 element of q. So, I should write it properly, x 1 element of q. That is also rational number.

Similarly, instead of rational number you could choose irrational number. Let me take a sequence of rational number converging to x naught. For if x naught is rational every f x value is 1. If you have a rational sequence x n converging to x naught where this x n is element of q then this is 1. But if I take a sequence of irrational number x n going to x naught, which you can do; instead of x 1 you can choose q, you can choose something which is in q compliment. So, x n is in q compliment, is in irrational number.

In that case, the function value at each of those - x n would be equal to 0. So, a limit of f x n would be equal to 0. But the equivalent definition of continuity says that whenever f is continuous at x equal to k, for all the sequence x and that goes to f x will has to go to f a, but here since x naught is rational point f of x naught must be equal to 1 that is the definition of the functional value at rational point.

Here, if I take a sequence of rational number going to x naught, the function, the limit value is 1. If I take a sequence of irrational number going to x naught, the limit value is 0. So, f 2 different limits values they are not converging to a same value. This function is not continuous.

In the next class we will give much more examples to illustrate all these points in much more details and we will take some examples and calculate them.

Thank you very much.