Basic Calculus for Engineers, Scientists and Economists
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Lecture – 04
Sequence-02

Welcome back to the study of sequences. We are going to talk today about subsequences. You are familiar with the idea of subset, so you could be familiar with the idea of subsequence also. Or, you can make it up in your mind. It is very good to make up concepts in your own mind.

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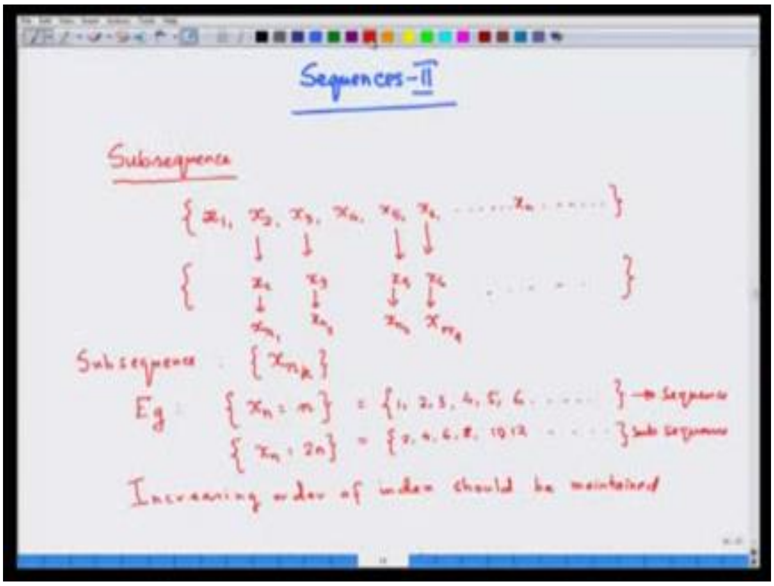

Suppose you take a sequence x 1, x 2, x 3, x 4, x 5, x 6. I hope I am maintaining 1, 2, 3, 4, 5, 6, correctly. And then, suppose I want to just; may construct a new sequence out of this sequence. So, what I would do? So, for example, I choose x 2, and then I choose x 3. So, I leave out x 1; I drop x 4 and choose x 5, x 6, I drop x 7 and I choose x 8, x 9, drop x ten, choose x 11, x 12 and so on and so forth.

Drop the first, take the next two; drop the next, take the next two and so on. So, what I have created is the subsequence which consist of x, 2 x, 3 x, 5 x, 6 x, 8 x, 9 x, 11 x, 12 x, 14 x, 15 and so on. But, remember that here I have to maintain the order of index 2, 3, 5, 6 and so on. The increasing order of the index has to be maintained. If I say first put x 6 and then I take, put x 2 and then I put x 12; it is not a subsequence. And, this is a very important point. If you do not maintain this point, then you cannot play with the subsequence. So, playing with the subsequence is a very very important game that mathematicians play. And, that has to be kept in mind.

Now, you might ask me what good is subsequence or sequence or any of these is in your studies, in Engineering because you have to differentiate, you have to integrate and do all those stuffs. The real issue is the following; without having a good understanding of the sequences, you might not be able to have a very good understanding of even differentiation and integration, continuity or whatsoever.

Sequences is some sort of a discrete version of the analysis that calculus that you are learning. But, so it might not be useful at the very moment, but it has some use as we go on.

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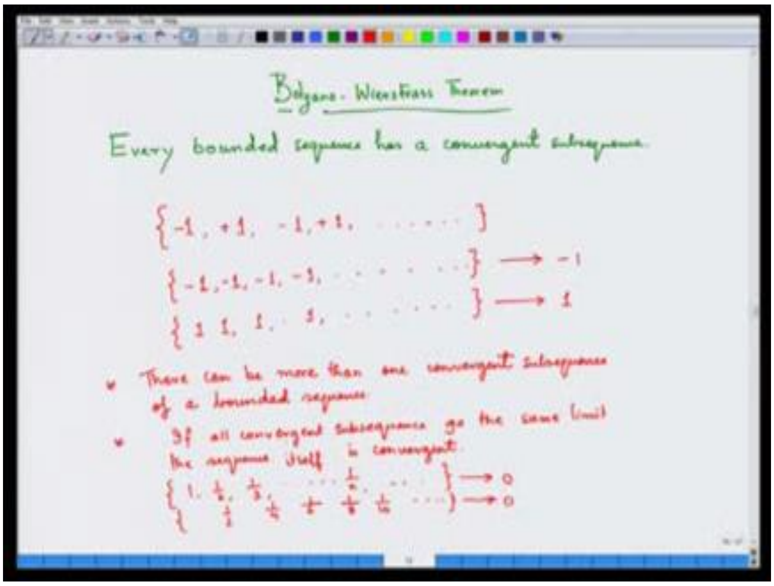

So, here is the famous Bolzano-Weierstrass theorem, which says every bounded sequence have the convergent subsequence. Let me take a sequence which is bounded; minus 1, plus 1, minus 1, plus 1. You are already accustomed with the sequence, pretty cute sequence. And, of course it is bounded by 1. And, now you; I take a subsequence minus 1, minus 1, minus 1, minus 1. So, I am taking x 1, x 3, x 5 and so on. This is a subsequent. Is this is convergent? Yes, this is convergent to; this is convergent to minus 1. And, I can also take a subsequence 1, 1, 1, 1, and this is convergent to 1.

So, you see the first sequence convergent is minus 1; the second sequence convergent is to plus 1. So, if you have two different convergent subsequences, both of them need not go to the same limit. If all the convergence of subsequences goes to the same limit, then the whole sequence itself is convergent. Then, the whole sequence itself is convergent.

And, remember if a sequence is convergent, then every subsequence must go to the same limit. These are some important points to keep in mind. There can be more than one; so, more than one convergent subsequences of a bounded sequence. This is something you have to keep in your mind.

In fact, there can be infinite such sequences. You are not going to get into this infinite term too much. You can understand there is not (Refer Time: 04:51) more than one. If all possible, if all convergent subsequence go to the same limit, then the sequence itself is convergent. This is a very fundamental fact. And, it will be good that when you look at Bolzano-Weierstrass theorem. You just keep in mind this factor. For example, you take this sequence which is also familiar. I am just trying to give you examples of easy sequences. I will give some harder sequences in exercises, which you can try out. But, they would not be so difficult.

If you take this sequence, you take the subsequence half, one-fourth, one-sixth, one-eighth, one-tenth and so on. Where does this sequence go? This sequence; all subsequences are going to be 0. So, whatever subsequence you chose from here, all of the subsequences are going to be 0. These are the lessons you keep in your mind about Bolzano-Weierstrass theorem.

So, once that is in your mind that every bounded sequence must have a convergent subsequence, then that allows you to prove something very deeper. It proves or resolved which is at the very heart of sequence theory. It tells you the true behavior of the sequence. It tells you that if a sequence is convergent, then it has a very peculiar behavior.

The behavior is that no matter that after a few number of terms of sequences, that is after the finite number of terms, no matter how long is your m and n, the distance between them will always lie in the two epsilon band. The distance will always be within two epsilons from each other. That is the individual distance. The distance between them should be always less than some epsilon; some given epsilon.

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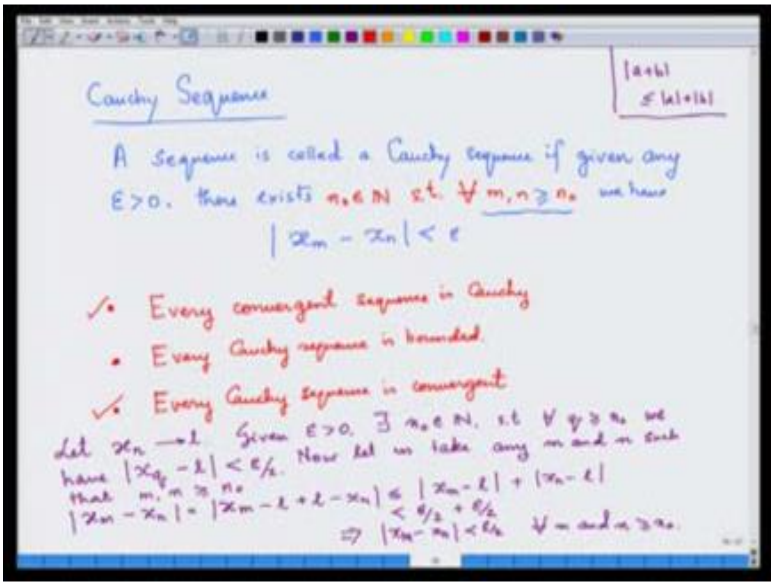

So, means they always cluster very close to each other. And, also clustered to number l to which they are going. This sort of behavior was first found out by Cauchy. There is a whole term called Cauchy sequence. So, what is the Cauchy sequence? A sequence is called a Cauchy sequence, if given any epsilon is greater than 0. No matter, how small. There exists n naught element of N. This is very important to note such that for all m, n, bigger than this n naught, we have mod of x m minus x n. It could be that your n is n naught plus one and your m is m naught plus fifty million, does not matter.

The distance between them can always be made less than this epsilon. Once the; once n naught is given. Sorry, I have made mistake here, for all m, n greater than epsilon. So, you have to keep in mind this write up. There exists an n naught; means you are counting from x 1, x 2, dot, dot, dot upto x n naught. So, for all m, n greater than or equal to n naught, this is always holding. This is a very important property that these points are clustering very near each other and also along the line. This is something of very fundamental importance.

So, why your Cauchy sequence is important in our studies? The reason is as follows: every convergent sequence is Cauchy; two, every Cauchy sequence is bounded; three, very very important result, extremely important; every Cauchy sequence is convergent. So, sequence; the real sequence is convergent if and only if it is Cauchy. Every Cauchy sequence is convergent. This is a very important result of mathematical analysis.

For example, if I just look at it as a problem and I want to prove every convergent sequence is Cauchy, how do I do it? So, what you do is that so let us take a convergent sequence, so let x of n go to l. Now given, which means given epsilon greater than 0, there exists some n naught; depending on epsilon, natural number, such that for all q bigger than or equal to n naught, we have x q minus l to be strictly less than epsilon or may be to be even better, we will take epsilon by 2. So, if I have epsilon greater than 0, then I also have epsilon by two greater than 0.

Then, for that epsilon by two greater than 0, there would exist some n naught such that this would happen. This is true. Now consider; so, I am trying to use the epsilon and delta language to solve problem.

Now, do not think it is some theorem or something. Let us try to solve a problem in calculus. Now let us take any m and n, such that m and n greater than n naught; such that m and n greater than n naught. Now, let us look at what would I have. The distance x m minus x n; I can now write this as x m minus l plus l minus x n, so I am adding l and I am subtracting l.

So, now by the standard fact about absolute values, that is, absolute values of a plus b is absolute value of a plus absolute value of b. Even if you have forgotten, just I am writing for your reference on the side; for any real number. This is of course true. Sometimes it called the triangular inequality. This; by applying this in equality we have; so, what we have got before that once I have given epsilon, I will consider epsilon by two. And for any, and I will find n naught such that for whatever q I take, which is the bigger than n naught x q minus l would be strictly less than epsilon by 2.

I have taken m and n such that m and n is bigger than n naught. So, for this n naught x of m because this is one of the q is strictly less than epsilon by two and x of n is less than epsilon by two. So, finally it will make me conclude that x m minus x n is strictly less than epsilon by two, for all m and n, which is bigger than n naught. Hence, you have showed that this sequence is the Cauchy sequence, which is of very fundamental importance.

Now, I will try to give you a scale of how would you prove the or at least have an understanding of this fundamental result that every Cauchy sequence is convergent. This is very important to understand. This tells you very important criteria. That is, if you look at, if you want prove a sequence is convergent, what is your job? Just prove the sequence is Cauchy.

When Cauchy was giving a lecture on this in Paris, he was not very well accepted by students. They found that it was a very big wastage of time. But later on, I will tell you little stories when we will talk about infinite series, in which also Cauchy had a major role that how important these ideas were in actually in Natural Sciences, in Physics, in Mechanics actually.

So, what would you do if you want to talk about sequence or you want prove that the Cauchy sequence converges? So, here you will see the application of the Bolzano-Weierstrass theorem. So, sketch of the proof is this. I am just giving a sketch.

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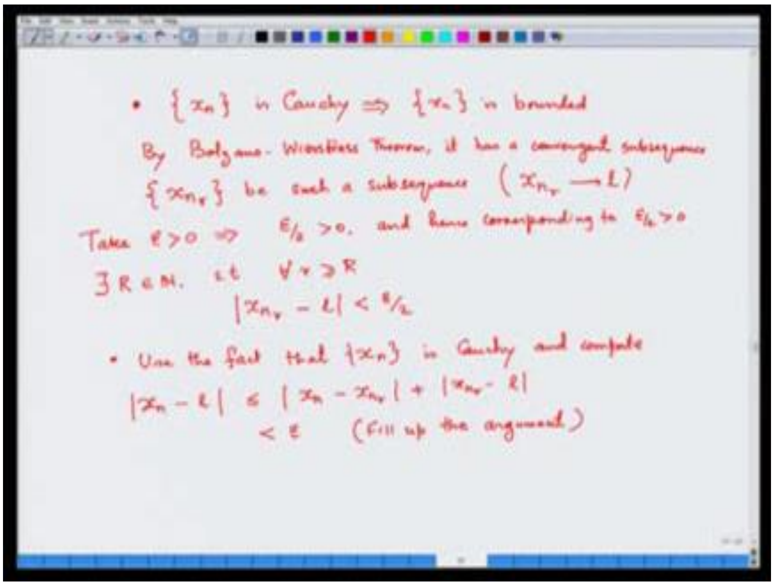

So, your first step is that you know that x n is given to be a Cauchy sequence; x n is Cauchy. So, by our second result on Cauchy sequence, it implies that x n is bounded. x n is bounded. And, x n is bounded means it has a convergent subsequence. So, by Bolzano-Weierstrass theorem; so Bolzano-Weierstrass theorem, it has a convergent subsequence. You could possibly denote it by x n r. So, let x n r be such sequence; x n r or x n k if you want be such a subsequence. So, what happens? Take epsilon greater than 0, so that would imply epsilon by two is greater than 0.

And, hence corresponding to epsilon by two greater than 0, there would exist R element of n, such that for all r bigger than R or equal to R, you would have x n r minus l; which is the limit of the subsequence. This subsequence is convergent. Let us assume that this subsequent goes to some limit l. This would be strictly less than epsilon by 2, for all r greater than or equal to R.

Now your next step is to make use of the; The next step is we will give the detailed proof and we will do your exercises. It will be asked in the exercise proof. So, I am giving you the steps of the exercise. So, use the fact that x n is Cauchy and compute mod x n minus l, which you can write it as following; mod x n minus l can be written as mod x n minus mod x n r plus, sorry I am just writing it in a slightly in a step jumped manner, x n r minus l.

Now, this could be also made strictly less than epsilon by two, by the definition of the fact that x n is Cauchy. And, this can be made less than strictly epsilon by two because I already know. But, how will you make this strictly less than epsilon by two? Needs some argument, so you can basically show that all this whole thing is less than epsilon. So, your exercise which will be a part of your assignment would be filling up the argument. Try it at home.

And, I am telling you once you can argue this out, you will see you will have a piece of joy or a thrill passing down your spine. I always enjoy when I can actually give a proper reasoning in a mathematical problem. I think there is no joy for me at least, much better than that.

Thank you very much and have a nice evening.