Basic Calculus for Engineers, Scientists and Economists

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Lecture – 03
Sequence-1

Welcome to the third module or third lecture, whatever you want to call it. So, we are going to talk about sequences. Sequences are nothing but looking at counting as a function. I had said that counting our set is same as putting it in one to one correspondence to the set of natural numbers. So, set is countable if (Refer Time: 00: 34) is finite or I can do this correspondingly.

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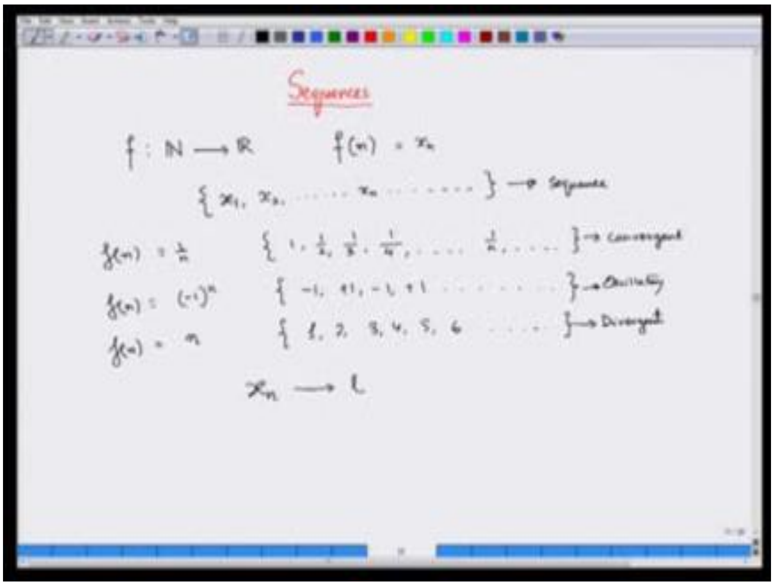

Now let me consider a function which is same as counting. But, I am trying to map the set of natural numbers to set R of real numbers. So this map, which takes an element n and maps it to a real number x n. And, if I list down all; for every, if I list down exhaustively the outputs that I get, then that exhaustive list is called the sequence.

Sequence, of course is not representing the range of the function. For example, if I have this function f n equal to 1 by n, then what I get is the sequence of this form; I start with n equal to 1, n equal to 2 and go on. So, it is 1, half, 1 by 3, 1 by 4, and so on and so forth. Of course you could have a sequence, which is slightly more interesting looking;

minus one to the power n, which just go on having this. But, you just have to list down everything. That forms a sequence.

You look at the range of this function; minus 1 by n. This is just minus 1 plus 1. You look at the way the values are moving in this particular sequence; one-half, one-third, one-fourth. This is decreasing and going towards 0. Then we can say that this whole sequence values are moving towards 0. This process of moving towards the particular value or a limit is called a notion of convergence of a sequence. And this one just does not converge, the values move around two points; minus 1, plus 1, minus 1, plus 1. This is an oscillatory sequence, a sequence whose values do not blow up or they do not go to a limit, but move between two values.

Another type of sequence, for example, this is just same repetition. So, here I can make the numbers as big as I like. So, we do not go to a fixed, towards the fixed number. This is called the divergent sequences. This sequence is called the convergent, this sequence is called oscillatory and the third one is called divergent.

Now what we intend to do is the following. We want to talk about what is the meaning of convergent, what is the meaning of a sequence x n moving towards the number l. So, when I was a youngster and was in my high school, so one of our Physics teacher started talking about derivatives and all those things, limits and everything. And, we were not understanding being students of class 11. So we asked him, “what is the meaning of the limit?”. So, he said that limit means coming, coming, but still not coming. Which is essentially the fact, but which can be cloaked in a much more intelligent language. x n goes to l means as I make n as large as possible, the distance between x n and l can be made as small as I like. That is the idea.

So, you say, if you tell me, I will give you a distance between x n and l or I will fix up a number, can you find x n? So that, all x n minus l would be less than that epsilon. That is, beyond some finite numbers of elements of the sequence will all the x n be such that the distance would be less than given epsilon.

That is the meaning of being able to make the sequence move closer and closer to l. That whatever we have distance, however small, I can make that, I can show that after finite number of terms all my sequence terms are getting inside the band; the x n would lie between l minus epsilon and l plus epsilon. This actually implies; if you say that this is less than epsilon, then it would imply that x n minus l is between plus epsilon and minus epsilon.

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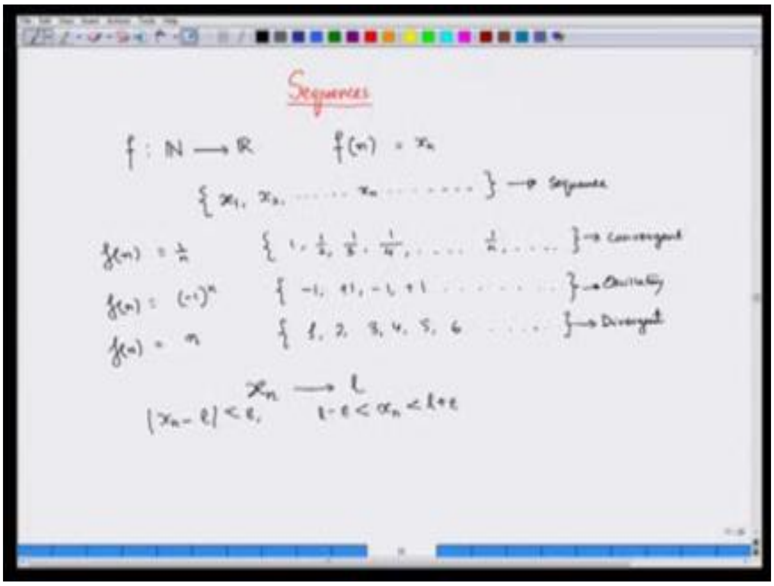

So, x n is between l plus epsilon and l minus epsilon. The idea is that you have to tell me if I give you epsilon and you have to tell me, what is the finite number of element that you are leaving off? So determination of the finite number of elements is the main thing. So given an epsilon, you have to tell okay. If I throw away this finite number of elements from the sequence, then rest are all within this band. That distance is all less than epsilon.

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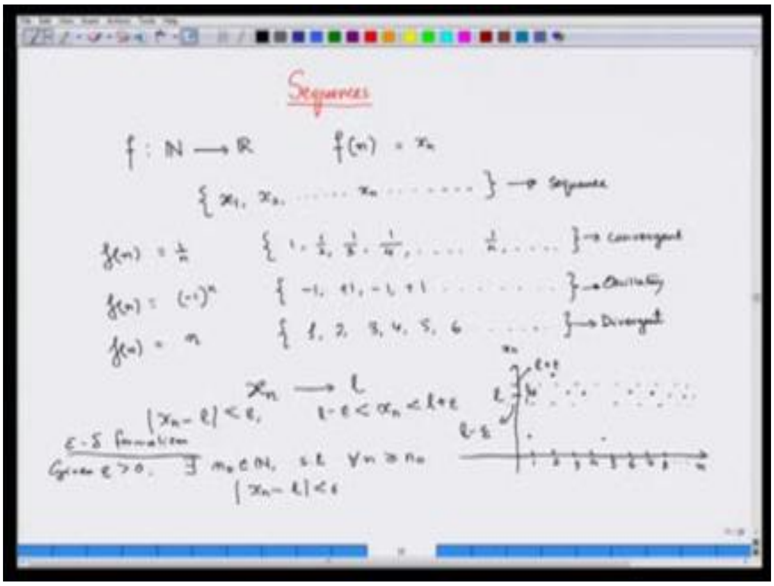

This is cloaked in a much more formal language, which we call the epsilon delta formalism. Though you need not be too much bothered about it at this stage. But just for sake of it, I am just writing the definition. So, given epsilon greater than 0, no matter how small, there should exist an n naught, a natural number such that for all n greater than or equal to n naught, I must have this fact true. So, x n going to l simply means that if you give me an epsilon that I should be able to tell you what amount of elements of the sequence I should forget. And, remaining every one of them should lie between l plus epsilon and l minus epsilon. That is the very basic idea which we can draw out.

Suppose this is n and this is x n. Now, we can just talk about; this is my l, which I think the sequence will go to. So, I will take some epsilon and create a two epsilon band around l. This point is l plus epsilon and this point is l minus epsilon. And, now suppose this is my 1, 2, 3, 4, 5, 6, 7, 8. So, may be this is here, second one is here, third one is here, the fourth one is there, the fifth one is here, sixth one is here, seventh one is here, eighth one is here and so on.

So, what I see that the first one is outside the epsilon band, and this two epsilon band - epsilon and epsilon. This length is two epsilon; two epsilon. So, it is outside this two epsilon band. And, so is 2. And 3 is inside, 4 is inside. So can we say, if we leave out two elements, everything is inside. No, we cannot say that because 5 is outside. But after 5, we see that everything is inside.

Let us assume that it remains inside, and then we can say that if I leave out 5 of them, then for this particular epsilon, all the other term starting from sixth, toward 6 towards infinity and going on that would remain in this epsilon band. This is the very basic idea of convergence, which we have written down in this language.

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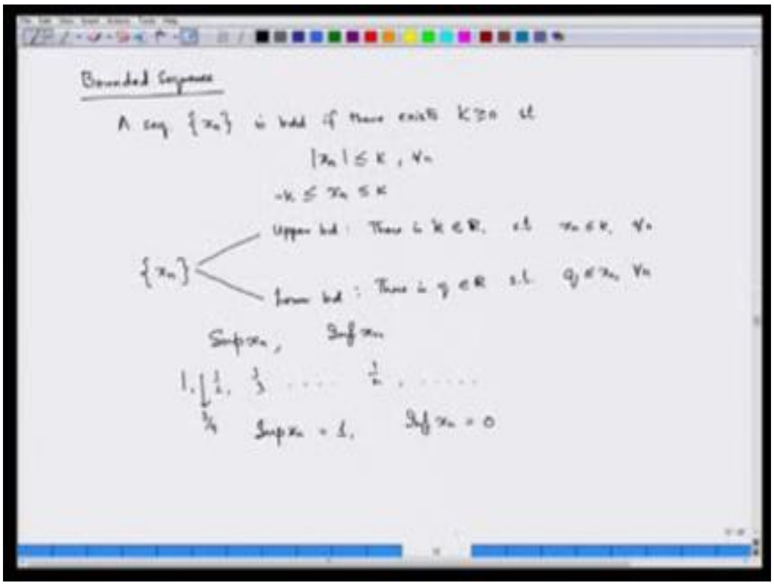

Now, a very important notion is that of the bounded sequence. So, a sequence I am writing “a seq” for short. This is a notation of the sequence in short. It is bounded if there exists K greater than or equal to 0, such that mod of x n is lesser than or equal to K. This simply means essentially K is the upper bound of the sequence and minus K is the lower bound of the sequence. In fact, you can separate them and say you have a sequence.

Sequence may have an upper bound, but may not have lower bound or may have both the bounds. So, for example, upper bound means there is K element of R, such that x n is less than or equal to K for all n. This is true for all n. Now, you can talk about lower bound. There is q element of R, such that q is less than or equal to x n, for all n.

These two concepts will become very important. But, a very important idea in Mathematics is existence of least upper bounds. So, if K is an upper bound, possibly there could not be any of upper bound lesser than K or there could be an upper bound lesser than K. So, finally we will find an upper bound, which is called the least upper bound or the supremum. So, supremum; we have to decide the notion of supremum of x n and the greatest lower bound, which is called the infimum of x n. These are two important quantities.

If you take a sequence and you say that say 1 is an upper bound to a sequence, then 2 is also upper bound to the sequence. So, I will just go back and show you. That is how you take a sequence. So, plus 1 is the upper bound to the sequence and 2 is also the upper bound to the sequence. Everything is bounded by 1. So, 1 is the upper bound to the sequence; minus 1 is the lower bound of the sequence. Then, minus 2 is also lower bound of the sequence, but plus 1 is the least upper bound in this case. So, because if I consider say anything less, anything less than 1, it cannot be upper bound to this.

So, you have to; you cannot get an element. You should be able to. See, what happens is that when you have this sequences, so among all the upper bound; means, what do I mean by is very important concept, and has to be kept in mind that is very useful in many things, including algorithms. See, what happens? You have taken an upper bound here and then you have considered a lower bound here. But, among the lower bound, there is something called the greatest lower bound; among the upper bound, there is something called the least upper bound.

It means if you drop that number, then that number, that will not become an upper bound anymore. That is the whole idea. That is if you drop that number by some quantity that should not remain an upper bound of the sequence. For example, I will tell you. You look at one, half; this sequence. What is an upper bound here? One is upper bound here. If you decrease it to say one minus epsilon, see if you take any number between one and half, you take three-fourth. And, is it three-fourth an upper bound? No because the quantity one, which is the element of that set is bigger than this. So, it cannot be an upper bound. It violates this definition.

Similarly, in this case this sequence. So, in this case the supremum of x n of this particular sequence is 1, while the infimum is 0; because you increase any number, 0 by any number, always find the n such that one by n is below that number. This is something that you have to give a little thought to. These are very important concepts.

So, I will come to now the concepts of increasing and decreasing sequence. Before that, let me tell you that every convergence sequence is bounded. Proof can be done by using epsilon and delta formalism, which I will not get into the details. If I have time, I will tell you a bit.

But, let me now tell you that every convergence sequence is bounded, but every bounded sequence need not be convergent. For example, the sequence minus 1, plus 1, minus 1, plus 1; it is not convergent. But, still it is bounded. So, bounded sequences do not need to be convergent, but convergent sequences need to be bounded.

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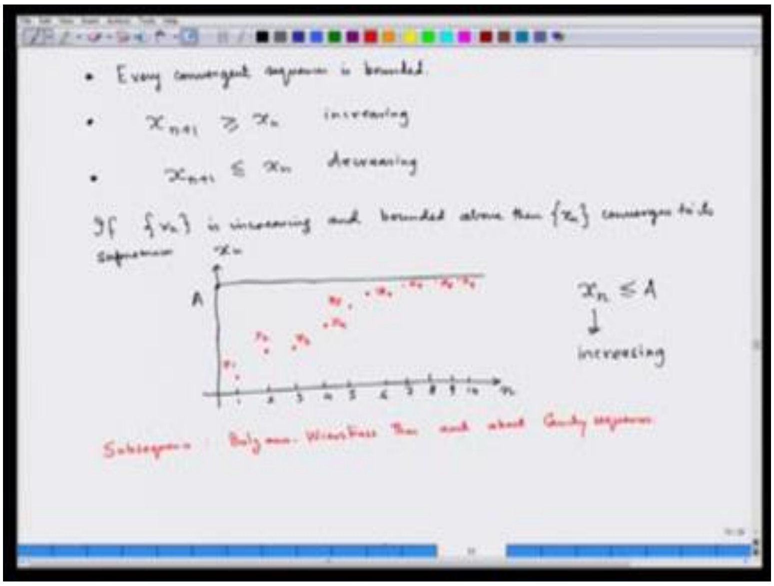

And, I want to repeat this term as I go on. That you have to know that sequence is much better, if you know that every bounded sequence need not be convergent, just like, I will tell you that when we learn about derivatives. You will know calculus much better, if you know at the very outset that every continuous function did not have a derivative.

Let us look at something called increasing sequence. You can understand what is the meaning of increasing sequence. It means the terms of the sequences increases, as I increase the value of n; that is, x of n plus 1 must be bigger than x n. This is what we call increasing sequence.

And then, something is called the decreasing sequence, that as I increase the value of n, the value of x n decreases. So, what is so fascinating about increasing and decreasing sequences? If I have a sequence which is bounded in increasing, then that sequence must be convergent and will converge to its supremum. And, if i have a sequence which is lower bounded and decreasing, then it will converge to its infimum.

If I have a sequence which is increasing and has upper bound, it will converge to its supremum. So, here I am giving that if I have a bounded sequence, it will not be convergent. But, if it is additionally having the feature that it is increasing or decreasing, then it is convergent. So, here we are adding some conditions on a bounded sequence to show that when a bounded sequence can be convergent.

So, if x n is increasing and bounded above, then x n converges to the supremum or the least upper bound. And, x n converges to its supremum. The idea; so please look at the supremum and infimum idea very well. We have already mentioned the text. And, I think you should look into the idea of supremum and infimum very well; because we are, of course doing the things in a very short time.

Now, let me explain to you this idea pictorially. And, by explaining to you this idea pictorially, we will end today's discussion. Let A be the bound. Let x n has an upper bound given by A. Let this be A. This is my n. I am plotting the n and this is, I am plotting the value of x n.

Now my function, my thing is increasing. So this, I assumed as increasing. Let us see. This is say x 1. Say x 2 must be bigger to or equal to x 1. Let us say x 2 and x 3 are the same. Say, then I had x 4, then I have x 5, then I have x 6. So, none of them can exceed A because every value has to be less than A. Then I have x 7 and I have x 8. You see this is almost becoming asymptotical to this line, x 9 and so forth. So, it must converge. It cannot go beyond, so it asymptotically if this A is my supremum, it asymptotically going towards A.

So, with this idea I would like to end today's discussion on sequences. For tomorrow we will, or in the next class we will concentrate on talking about something called subsequence, Bolzano-Weierstrass theorem and about Cauchy sequences. These are very important aspects of calculus, advanced calculus rather. Now, we are not doing high school calculus. Just I want to keep in mind this thing.

So, I hope you have a very basic idea about what is going on. You have to remember convergent sequences are important; every bounded sequence is not convergent. But, if the sequence is bounded above and is increasing, then it converges to its supremum and if the sequence is bounded below and is decreasing, then it converges to its infimum.

Thank you very much for your attention.