Basic Calculus for Engineers, Scientists and Economists

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Lecture – 29
Multiple Integrals – 2

We are going to talk about Line in line surface and volume integrals. Means line integrals, surface integrals and volume integrals. Please be sure that these parts have tremendous importance in the Physical Sciences and tremendous importance in Engineering Sciences. You just cannot do without what we are now discussing.

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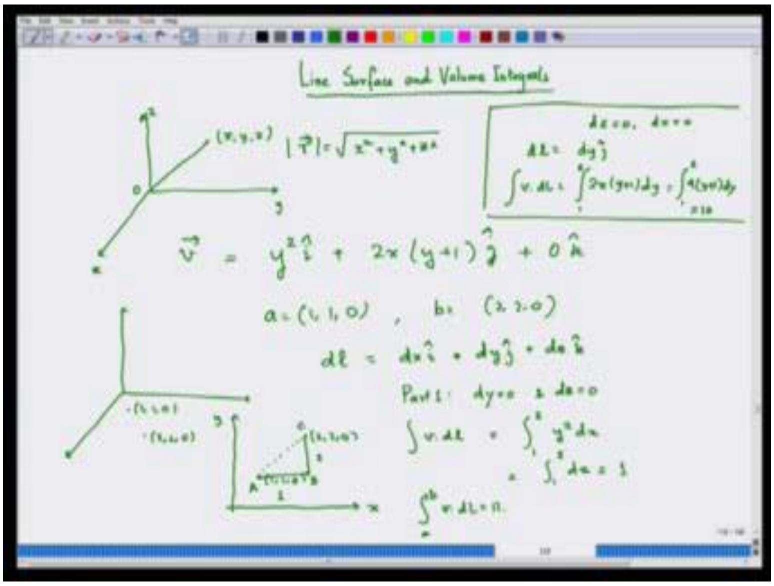

Essentially we are going to stick to what is a three-dimensional space. We have been only talking about two-dimensional spaces for essentially function of two variables. So, but if you look at the place we live in, we always are aware of length, breadth and height. Basically we are aware of 3 numbers length, breadth and height. Basically in a three-dimensional space, any number, any point can be expressed by its length, breadth and height; which is the which called the coordinate of the point and it is given by three real numbers say x, y and z. This is the x-axis, the y-axis and the z-axis.

So, any if I connect it with the origin o, of course, we have to fixed our axis coordinate axis, the reference frame with some origin o, which can be arbitrarily chosen. And this one this distance is computed just by using Pythagoras theorem, and just like in the two-dimensional case, the distance from 0 is just this quantity. If you look at this point x, y, z, if I have a point x, y, z here, it has one direction, if this is the origin.

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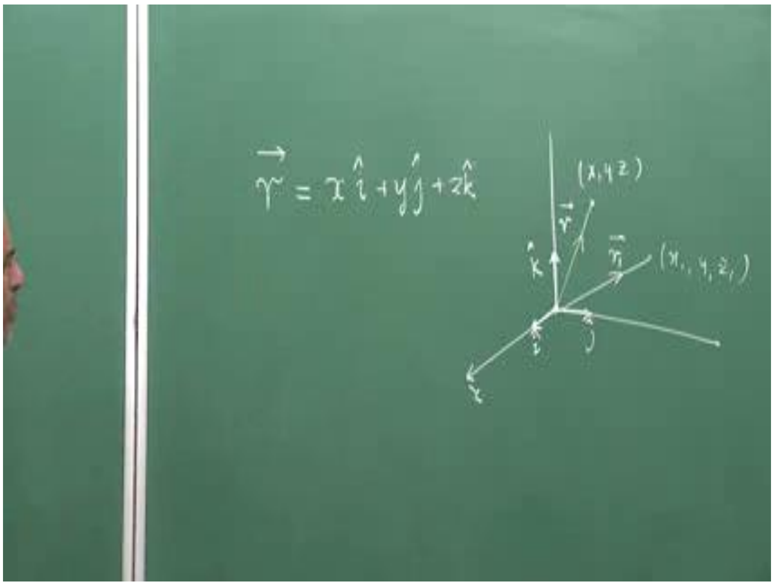

If I take the origin at some point, and if I take a point here, it gives me (x, y, z), and then if I take a point here it gives me another x 1, y 1, z 1. These two things have there may might be of same distance from the origin, but are different objects because they are in pointing in different directions and these objects which has both direction and magnitude is called a vector. This is of great importance that we note this fact. So, here along the x-axis, y-axis, and z-axis we consider three vectors, which have which are of length one, distance one from the origin.

Along x-axis, I considered i vector; along y-axis, I consider the j vector; along z-axis, I consider the k vector then any vector, which I now denote as r vector. This could be denoted as the r 1 vector. This r vector can be always written as in terms of the coordinates x i plus y j plus z k; those who know some linear algebra would immediately recognize that i, j, k are nothing but the canonical basis vectors.

But let us talk in terms talk in the way if this is would actually talk about. And in this, so here we call mod r the distance vectors here, among the vectors i would have some important way of looking at things. There are set of important operator, which I will again go to the board and do, which will be much better.

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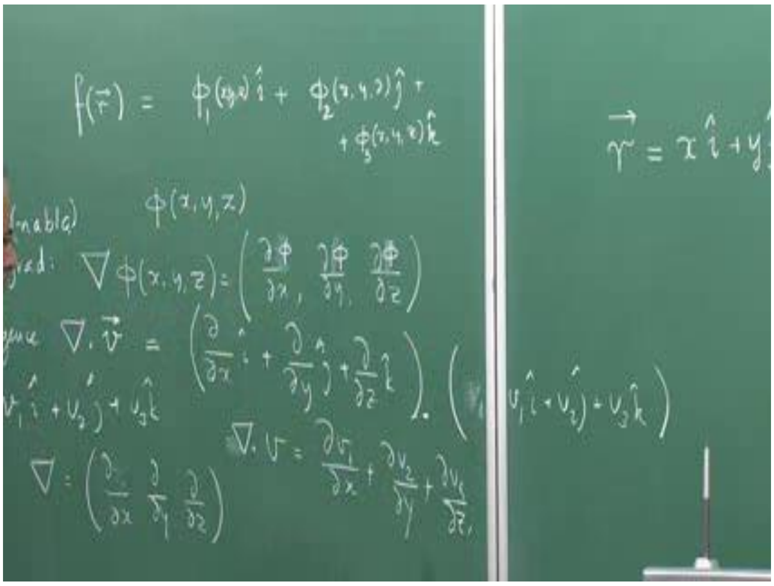

For example, if I take function of some r vector, it could be say some function of x into I vector some function of o or may be x, y, z it does not matter x, y, z into i vector some function phi 1 and phi 2 x, y, z of j vector. If you look at the vectorial part, this is nothing but a scalar composition of a vector. So, you can always write a vector in this form, because the length of this would be exactly this x square plus y square plus x square root.

So, many of you have possibly know this we have writing for many of you have studied physics. I am assuming that you have this basic knowledge that I can always write a vector in this form that scalars multiply the vector i with this, vector j with this and vector j with k. Then I make vector addition it will give me these vector; the vector r. I can have, you can also write the thing in this particular fashion right.

Now, if I have function of 3 variables, so this is a function. These are these sort of functions pretty often common suppose, I just have function of three variables. And then there is important operation called the gradient operation, which is nothing but a vector collecting all the partial derivatives. This is nothing but a collection of a partial derivative, sorry del phi. So, will be only concern you can now also extend it to n variable what we will be concerned only with 3 variables. Why I am writing this, because you have already aware of these for 2 variables.

But I am doing this to make you aware of that we can make an extension to three and more, what we will be just concerned with three here. There is a very important operation that given a vector v right, there is operation called this. This is the grad or grad operator we just called, also call it the Nabla operator and this is the divergence operator; how do I do it. So, just like the vector function v it is possibly a function of (x, y, z). So, any if at any point in (x, y, z) I can have a vector function like this. Basically, this is a r vector is basically a (x, y, z) f of (x, y, z) is this.

Then these represents of vector field this is what is called a vector field. So, v vector is written as v 1 i vector v 2 and these are scalar v 2 j vector and v 3 k vector. That divergence operator is often written as del del x, del del y, del del z. this is usually written as divergence means, from a point lay let say a light source; how the light is diverging in all directions. That is that is the physical meaning or trying to look at a spread of a physical quantity from a given point.

This so divergence actually now is can be written in the following way, basically we were looking at the dot product of two vectors, dot product is some and with you do the operation like this (Refer time: 07:46). These are some quantities these are composition is a coordinates of a vector. That can be also written as del del x i plus del del y j del del z k. And this you take an inner product dot product with this part v 1 i vector v 2 j vector, v 3 k vector v 1, v 2, v 3 all this can all be functions of x, y, z. So, finally, what will end up is you multiply take d d x of v 1, d d y of v 2, d d z of v 3. So, del dot v del dot v finally, becomes a scalar, which is del v 1 del x plus del v 2 del y plus del v 3 del z.

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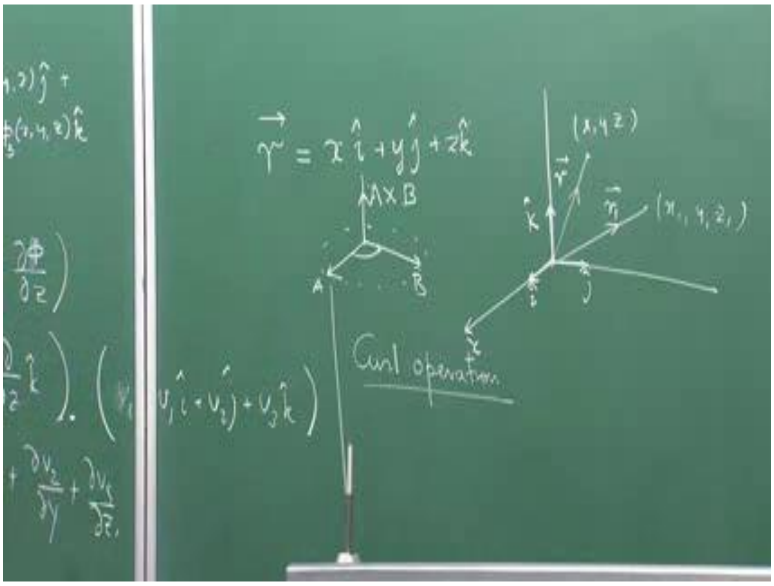

Similarly, you can talk about another operation called the curl operation. So, curl operation is like putting screw. Basically of two vectors a and b. Basically this is the base of screw and you are not try to rotate this screw, you are trying to rotate the screw from a to b in this direction. Then the screw would actually get inside a wall.

For example, there is a screw and you want to rotate the screw; here it will get inside the wall. The vector the movement of the vector you along these directions, these gives me what is called a cross b that this called the cross product. The curl operation is based on the cross product.

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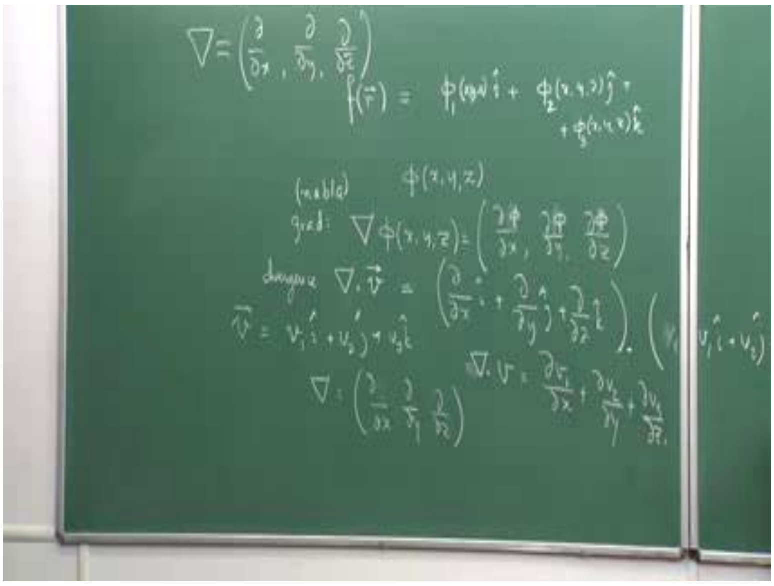

It is a same operator del, del operator is this operator del del x del del del z. So, (Refer Time: 09:51) are always using this theoretical features are always using this this operator. Mathematician would not like this sort of symbolisms, because this does not mean anything for mathematician, but to a physicist, it has a lot of meaning it is a del operator, where you can play lot of games with this. So, and you get conclusions about physical things, so this is very important.

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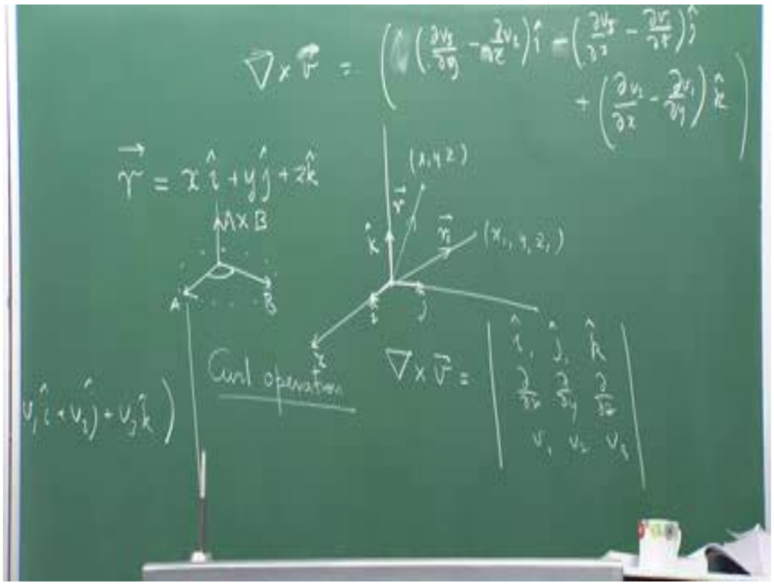

So, curl over vector v, what it means. This is defined as follows i, j, k again it will not make sense to mathematicians, because you are talking about determinant and you are talking about matrices in, you are talking about basically vectors in them i vector, j vector, k vector, del del x, del del y and del del z. But it is now not harm - no harm, if you think like a physicist. Now you obviously, calculate the things out.

Now if you look at this then this vector, the v vector I should may be put putting an arrow. If I am trying to behave like a physicist, then this is what, this is nothing but i into this, this, minus this. So, you will have i vector or rather I should put this first del del y of v 3 minus del del z of v 2, del del v 3, y minus del del z of del del z of v 2 this into i vector. It is vector. The curl operation gives a vector grad operation of because the vector all the divergent, divergence operation sometimes written as div v this gives you a scalar.

Now, if you have to take this and this. It becomes del del x of v 3 minus it just taken the determinant in the standard way, but it is really not taking a determinant you can write it in the form of determinant, but this is actually the definition. So, cyclical movement of the vectors del v 1 del z, z vector, j vector plus del del x, v 2 minus del del y v 1 k vector this is actually the definition of curl. But physicist have invented this short hand of writing this whole thing in the standard form or determinant, but obviously, I have addition found really like this sort of things yes. Is like a standard taking of determinant. Is like a scalar 3 2 3, 1 2 1.

So, once this is done you can actually compute out this sort of things this objects. Now, once we have this ideas in mind you can compute out very simple examples of these this is just two simple for me to waste time on it this format, because we have we are we always run short of time. So, we are I am going introduce what is called the line integral and then we are going to talk about the volume integral, in the surface integral and the volume integral.

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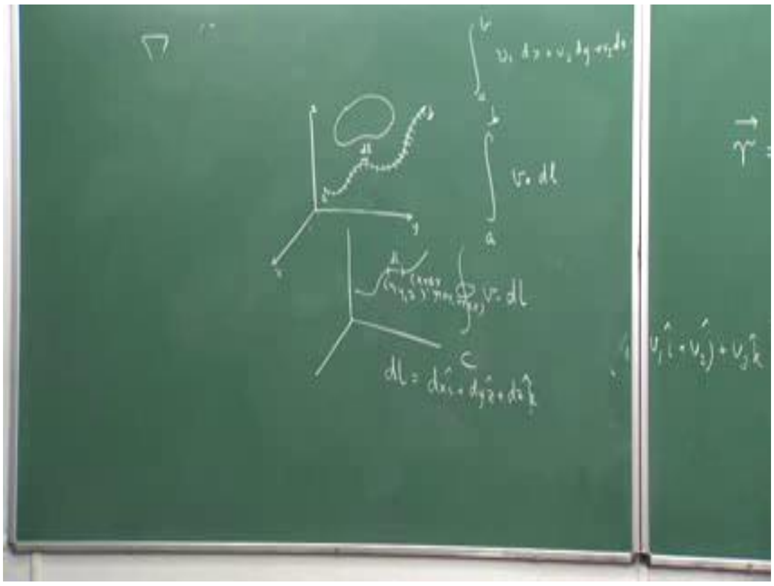

So, what about the line integral? Basically, you line integral is nothing but you want to integrate of vector over a line in a space say starts from point a and this line goes to point b. So, you can break this line up into small partition. Integral is all about partitioning constructing, and taking this times stuff over those small partitions and then moving on and summing up. This is elemental interval, which I called del l d l, it is the line element. Basically I am looking at the integral v dot d l from point a to point b. This is what I am retrying to figure out.

Now, suppose I am in a situation, where I am talking about, so v dot d l. So, d l then essentially consists of what. This is line element and basically this is nothing but is a piece of an arc. If it is x, y and z you know how do you write the piece of that arc. So, each length is basically this is nothing but d x square plus d y square plus d x square, square root. So, you take one of them out and then you keep the rest. So, we will just show you by an example how to calculate this.

But if you are trying to now consider a path, which is close like this I am trying to take an integral from this end to this end. If it is over a closed path right then you have over the close path you essentially you are talking about line integral in this form. Then we put a circle here to say that we are moving over closed path, but the moving around close part does not always give you integral 0.

So, what if you look at d l vector, what is the d l vector the length vector, look at d l. So, what it has it has this small length as has three components. There was some (x, y, z) and then you have made a little change to x plus d x y plus del y z plus del x. This was (x, y, z) coordinate and then you have changed over two x plus del x, y plus del y, z plus del z that sort of thing.

Basically when these are very small del x is d x, del y is d y, del z is d z. So, we have basically you know that the differential is nothing but for the independent variables nothing but the change the very little change that you make of the variable. So, what does d l consist of, then d l actually is d x of i vector, d y of j vector, d z of k vector and that is exactly v is actually if v d x plus v 1 d x plus v 2 d y plus. It this whole thing is actually v 1 d x plus v 2 d y plus v 3 d z that is exactly it.

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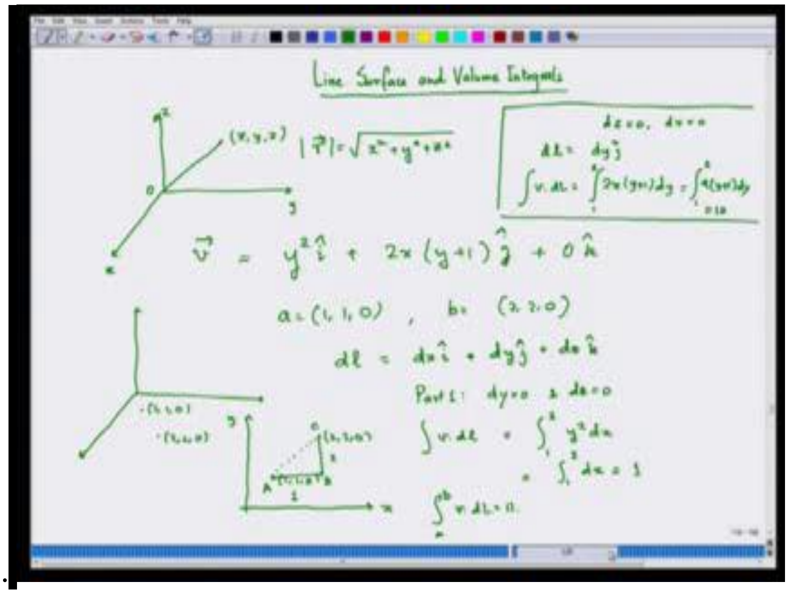

Let me just work out an example here. Let me now consider one example. Let v vector it will be given as y square I vector 2 x into y plus and you see it can many functions of (x, y, z) or just takes just y whatever you want. I did not have anything say. This is in two dimensions just for first introduction. Now, I want to take. I can put 0 into k vector. This is just essentially a two dimensional stuff, but if you look at in three dimension you actually go in three-dimensions also. Basically then let me talk about the two points that I am choosing a and b. So, a is (1, 1, 0) something like one and one and 0. This is the point (1, 1, 0) from here I want to go to (2, 2, 0) this point say essentially it is a two-dimensional stuff.

The vector is essentially a two-dimensional vector, but you can look at it from a three-dimensional perspective also it does not matter. So, b is (2, 2, 0). So, you see. So, here I have written a to b, a to b actually means basically you have to integrate, but putting the proper limits on x and y. Then here what would happen here I would have now suppose we want to move along this bold green path. So, we want to go from (1, 1, 0) to (2, 2, 0) on this path. So, we are instead of I am looking at a three dimensional thing in this two dimensional picture. Along this, so d l is this 1, which we have already also written there.

Along this path if from a first move from x equal to 1 to x equal to 2 and I do. I do not make any change in the levels of y and z. This is the first part of the path; this is the second part of the path. In path part one of the path d y is 0 and d z is 0, there is no change in x and y. Basically I have the d x part. Basically integral v dot d l. So, d x is changing from 1 to 2. Is 1 to 2 so, x square d l. So, v dot d l is y square d x. So, what is y here, so y has maintained the value 1 it has not changed, you see y is all y is in the; we are changing horizontally. There is no change in the y value. We define x is changing from 1 to 2 y remains to take the value 1. Basically it becomes 1 to 2 d x, this becomes 1.

Now we go from the point this say if I am make it a b c I have come now from a to b and I have to go up to c see if I have to do that let me do the calculation here on this same page, so that you do not lose the track. So, here I am changing only y, I am not changing z, I am not changing x. So, my d z is 0 and my d x is 0. So, my d l is d y z vector. So, what does this mean what is my v dot d l, but along this path, what is the value of x, that see you know this. So, my v dot d l. So, my integral v dot d l this will be integral what is what was my second part 2 x into y plus. So, y is now changing from 1 to 2 into d y, but what was my x, x was x is held at 1. So, when y is changing from 1 to 2 my x value is 2. Basically I will put 1 to 2, 4 into y plus 1 d y and this will give me the value 10.

Finally, I will add these two things. To write that my integral a to b a is (1, 1, 0), b is (2, 2, 0) this is 11. The question is what will happen if I take it from point a to point c you can try out by doing. So, from point a to point c you observe that only z does not change, but x and y both changes right. Basically you have to write it in this format. So, d l would be exactly the same way we have written. If I go from now I want to go from on the straight path.

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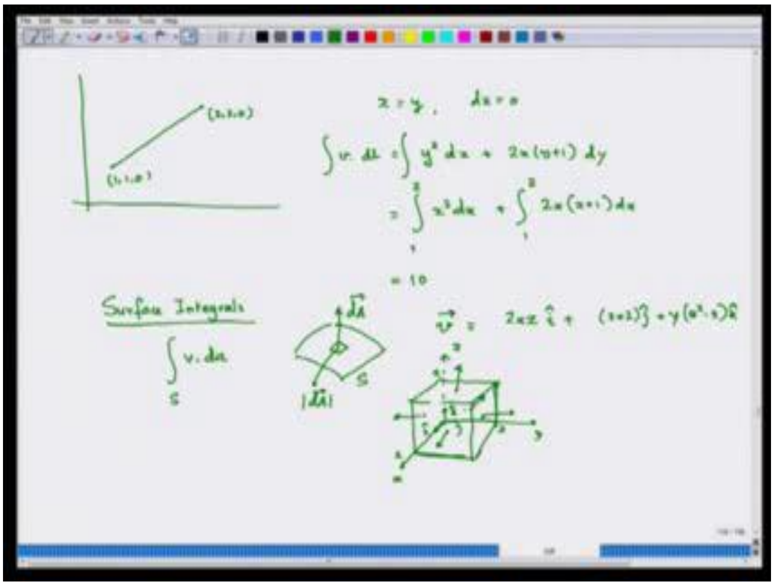

So, from (1, 1, 0), (2, 2, 0), so here observe what is the important thing, the important thing to observe we had there is at x is equal to y and the second thing is to observe the d z is equal to 0. So, my integral v d l will have. We have to go from basically I can put d x equal to d y here. In this case my v dot d l I will put y square d l is d x d z is 0 plus the next part is 2 x, y plus 1 and I will put the value here as d y, but d y is equal to d x, because x is equal to y. So, x is equal to y this will become x square d y plus same going from 1 to 2, going from 1 to 2.

Here I will put x, 2 x into x plus 1 d x. If you evaluate this you will get finally, the answer 10. The message there is that if you move from (1, 1, 0) to (2, 2, 0) along path one we get the line integral to be 11. But if you move along path two we get the line integral will to be 10. So, but the there are certain cases, where the line integral does not depend on the path on which we integrate they only depend on the n points that sort of vector fields where it happens is called set to a conservative of vector field and it is of prime importance in physics. For example, gravitational field is a conservative of vector field.

Now, you are going to talk about surface integrals and volume integrals. So, surface integrals is that OK; your surface an over which you want to do the same sort of integration. If you have a surface like this then da would a small area the elemental area. This is a elemental area on this surface s and the surface area if you want to express the areas of vector it is a vector pointing outward from the surface. This is the d a vector then this is the magnitude of the vector this is the d a, this is the surface the whole surface.

So, of course, there is little bit of ambiguity about which way is the outer one which way is the inner one if the surface is like this I can say I can think that a way as outer one. But usually there is understanding, that you for example, if the surface is a closed surface, then you can understand what is the interior and you know what is the exterior. So, anything which is pointing towards the exterior is the one is the direction of the elemental area. For example, you can we will just take an example here. So, you take a example of v vector. Now I want to take it out on a cube, I want to find the integral of this on a cube on the surface integral, a cube which is of length breadth and height 2.

In the next class, we will first start by computing this one. Once we start by computing this one we will then go about to volume integral show you how to compute and mention several important theorems using these integrals and that will be the end of the course. It is the same game here you now if take elemental area you first you will take one area by one surface by another; I want you know about the whole surface. So, you take this 6 surface. The elemental areas are like this. Suppose, I want it, I do not take 6 side, I take it over 5 sides, I will I forget the bottom I will take it over say this side, this side, this side, this side and other side outer.

This is how I think about the elemental area; I know what is which is inside of this cube and what about, which is the one which is for which part of the cube is outside one. If I know this, I can easily draw out. You can also take the six one you can go the outward curve would be like this, but you can just do it for the five if five once. Try them out. So, here what is what is the area an elemental area, if a small elemental area is here, for the (Refer Time: 28:17) very it for one facing us, the very first one facing us the elemental area is dx dy, the dy d z, because essentially it is the change you are looking the length and breadth, this length is y and the breadth is y, and the length is z.

It is dy dz into x into i vector the vector is here. It is the vector along i vector along the x-axis. So, you have to just find the elemental areas like this and just go on doing it step by step that is all here you to have a double integral, because you will have in terms of dz dy, dx dy, dz dx and all those things.

We will just try to do the calculation very past in the next class; and go to volume integral do a calculation, and mention the theorems of Stokes, Green and Gauss. And with that, we will end; we would not have time to really go through doing examples then we need much more classes to finish it. So, we will just end it here.

Thank you.