Basic Calculus for Engineers, Scientists and Economists
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Lecture - 28
Multiple Integrals – 1

We are absolutely in the last part of the course. And we are talking about multiple integrals and for this you need a maturity level, which many people in varied audience might not have, those who do not have you may be skip this part without any problem.

Those who have some maturity in thinking, they can take a very good look, they can run the lectures repeatedly or even take a look at the books you know lectures there about books cannot replace, otherwise they would not have been any books and everybody would just lecturing and giving notes. So, because books are organized, so thoughts are already organized in a book and you can give a proper discussion. For example, we are going to first talk about how to integrate a function f (x, y) over a rectangular domain.

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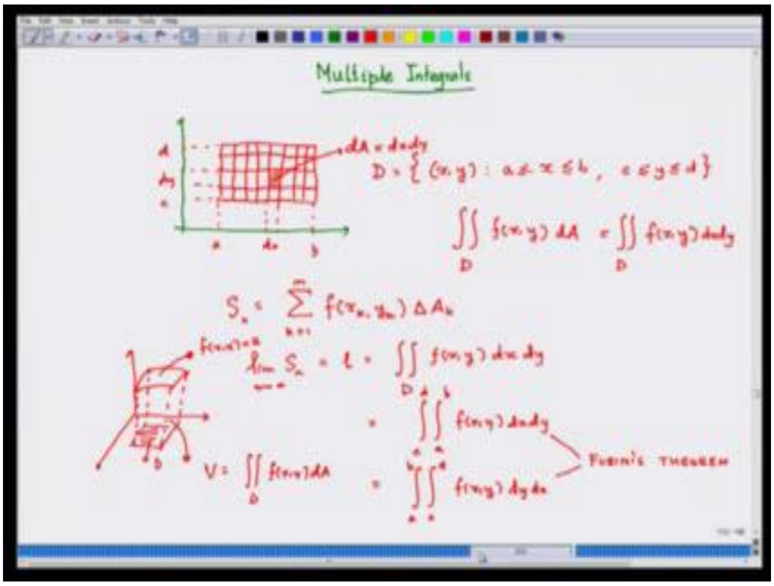

Now, we are in say two dimensions, and I consider rectangular domain. So, here is my 'a', here is my 'b', here is my 'c', and here is my 'd'. The rectangular domain D - this one consists of all (x, y) where x lies between b and a, and y lies between c and d. This is my domain. And I want to ask the question what is the meaning of this integral, can I if I write a symbol like this that f (x, y) is the function of (x, y). So, z equal to f (x, y) and dA is a elemental area which you can write is dx dy also dx into dy you have an elemental area taken out of the given domain D. So, what we do is you basically divide the domain into rectangular grids.

So, we will take this one for example, and if you say this part is d x and this part I mark as d y and this elemental area this 1 d A, these are very small area the picture it looks very bigger does not matter. This is actually d x into d y it is a rectangular area (Refer Time: 03:04) basically you are asking what is the meaning of this integral.

Essentially what it means, that given any such grid a grid is just like a partitioning that we have done on the d 1, when we started d 1 integral integrals partition the line here we partitioning the domain. When we partition the domain, what we do is the following we construct the following things. You take a say the midpoint of every domain you take as a as a reference point and then you contract this sums. So, at the kth domain and del a k is the area of kth domain, basically do if it each and every one of them.

Then you start taking partition finer means you are taking the domains with smaller and rectangular of smaller and smaller area. If you take the limit of the area s n, n tends to infinity when n tends to infinity, because that mean a number your partitions number of your rectangles here becomes very large. As a result of which you have more and more, each sum is a bigger sum basically, you have more terms in that sum. The you want to compute this and suppose these are the finite value say l and then this finite value is assumed now written as is defined to be the integral or double integral of f (x, y) over the elemental area d A. You can also write this as in terms of the limit.

In fact, we need to show that if f (x, y) was continuous you can actually swipe the limits, you can change this order of integration; these only for a rectangular area for more general area you have do something else. This is this is this fact is what is called the Fubini's theorem. Now, what does this tell me, what does this double integral tells me, it tells me that just like you find an area under a curve, under a surface f (x, y), z equal to f (x, y) represents the surface the graph is a surface.

Essentially, when you are integrating over a rectangular zone you are finding the volume the solid that can be placed under that particular area; this is my domain d. And suppose this is my surface s then the volume. This is my surface that you have f (x, y) z equal to f (x, y). Now this surface is a graph this function f (x, y) and now this is my domain d and here suppose you fill it up with say solid l or some solid inside. Then you have to going to find the volume of this region enclosed by the domain d and the planes on the 4 sides here. This integral if you look at the volume V this is actually integral d f (x, y) dx. Let us just go in and try out one such example at a same time verify the Fubini's theorem.

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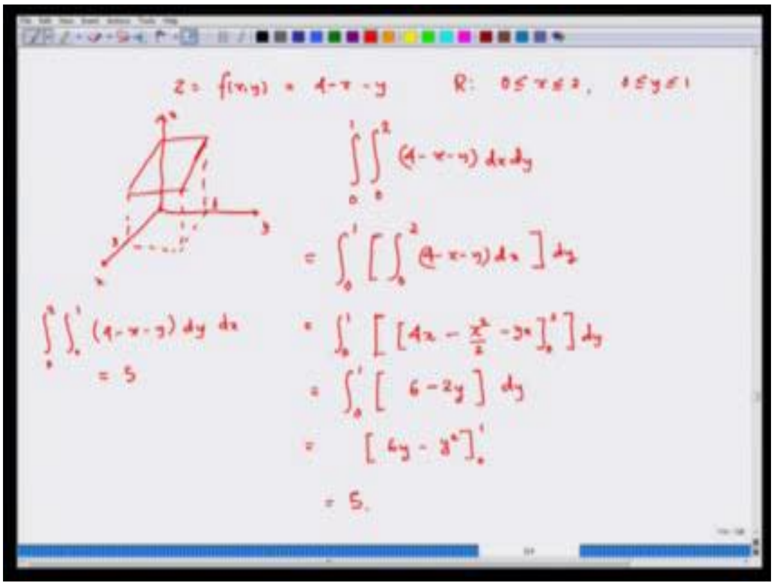

Suppose you want to calculate the volume under the plane, equation of plane in three dimension is given as x plus y plus z plus constant equal to 0, so this defines a plane. Now I want to find the volume under the plane. So, what should one do? So, here is this thing something like this and the domain that is given to you is the following rectangular region. It is up to 2 and that is up to 1, y is up to 1, y axis, x axis, z axis. So, you are basically trying to find the volume. There many where of looking at it.

By Fubini's theorem what we can do you can verify Fubini's theorem. So, find the volume, basically you now have to compute the integral x is from 0 to 2 and y is from 0 to 1 and this is my function 4 minus x minus y d x d y. As the first step you would just integrate the first integral in terms of x where you will keep the y constant.

If what you will get back is the function of y. It 0 to 2, 4 minus x minus y d x and that will give you suppose this will this give you. It will give you 4 x minus, here I will get x square by 2 and here I will get minus y x and this has to be put within 0 to 2 d y. If I put it within 0 to 2 d y, it is simply going to give me the following function, is going to give me; if I put to here 8 minus 2 square is 4, 8 minus 4 minus 2 y. It is 6 minus 2 y right. Now, calculated this integral and then I will get this one.

So, what I will do, now I will just now have to do this. It will become 6 y minus y square 0 to 1 and you know that this is this will give the answer 5. So, you can check it up in the other way also you can try then the other way you can do 0 to 2, 0 to 1, 4 minus x minus y d y d x. So, you first integrate with respect to y holding x constant then you will get a function of x then integrate with respect to x; you can verify that the answer would again be 5. This is a very basic idea which come and this two are same. It also verifies the Fubini's theorem.

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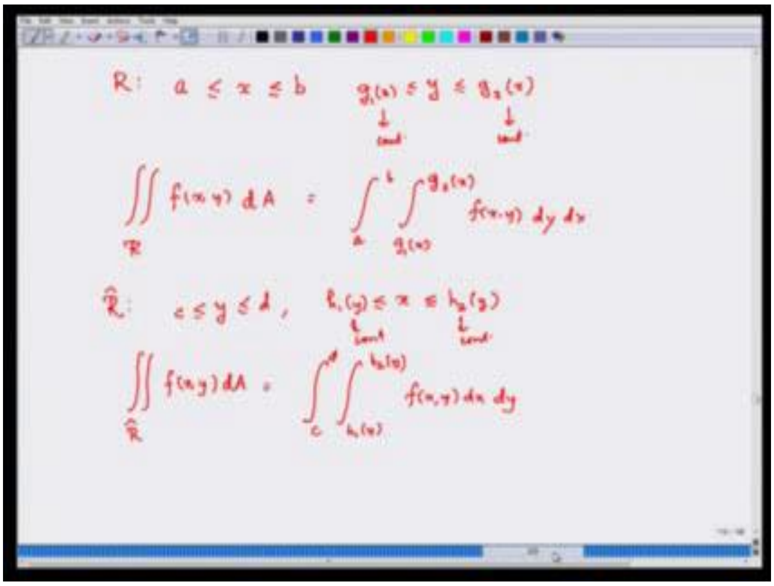

There is some more general thing in the Fubini's set up is that sometimes you know this y does not vary between a fixed number it is not a rectangular 1. So, y varies between two functional values of x, that is you have x varying between a to b and y varying between the values of two functions g 1 and g 2, which are continuous. So, g 1 and g 2 are continuous functions.

In that case, how would you really what is your my domain now is this my rectangular region or the domain is this. Then in that case what is the meaning of this integral how do you compute it. It will be because now x is fixed (Refer Time: 11:43), if that as outer one, do the functional calculations, because y is in now terms of functions of x.

So, you have do, put y as the in our calculation, that I integrate over y first and then x. So, you can do this. It could be also different you mean R could also be for example, if I have region R say R had, where y is given in y is given the region y is fixed. But my x is varying within two functional values of y given by two functions which are continuous. This is also all to due to Fubini and this Fubini was the Italian mathematician we gave this results in 19 100 7, so a very pretty old game as you see.

Now, your y should come out side, because that is outer variable, which is fixed. Now this would remain is ultimately you have to get a function of y to integrate over y and here I integrate our x first and then y. This is what we will happen if I take for sometimes, but you know it is not so easily, that even swipe the things is not so easy things can be become slightly difficult. I will just show some examples of this and we will try to go to some other thing.

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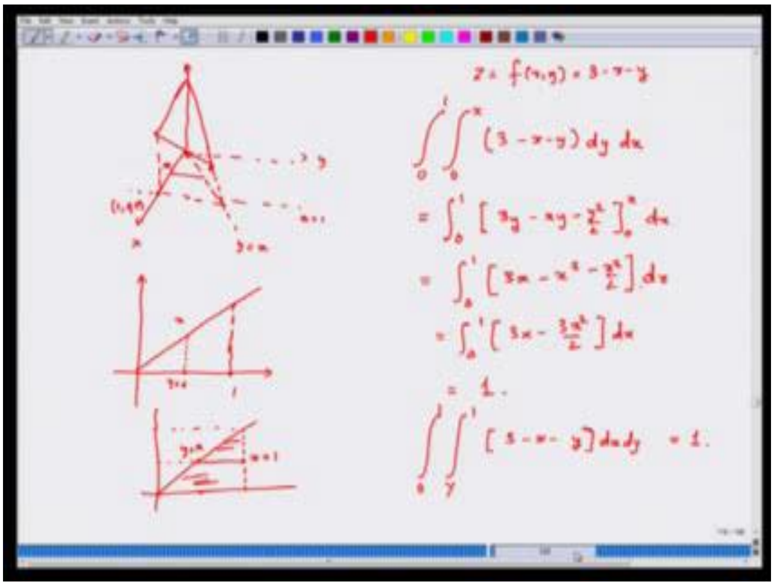

So, basically I am supposed to find a volume of a prism. Basically, I am having a given plane which is of the form z equal to f (x, y) equal to 3 minus x minus y. And then if I have this I am asking you to basically find the volume over a triangular region and what is the triangular region I am fixing the triangular region at x equal to 1 look at the line this and I look at the line y equal to x. This is the line y equal to x and this is the line x equal to 1 basically this is this is point (1, 0, 0).

And if I look at this expression this would be some this would be a plane, which would be of this form. You come down, here and meet basically then I have to find the volume under this. Of course, here I have to be to be slightly more may be the plane will not meet like this plane would slightly more sharper.

Basically now, I am telling you to find this volume. This is presuming now. I am now trying to now here you have to be very careful, because when x is varying from 0 to 1, y is varying from 0 to x, because the in the on this line because y can I now vary from here to here as I draw the line. As x rise from 0 to if x is at certain point x then y when x is varying from 0 to x, y is varying from if because if this point is say sum x at the point y is equal to x. If x is varying from 0 to 1, y is varying from 0 to x. For any for any x that I take y can take values from 0 to x. That is the key idea for any x that I choose here y can be anywhere here.

I have to swap up this part of the domain also. So, basically now, my integral would look like 0 to x. These are my x functions and here this is x is from 0 to 1. So, here I will have 3 minus x minus y and we shall first integrate over y and then I integrate over x and the if you finally, calculate in the same way this is 0 to 1 this will become if you do it. So, 3 y minus (x, y) minus y square by 2 here you put 0 and x. So, when you put 0 everything will be 0. This will become 0 to 1, 3 x minus x x square minus x square by 2 that d x. So, 0 to 1, 3 x minus 3 x square by 2 and you calculate out the answer turns out to 1(Refer Time: 17:05).

I am giving this example from Thomas and (Refer Time: 17:08) just for your illustrate might the book by Thomas calculus, for a illustration. Now you also to understand that there are certain cases, you can in this case you can reward back, because you see when a if I now vary if I rewards the region of integration and if I say that now if I look at this triangular area. So, how did we get the region integration? The region of integration is obtained in this way that, when you are at any point (x, y) can be anywhere here y can be between 0 to x. Now if I am now varying I can do a different sort of variation.

The variation can be of this form that I actually I am varying along this line y from, because these a line y equal to x I am varying y from 0 to 1 the domain. But I am varying y from 0 to 1 as I vary y from 0 to 1 at this point y is equal to x. So, x can vary from y equal to x to x equal to 1.

So, basically if I want to use x instead of (Refer Time: 18:48) when I first indicate with x and then is indicate with respect to y. So, when I have a particular value of y then x is varying from y to 1. And then I can do this calculation in this following way and then x is of y is of course, varying from 0 to 1 and you would compute out and you would have the same answer. But this might not be always the case you can. For example, so here you have to understand the things may not be always very simple.

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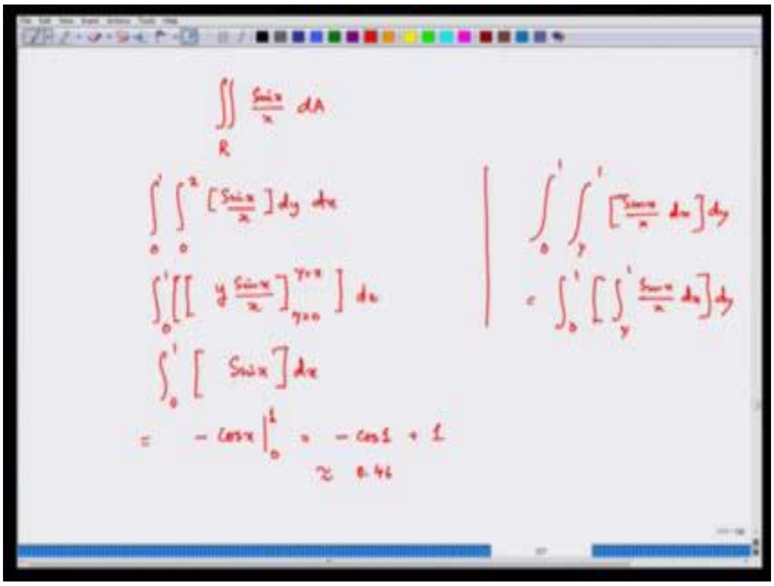

(Refer Time: 19:27) For example, if you want to integrate this function for just you have to do it over the same region. Now first I do it over keeping x fixed at 0, 1 and varying y from y equal to 0 to x. If I do that, then I am first integrating with respect to y then I am integrating with respect to x if I am doing this then what I am getting here. I am finally, getting y sin x by x and from y equal to 0 to y equal to x.

So, once you do this. So, once you integrate you have integrated this part, because now your x is holding in hold is held fixed and you are just integrating it with respect to y. So, you just have y and then you integrate within the limits. So, 0 to 1 and this when you put y equal to 0 it will become 0 and y equal to x will become x. It will become sin x d x. The answer is minus cos x 0 to 1. So, essentially it is minus cos 1 plus 1. So, some it will be the answer will be around 0 point 46 approximately.

Now you cannot do the reverse plot, when you cannot say that I can I can do it with the different sort of thing. Then basically, then you have to compute it as follows 0 to 1 y is being kept fixed and then just as before you know that x can move from y to 1 and then you write that this is sine x by x d x and then. The probes, then you have to essentially do this.

The problem is that you cannot calculate this and get this answer, because this sin x by x does not have anti derivative. So, every time that you will be able to get this from back is not always true. So, for that you essentially need an anti derivative of the function, but what Fubini's theorem says that it is continuous in the given region we have to really figure out whether this function is a continuous in the given region actually the problem would be coming at 0. So, you cannot really say that because y can take the value 0 also.

Whether it is continuous and then x can take the value from 0 to 1. So, basically, because of that, because of non-continuity, you are been able to unable to handle this situation while here you are lucky, because you are first doing it with respect to y. This is one over the cases where things can just go back.

With this we stopped here; and the next thing we will essentially give a very brief idea what is a volume integral, and then we would immediately go over to what are vector fields and other issues. So, we can end with our end our discussion with a very important data called Gauss divergence theorem. The Gauss divergence theorem is central to many, many things, for example, those who are from physics and electrical engineering Gauss divergence theorem is fundamental to their existence. Let us end it here and we will come back to the next class very soon.

Thank you very much.