Basic Calculus for Engineers, Scientists and Economists
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Lecture - 27
Infinite Series-3

This is the last part of Infinite Series, and here we are going to look at series representation functions. That if a function f is infinitely differentiable then can be represented as power series.

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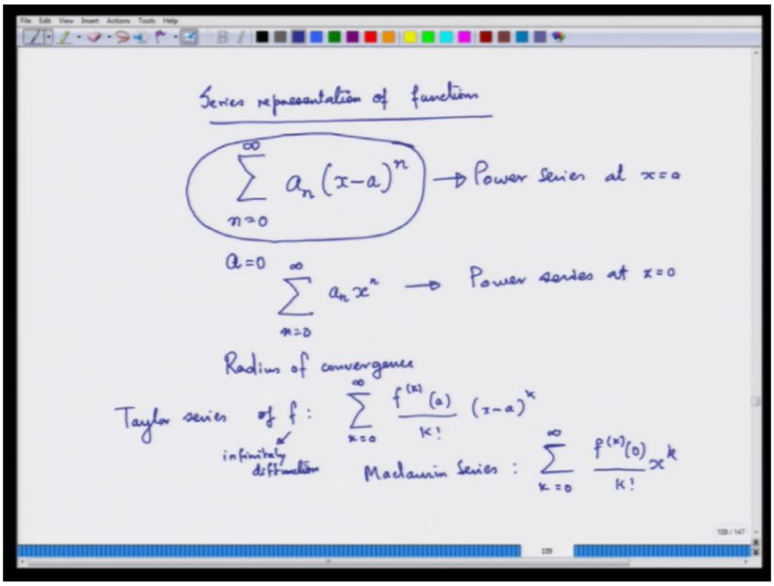

So, what is the meaning of power series? The power series is some sought of infinite polynomial. Here, you see this is the power series represented around the point x equal to a, a to the power n equal to coefficient nth term which is a power of x x or x translated by the something the power the nth power of that. So here, also look if I put a equal 0 it is become summation a n x to the power n.

Now the question here lies is, what do you mean by the convergence such a series? Here, the convergence would be that if I put a particular value of x then we will get a series of numbers, may be positive negative does not matter so then you have to decide the convergence of that. If I change the x I might be divergence of series.

The question is that we have to find x or an interval of x in which for every x this series would be convergent and these beyond series would not be the convergent. So that particular interval of x where that series would be convergent is called Radius of Convergence. We will not going into that detail I will refer you to power called even your Wikipedia if you want. So, we are going to now talk about the following that can we take a function which is infinitely differentiable and represent them in terms of a series where the coefficient are the given as a derivative of a certain order.

For example, there is something call that Taylor series or Taylor series of a function. If you function f, and then Taylor series of f, function f is infinitely differentiable then it is not for every function this is for infinitely differentiable functions. (Refer Time: 02:37) just keep on differentiating to whatever order you like, 3 infinity function basically. So the kth order the derivative at the point a by k factorial x minus to power k, so this called the Taylor series of f.

Similarly, there is something called Maclaurin series of f. Maclaurin series is much a easier representation, this was a Maclaurin student of Newton also just like Taylor was. Maclaurin series is of this form. So, 0 factorial is assume to be 1 please not this, because k is starting from 0, we just have to get the function f 0 f of 0 a is f a. Here, a is 0 so it is just x to the power of k. Sometimes it is easy to represent functions in the formula Maclaurin series. Now, what does the essentially means? How do I find out that Taylor series? See what we know about Taylor this fact we know what Taylor’s polynomial.

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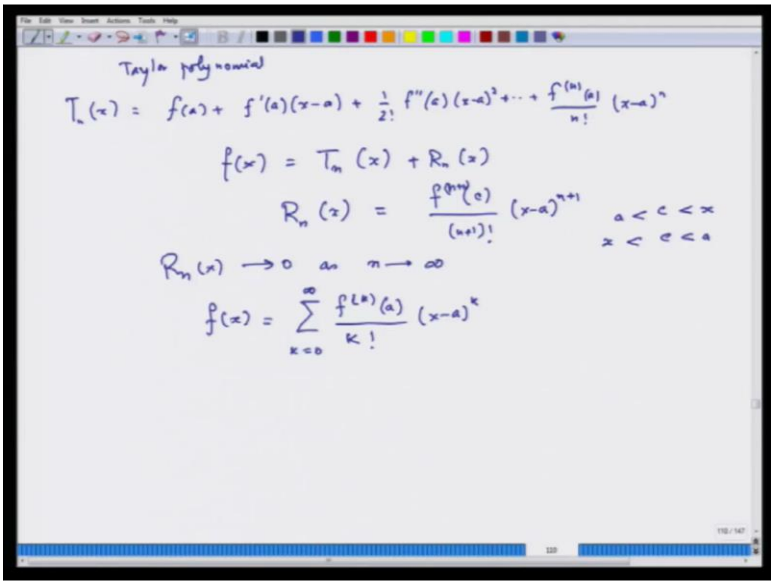

The Taylor's polynomial of order n is given as at say time a so Taylor polynomial. So, Taylor polynomial is we have already studied about Taylor polynomial earlier, you see this course everything is linked and everything is used, we are not doing anything out of the glue f n a by n factorial so that is the Taylor polynomial. What is a function? A function f x can we always represented as the Taylor's polynomial T n x plus R n x, when you in that term or their term. So what is R n x here? R n x is f n plus 1 c by n plus 1 - that is remainder after the nth. What is c? C is something lying between x and a. I am taking x bigger than a or it could be like this c is.

So, interesting part is that if you show that this reminder can be made smaller and smaller and smaller and smaller, then if this reminder can be met smaller and smaller and smaller that is if R n x tends to 0 as n tends to infinity then means I am making there as n is becoming larger and larger my error goes to 0, then we can show that for whatever x you choose within the radius of convergence or for a most cases for every x f of x can be actually written as a Taylor series. We can write f x as. For this, the reminder term has to go to 0 as n go to infinity. This is the very very crucial term.

We will show one example of how we can show this reminder term goes to 0. Of course, if x is a very near a, and of course if am only looking on an interval codes and interval and I only looking for those x's within that closed interval, so function has defined about that closed interval. Then over that closed interval if f k, so f k is continuous that we bounded; basically, you can bound that in way if we name x is very very near a for that particular x as k goes to infinity if this mode of x minus a is less than 1, then you are through at the limit would be 0. So, R n x it takes little bit of work to show that in some cases R n x can be easily shown to be going to 0. Let us check on e example.

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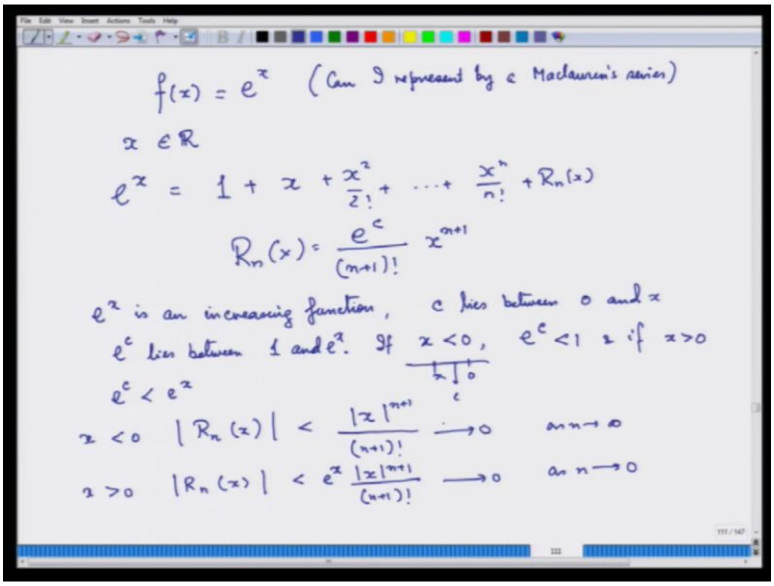

Let us check f x is equal e to the power x. So, question is can I represent it by a Taylor series or by a Maclaurin series. Maclaurin series is nothing but when you put a equal to 0 in the Taylor series. I understand I am not been very rigorous here, but if I want to be truly rigorous here I think I should have spent half of the course talking of the infinite series. Infinite series is the very very bigger area it cannot be starting 3 lectures, but I am also learning how to give crash course is to large group of students. We are going to now try to estimate the error of this function when you write a (Refer Time: 08:43) Taylor polynomial had around x equal to 0 and then see what happens.

Now, we will consider any x in R and let us see what we can do about the error. So, e to the power x the Taylor polynomial at 0 can be expressed like this. R n x is given as e to the power c because e x is always derivative is e x, e to the power c n plus 1 factorial into x to the power n plus 1.

Let us see what happens to e to the power. I am looking for particular x I have taken particular x here now, I am not x is not wearing x is fix my game is will be played by the number n, so x is now fixed. I am looking for this whether for this particular given x I can write it as the series, I can remove this R n x and I can keep on writing them, so whether the whole thing will converge. I know that if these go to infinity I can write, again this was to zero then I can write it like that.

So, e x is an increasing function. Now, the c lies between 0 and x, and e to the power c lies between 1 and e to the power x. If x is strictly less than 0 then c is strictly 1 and e to the power x. If x is strictly less than 0 what happens x. So, e to the power c has to, now c is lying between 0 and x which means e to the power c has to be less than 1, c has to be some negative number because x is less than 0, so x is here, 0 is here, c is lying here. E to the power c in this particular case has to be lesser then 1.

And if x is strictly greater that 0 then, I am not taking 0 because it is 0 then is 1 then I do not have to bother about this and is 000 and it is obvious I can write series, but I am just bother about strictly greater than 0 and strictly less than 0. Strictly greater than 0 then e to the power c is strictly less than e to power x. Then what is my R n x? I consider the case R n x and I have to take the mode of this. Mode of e to the power of c is any ways strictly less than 1 it is less than mode x n or n plus 1 divided by n plus 1 factorial. When x is greater than 0, then mode R n x is less than equal to.

And both this cases n becomes larger and larger x is a fix number, and so it is n is becoming larger and larger so x square x whatever is going up. Now, if you start from n equal to say 1 then you have - x square, x cube, x 5, x 6, x 7, but the factorial, because you multiply numbers it grows much faster than the powers. Because per timing you are multiplying a bigger number, here you are multiplying the same number. When you are doing a power you are multiplying the same number, there you multiplying the bigger number when you take the factorial.

Factorial is going to infinity much faster them mode x to the power n plus 1. Mode x to the power n plus 1 means what, n plus 1 we have just multiplied by a x n plus 1 times. Or do an x is, my n is sufficiently large then n plus 1 factorial is much larger than mode x to the power n plus 1. In both the cases this goes to 0 as intense to infinity. My R n x actually this term goes to 0 and which means I can make this term as small as I like, can we made less than x R. This is how you look at the rate of growth of functions and that is the way you do the things.

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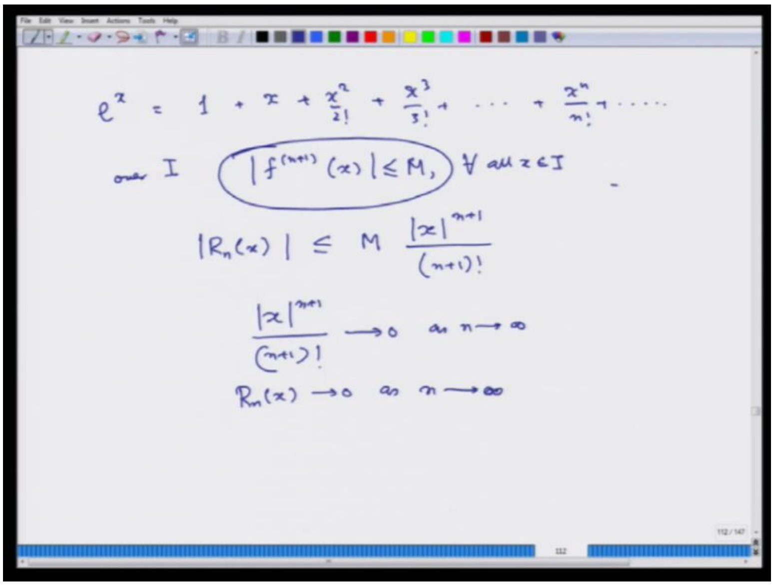

And here then you are able to now write e to the power x in this form 1 plus x plus x square by 2. Now, how will you estimate the reminder? Suppose, I am an over a interval I which is closed and bounded and over which a continuous function say if is a continuous or any derivative con continuous here because infinitely differentiable, then it will have a bounds. Say over I, mode f n plus 1 x less than equal to sum capital M for all x in I. Some interval which is unclosed interval would be closed and bounded interval, but it could a open when a code be closed but not bounded like r, but some of these is bounded. Here also for the e to e to the power x k's we have shown their bounds.

When x is given to me and I have this for a given x then what do I have, I have mode R n x is less than equal to M times, I am just taking Maclaurin series you can take the Taylor series does not matter even at x minus a also here if you want to. But what happens?

Whatever be your x it does not matter if this thing is bounded you know that mode x to the power n plus 1 by n plus 1 factorial, does not matter what is your x this will go to 0 as intense to infinity (Refer Time: 15:55) discussion. Which means that under this condition R n x is going to 0 as intense infinite. If you then show that this is bounded for every x in the interval you are showed that you can write the whole thing as the Taylor series.

For example, if you look at the sign f thing. Sorry. I am writing a (Refer Time: 16:19) mistake writing n goes to 0 it should be intense to infinite always. So you look at sign x, if you look at the sign x function if you take the nth derivative of it. It does not matter what is your nth derivative. Nth derivative would be always less than 1. Again you will have same sort of thing and you can always right sign x as the power series, I would leave it to you for that I will not just go and do all these things, so you can easily do it.

With this I would like to finish my lecture in infinite series. This was just given you a very brief idea that how can function, we can represent and infinite series and lot of things can we done after that. So, after this we are going to multiple integrals which we will start and post three lecture would be end of this course. So, I hope you have got some idea what was going on, may not be everything that I am speaking about but as some if you get a little bit of idea you even get 30 percent of what I have been telling you there would be enough for you.

Thank you very much.