Basic Calculus for Engineers, Scientists and Economists
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Lecture - 26
Infinite Series – 2

I will just continue with what I had left of, before I go to this (Refer Time: 00:21) ratio test.

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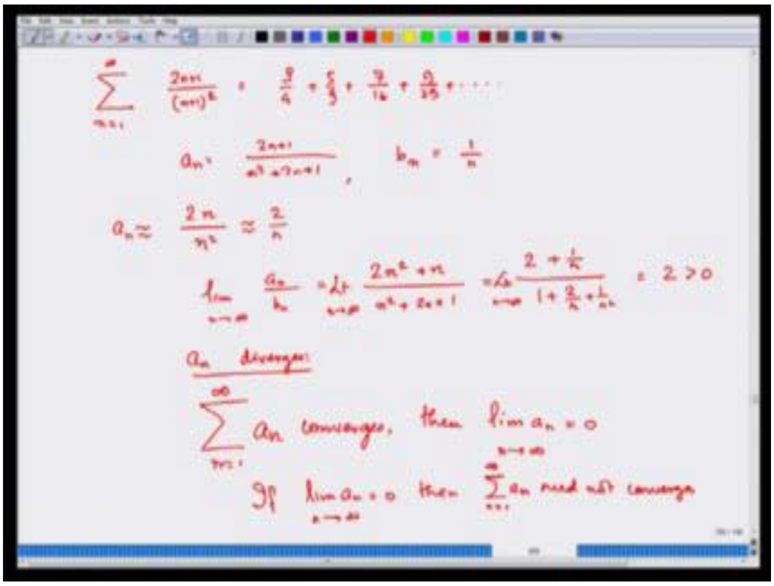

I just want to remind you that if this summation a n converges then limit of a n goes to 0, this is the simple thing you can figure it out yourself. But if limit of a n goes to 0, then summation a n need not converge. So, an example is summation 1 by n, so limit of 1 by n goes to 0, but summation 1 by n does not converge.

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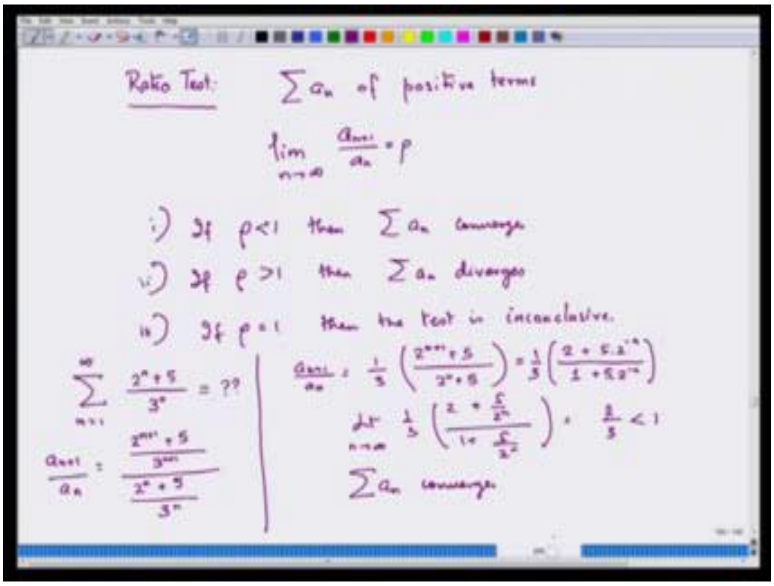

Let us now look at another more test. It might be slightly boring that we are just doing test after test, but these are useful, because every series cannot be checked up in the same way. So, again you have a sequence summation a n of positive terms then you have to find that is why it is called a ratio test, you are looking at rate of the movement of the sums, I look at the ratio of the terms.

So, sum of these rho, some number, if rho is strictly less than 1 then summation n converges; number 2, if rho is greater than 1 then series diverges; number 3, if rho is equal to 1 then the test is inconclusive. Then you have look into for look for other tests or other methods to show convergence if you think there is a convergence or show divergence. So, we cannot make any conclusion.

Let us try out an example that will show us using this test that yeah this is an effective test, you can at least check out. So, you can write, so you are asking for a sum of this or whether it converges. For the first step is writing a n plus 1 by a n and that is. So again you will have a n plus 1 by a n is equal to one-third mean by so divided by 2 n. It is 2 plus 5 into 2 to the power minus n on 5 by 2 n that is 1 plus 5 into 2 to the power minus n. So, ultimately the limit of, so this limit, so one-third is outside sorry.

Limit of n tends to infinity one-third 2 plus 5 by 2 to the power n 1 plus 5 by 2 to the power n, this is nothing but two-third, and this goes for 0, 5 by 2 to the power n part goes to 0. It is two-third, so two-third is strictly less than 1, so which means now that summation a n converges, but note that two-third is not really its sum, it is just this number rho, how do I find the sum of this actually find and actually say what is the sum.

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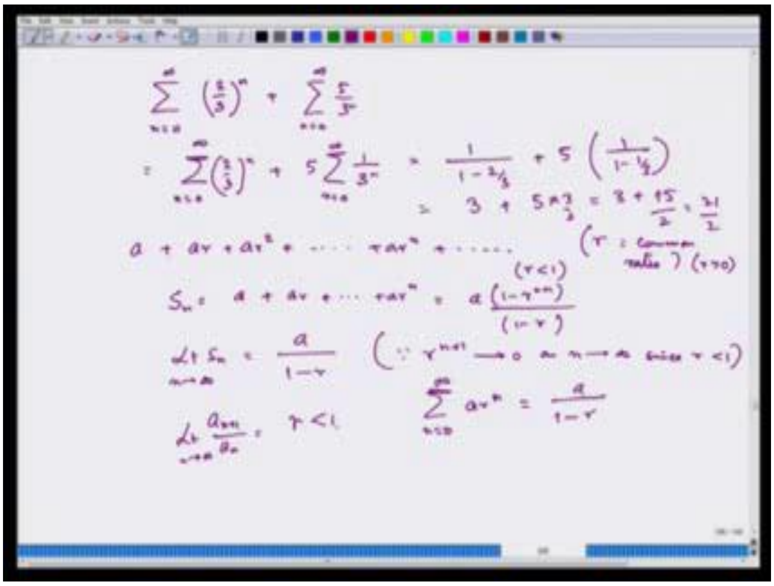

The sum now because this series is convergent, the immediate all the parts itself has to be convergent. I can break it up into two parts. I can actually write this as, 5 is a common number, so it can be taken out. So, we will look at these two parts series, this is what is called a geometric series, because the first term of this series sorry n has been zero, so it does not matter. N equal to 0, it is 1; when n equal to 1, it is two-third, then two-third squares.

So, multiplying with every number two-third and getting the next number this is called a geometric series. So, how do sum of geometric series. So, geometric series of the form, so now if I look at s n our geometric series where this r is called the common ratio; a is the first term. Because if I divide a, any term by a, if I had made a n plus by a n then I will simply have r, two ways of looking at this.

Now, let me consider the case of geometric series, where r is strictly less than 1, basically series of this form; here r is two-third, here r is one-third. So, here you will have a f. So, a plus a r plus a r n then this is nothing but a into 1 minus r to the power n plus 1 divided by 1 minus r. Now limit of s n as n tends to infinity is what it is as n goes to infinity, because r is strictly less than 1, this goes to 0. It is finally, so this is since r to the power n goes to 0 as n goes to infinity, since r is less than 1. Of course, here we are taking the common ratio r to be strictly greater than 0.

You can also use in this particular case, where if a is positive term here we are really not bothering about a being a positive term, but here if you consider a also has a positive term like here, then you can actually do the ratio test. Because a n plus 1 by a n here is nothing but r; and if you take limit n tends to infinity, this is also going to give you r and r is anyways strictly less than 1. So, you can guarantee that it converges, but here you know that the sum summation n equal to 0 to infinity a r n, so this is nothing but a into 1 minus r.

Similarly, so you apply this here. So, what you will have, the first term is 1, so it is 1 by 1 minus two-third plus 5 into the first term is 1, so 1 by 1 minus one-third. It is 1 by one-third. It is one-third, so it is 3 plus 5 into 3 by 2 that is 3 plus it is 15 by 2. It is 21 by 2, so that is the final conclusion.

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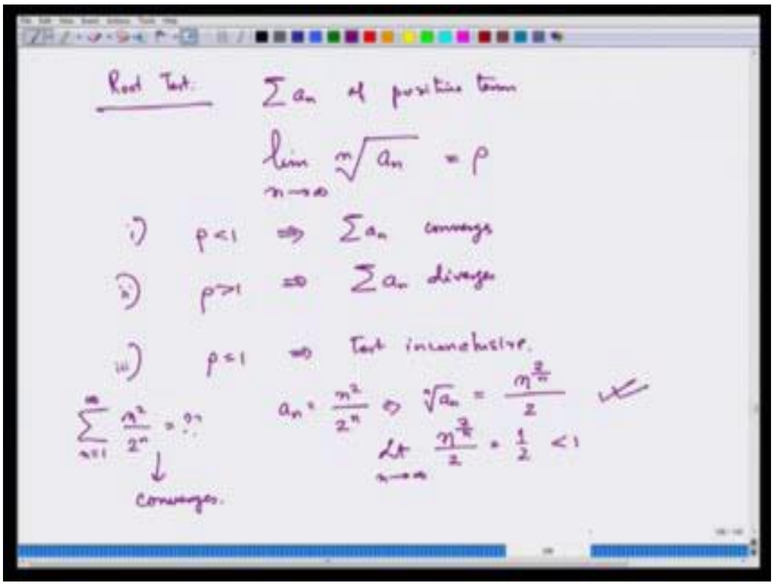

Now, there is some other test called the ratio root test. Let us see, let us do the root test. So, you are taking a series of positive terms. Once I take this term, and what I do I take the nth root of the nth term, then I take the limit as n tends to infinity and suppose this is rho. Again the same thing as a root test, rho strictly less than 1, it imply summation a n converges; rho greater than 1, implies summation a n diverges; rho equal to 1, implies test inconclusive. One, we have this information.

Now how do I make a check right, how do I check it through root test? Suppose, when you do root test, root test are done when you have powers, when you have powers on your terms then your root becomes useful say your nth power in nth term then your root term test becomes useful.

So, various tests they are designed for handling various types of series. For example, if you look at this one, what does happen to this? So, a n here is n square by imply the root a n, nth root of a n here is n to the power 2 by n divided by 2. If I take the limit n to the power 2 by n divided by 2 n tends to infinity, what happens, n be becomes bigger, but 2 by n goes towards 0, so how.

So, basically whatever how large your number a is n 2 by n is going towards 0, then finally, the number has to stabilize towards 1. So, basically the limit of this is half, and this is less than 1, and hence we conclude that this series converges. So, why we have just spoken about sequence of positive terms, what about if you have sequence of alternating term, one term plus, one term minus, one term plus, one term minus what will you do.

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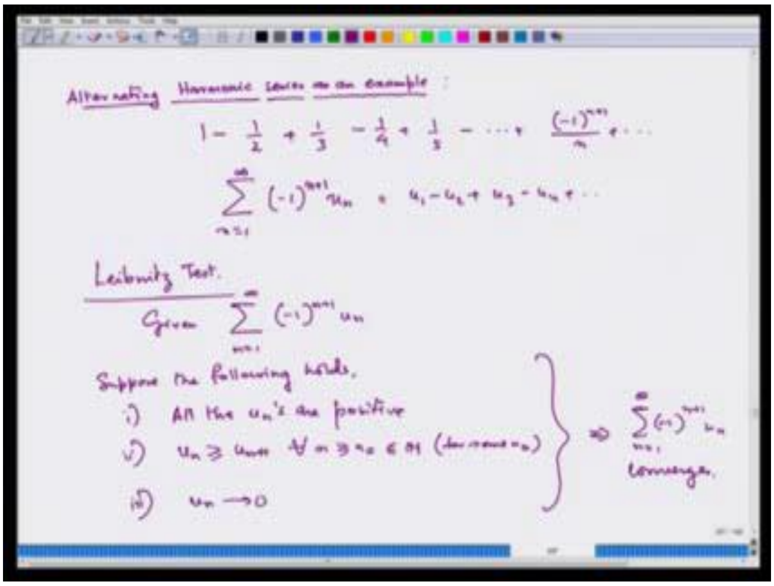

For example, you take the alternating harmonic series as an example. Let us see alternating harmonic series. It is a very interesting thing, which happens here. In general, an alternating series is given in this term; instead of a n, we will write the terms as u n; it is u 1 minus u 2 plus u 3 minus u 4 and so and so forth.

What is the meaning of convergent? The convergence again is same; take the sequence of partial sums and see whether they converge, but here you cannot guarantee increasing of sequence of partial sums that is a problem. If you cannot guarantee that then is there any way to test it test that the sequence actually converges. Basically can we use the then test under what sort of conditions the limit of partial the sequence of the partial sum converges.

So, one of the most basic tests that you take for this alternating series is called Leibnitz test; whenever in those days people had thought about this sort of idea, so which people get confuse even now a days with these ideas and those days at those days this was actually very advanced research mathematics, which we are learning at this very simple basic stage. So, you have given, now suppose the following holds, all the u n(s) the u 1, u 2, u 3, so there is sign in before them does not matter; all the u n(s) are positive. Two - u n is greater than u n plus 1; for all n greater than equal to n naught some n naught, some n naught.

Basically this u n should itself has to form a decreasing series; if I just take out the u 1, u 2, u 3, u 4, it will form decreasing series. And third, u n goes to 0; if all these things happen will conclude that summation converges. Let us now test it for the alternating harmonic series example itself.

The alternating harmonic series example of course, has one-half, one-third, one-fourth, one-fifth, they are all positive terms; second of all u n is bigger than u n plus 1. So, 1 is bigger than half, half is bigger than one-third, one third is bigger than one-fourth, and so forth. U n goes to 0 that is does here 1 by n goes to 0, so straight it goes to 0 finished that game is over. So, by Leibnitz test, we can conclude that it converges. There is something very interesting about how to show that whether alternating series converges is to use the idea of what is called an absolutely convergent series.

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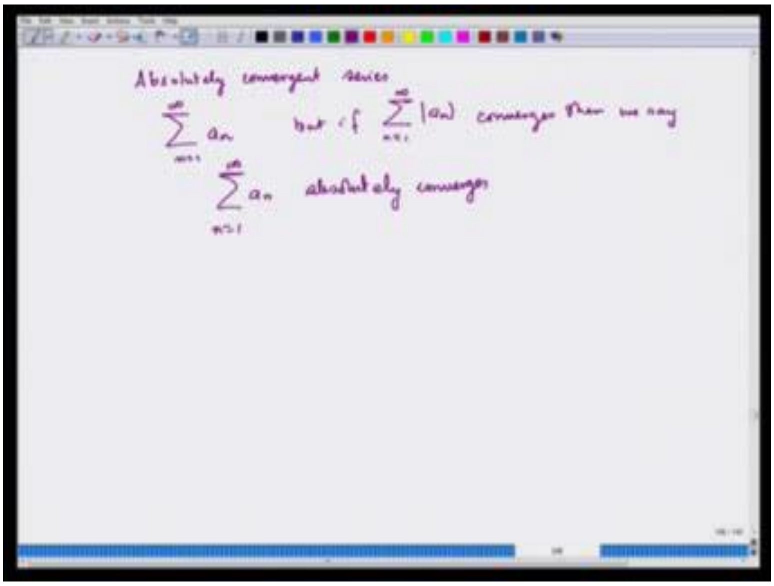

Absolutely convergent series is, so you take a sigma summation a n. It could be an alternating series; I am not writing that it could be an alternating series. But if here a n could be positive negative anything, but now here we take mod of n, they are all positive terms.

Then if this sequence, but if this converges then a n equal to 1 to infinity, then we say summation a n absolutely converges, but there could be an alternating series which need not absolutely converge, but just converging by the very basic definition of convergence, such a series is called a conditionally convergent series.

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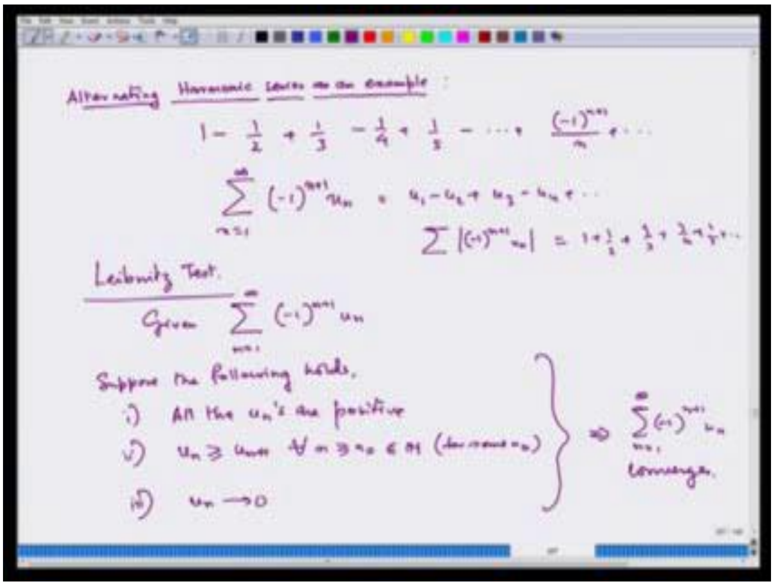

And example we have already given in the starting. See, if you look at the alternating harmonic series, and what is this, what is summation mod of u a n here; it is 1 plus half plus one third plus one-fourth etcetera, so that will give you the harmonic series right. So, harmonic series is divergent.

This series alternating harmonic series is conditionally convergent, it convergent by Leibnitz test, but it is not absolutely convergent, because if you make summation a n here, so basically in for the harmonic thing this particular case, you would take the mod of and this is a nothing but. So, what this is a convergent is already shown that is there is the harmonic series.

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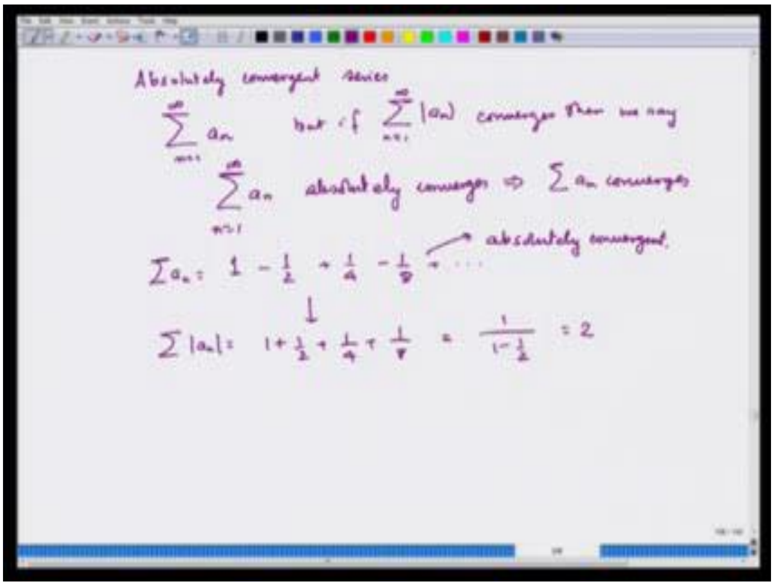

There is an example of series which is convergent, but not absolutely convergent; it is called conditionally convergent. Say any series, which is convergent, should be conditionally convergent. If a n absolutely converges, it implies that summation a n converges, but it is not so simple to show these facts, it needs a little so this is this is absolutely converging this is summation mod a n is converging, but that means, summation a n also converges. If we do not prove it, but it is not a straightforward fact; so now observe this particular alternating series 1 plus half plus one-fourth minus one-eighth and so forth.

Look at the series, this is summation a n; it is a geometric series basically when you write summation mod n. This is 1 plus half plus one-fourth that is (Refer Time: 19:47) two square one-eighth. In fact, it has a sum 1 by 1 minus half the sum is 2. So, here summation mod n converges, so a n also converges. This is called a n is an absolutely, so this is an example of alternating series which is the absolutely convergent. So, end our second talk here. The third part of infinite series would essentially would be bother about something called power series, and rather we would be not getting too much into the details of power series, but going largely into the issue associated with Taylor’s theorem.

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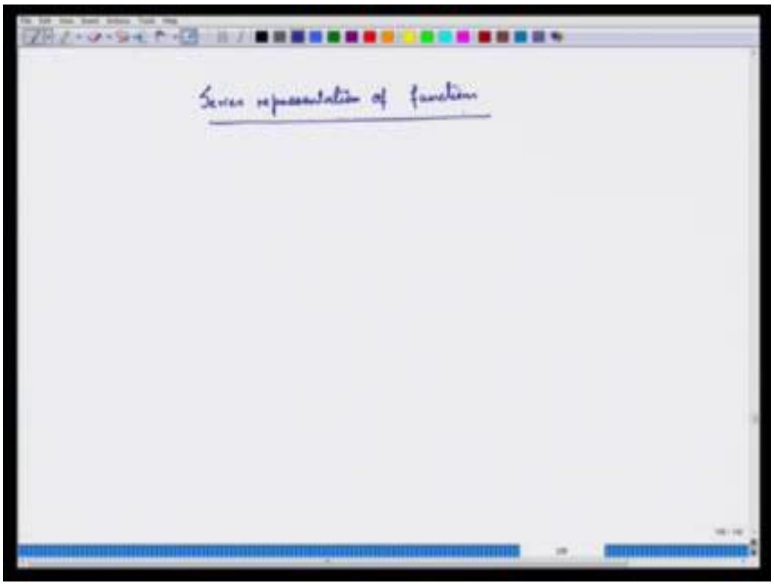

How do we represent a Taylor's theorem as a power series that would be our goal in the next chapter or next section, so that will be lecture 3, which we will start after sometime, next lecture. In which, we will talk about series representation of functions; anyway Taylor series will come where you basically make the Taylor polynomial into a series.

Thank you very much.