Basic Calculus for Engineers, Scientists and Economists

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Lecture - 25
Infinite Series-1

We are back again, but this is the last week, good news I guess, and last week of the course. But we are going to learn some of the very important aspects of calculus, one which is called the Infinite Series, which we will have 3 classes followed by multiple integrals.

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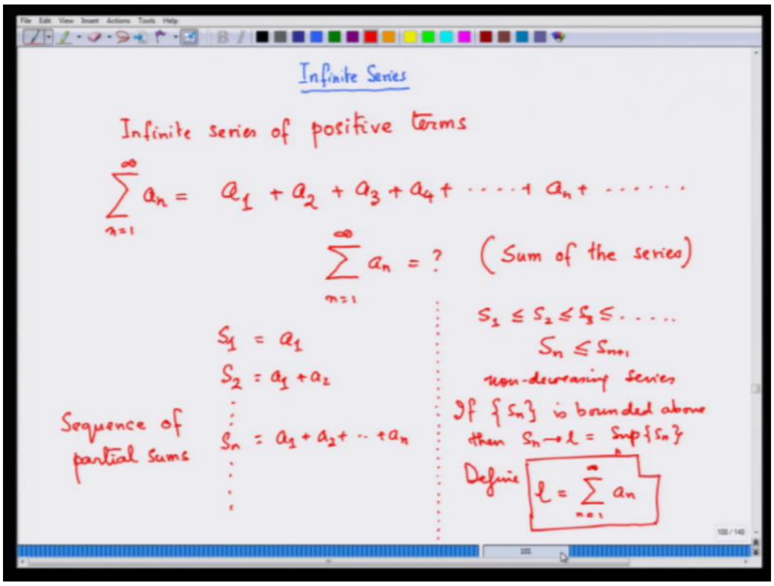

We will start with Infinite Series. Infinite series is required at many, many places where theoretically you can think of everything being happening in infinite up to infinite time.

Let me write down what is an infinite series, but I want to just now talk about infinite series of positive terms, infinite series of positive terms. It is written like, so it just goes on, it never stops. This can be written in a shortcut way as follows; summation a n, n starts from 1 goes up to infinity. The symbol infinity does not mean some number, of course, infinity is not a number, it is symbolism, which says that the sum does not stop, does not end, and it is not a finite sum that is all.

Now, is there any meaning to the sum of the infinite series, sum of the series? How do I even talk about the sum of the series? Now let us note something. How do I actually approximate this sum; if I take just a 1, then it is extremely crude up approximation of whole sum, was it positive sum so the whole term if there is something like a sum it must be bigger than all a 1. And I put a 1 plus a 2 slightly better approximation, a 1 plus a 2 plus a 3 slightly better approximation.

Essentially, you get a creating an increasing sequence. I will take say S 1 is equal to a 1, S 2 is equal to a 1, and S n is equal to a n and goes on. So, S n this term, so I formed a sequence, so this is called a sequence of partial sums. If I look at this series, because they are positive terms; S 2 is bigger than S 1, S 3 is bigger than S 2 and so and so forth. So, basically it is like this S 1 is smaller than S 2. So, basically what is happening, S n is smaller than S n plus 1. It is increasing series or non-decreasing rather I should say. But actually it is a strictly increasing series, because if these are all positive terms and then we are strictly increasing ok just.

Now if this series is bounded above, if S n is bounded above then S n converges this to l is equal to supremum of the sequence S n. So, what we were telling is that now if S n goes to l, if l should be the supremum, and S n will go to l, because S n is bounded above. If S n is bounded above then we define this l as the sum of the series essentially some people want call it a limiting sum does not matter, it is a limiting sum, if you say in some sense.

But this is the sum of the series this is what is defined as a sum of the series and we have to keep that in mind where we have to very careful about that. So, when there is the sum or the sequence of partial sums converges of course, if the sequences of partial sums are bounded above they will converge.

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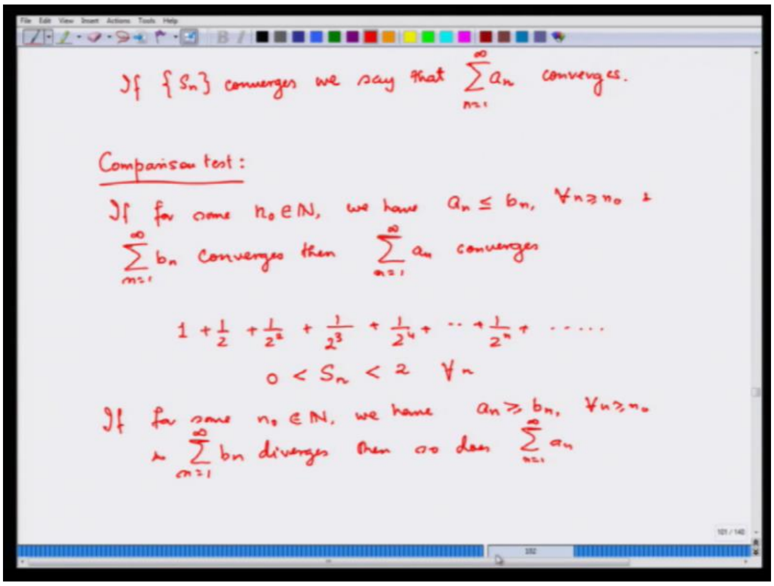

If this thing whole thing converges, and in the sequence of partial sum must be bounded above; the sequence of partial sum converges and if there is a sum of the series then we say that the series converges. If S n converges, we say that now it is very important to know how do I test, whether there is sequence, you know every time you are not going to write down partial sums. If you are not going to write down say sum up, take the partial sums, and try to find the converges, it is always not a good idea. There should be some easy way to figure out whether our sequence is convergent or divergent.

One of the if the sequence does not converges is either S n is oscillating or S n is going to infinity in both cases we say that the sequence and the sum a n does not converges, this does not have a sum or it diverges. It is important to now understand that there must be some ways to easily determine whether this sum a n has a sum or not or whether it converges or not. So, first one of the easiest tests is called the comparison test.

This comparison test only is working for series of positive terms; and it cannot work for other things or series one negative term, if you want to say. If for sum n naught element of n, we have a n less than equal to b n, for all n bigger than equal to n naught; and summation b n converges, then summation a n converges. It does not matter if you have left out finite number of terms; if that remaining part converges you can add the sum of the finite number of terms and can have the total sum.

Now, for example, consider this sum of geometric series, geometric series is something that which we have already seen in high schools. So, do you understand that S n here must be less than 2, it does not matter; S n should be always strictly less than 2 for all n. So, here S n is always less than 2. So, you see immediately you have you see in this very simple case, S n the sequence is bounded above and so you can immediately say it converges.

Of course, I leave to you to decide that whether you, but these for example, these one can be used as the sequence b n to show something else is convergent. A second one is the following; if the same thing, if for some n naught element of n, we have a n greater than equal to b n, for all n bigger than n naught. And summation b n diverges then so does a n.

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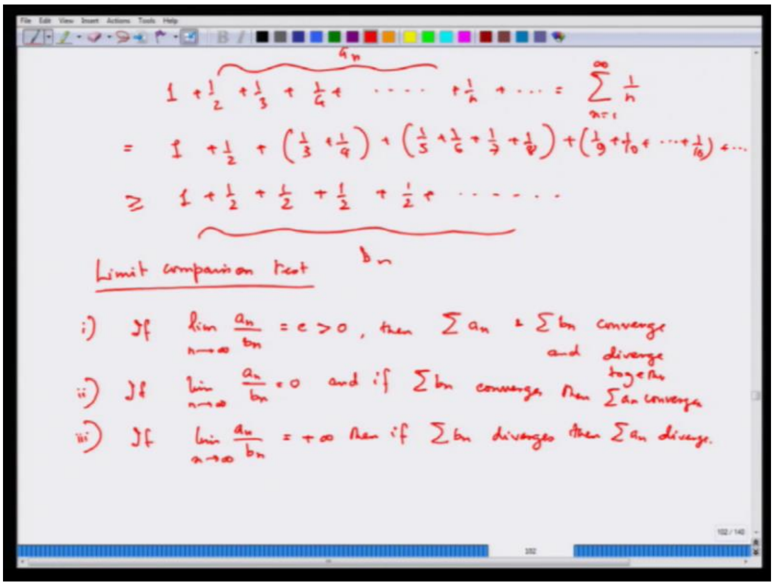

See we will now you have proof, we will use this for example, we will use this idea to prove that if I will show mean will show that the harmonic series which is consisting on the following terms 1 plus half plus one-third plus one-fourth. This is sequence 1 by n series 1 by n. Now, we will use this comparison test - a second thing to show that this actually diverges.

See what we can do is the following; we simply have to regroup this as follows 1 plus half plus one-third plus one-fourth plus one-fifth plus one-sixth plus one-seventh plus one-eighth. So, you group up to 4 and then group is n group here is up to 2 cube here is 2 to the power 1, here the n group is up to 2 square, the n group is up to 2 cube the n group is up to 2 to the power 4 that is 1 by 9 plus 1 by 10 plus. If you look at this, what does this become? This is become bigger than 1 plus half plus you see one-third is bigger than one-fourth, so it is one-fourth plus one-fourth half.

So, one-fifth is bigger than one-eighth, one-sixth is bigger than one-eighth, one-seventh is bigger than one-eighth, this is one-eight, so 4 times one-eighth, this is again half. This whole thing is greater than half. Then similarly, this is also bigger than half. This sequence 1 plus half plus half plus half this sequence diverges, because the sum of partial sums is not bounded above. As a result of which, we have a divergent series which is b n here and my series was a n here. You see that from this term, there is a strict when this b n is strictly less than the terms these terms.

Basically immediately you see (Refer Time: 12:20) sequence b n, and this b n is converging. So, by this comparison methodology, so I basically regrouped, basically I have written instead of writing this sequence, I am this sequence, I am writing this sum. I am telling that now what I am telling that this is the sequence. Now one the next sequence is next term is series. So, first this term is one, then the term is half then term is one-third plus one-fourth is.

Each term, so this is one term, this is one term, this one term of the Riemann series. Here rearrangement makes a difference because here are all positive terms. So, here you see this term is bigger than half, this term is bigger than half, this term is bigger than half and so on and so you have a sequence and which is showing that this sequence is convergent by the divergent. And this actually if it should be equal to this sequence or this sequence itself is also divergent.

For that is an little application of those ideas, there is something called integral test which I do not want to explain want to you because that would get you into some sort of more detail things. There are several comparison tests, which come so one is called the limit comparison test. So, again it is only for sequence of series of partial terms. If you have limit of a n by b n equal to c n tends to infinity and c is strictly bigger than 0, then summation a n and summation b n convergent diverts together.

Number 2, if limit n tends to infinity, a n by b n is equal to 0, and if summation b n converges then a n is then summation a n converges. So, you can use the limit comparison test. If given series, you can construct another series b n and do the ratios a n by b n. If limit a n by b n, n tends to infinity is equal to plus infinity; then if summation b n diverges then summation a n also diverge.

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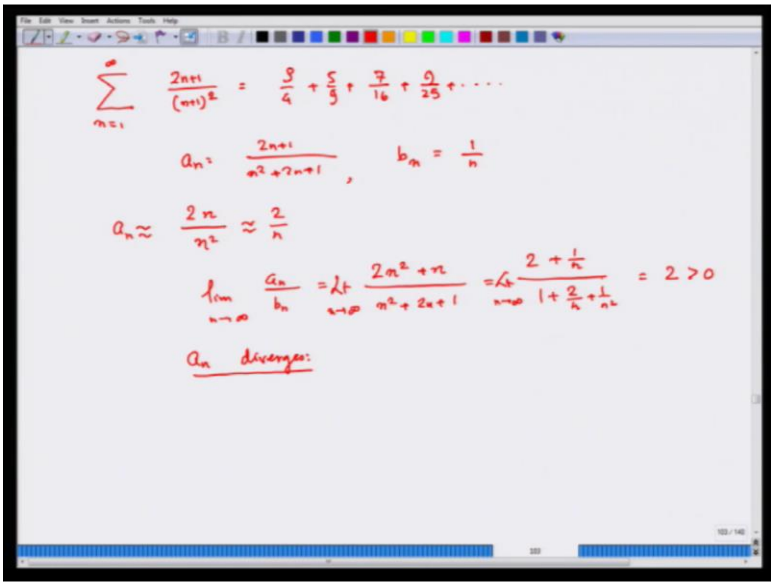

Now, now let us take an example; example is like this that you are considering the following series. So, each terms so if I put n equal to one it is 3 by 4 then the second term is 5 by 9 and so on and so forth. It is 3 by 4 plus 5 by 9 plus 7 by 16 plus. It does not seem on the very look of it that it might converge.

Let us if I have having by looking at it if I am having some doubt then I can easily consider this test, try out the limit comparison test. So, what should be the let me take b n. If a n is equal to 2 n plus 1 by n square plus 2 n plus 1, let me because I want to check out divergence, because it looks like that it will just go up, blow up then let me take b n is equal to 1 by n. So, b n is actually summation b n is for the harmonic series which we have just saw to be divergent, so then what happens because what happens you see that if you have why I am suspecting this just observe the following.

For n very large, n square will dominate to n plus 1, so it is something like 2 n by n square, it is almost like this. So, because of this, you have a suspicion that it might converge.

Now, let us do this a n by b n; 1 by n - a n is 1 by n. N will go to top, so it will become 2 n square plus n by n square plus 2 n plus 1. So, you divide by, n here you divide by n. It will become 1 by n, here it will become. So, you divide by n it because I divide by n square, then it will become 1 by n, then this sorry limit n tends to infinity then it will become 1 plus 2 by n plus 1 by n squares for the limit actually it gives me 2. If the limit is c, which is strictly bigger than 0, so which means a n and b n both diverge and converges together, but summation b n already diverges. So, we can now conclude that a n diverges.

We end a very simple test procedure here. We will come to our other test, how to do some important test like the ratio test and the root test in the next lecture. So, we will talk about root test and ratio test in the next lecture; give you some idea of a series which is not found of just positive terms as a plus minus alternating sign, thus very basic idea we will give. And then in the last one, we will talk about power series, and largely the power series inverted by the Taylor series rather than McLaren series.

Thank you very much.