Basic Calculus for Engineers, Scientists and Economists
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Lecture - 24
Constrained Minimization and Lagrange Multiplier Rules

Let us now speak about the Lagrange's multiplier rule. The Lagrange multiplier rule is associated with something we call constant optimization problems.

In the last class that we had or the last lecture that I have given in there we had been seeking x and y, which minimize maximize a function x, y over whole about (Refer Time: 00:35). We were not putting any restrictions on x and y, but now we can put a restriction on x and y.

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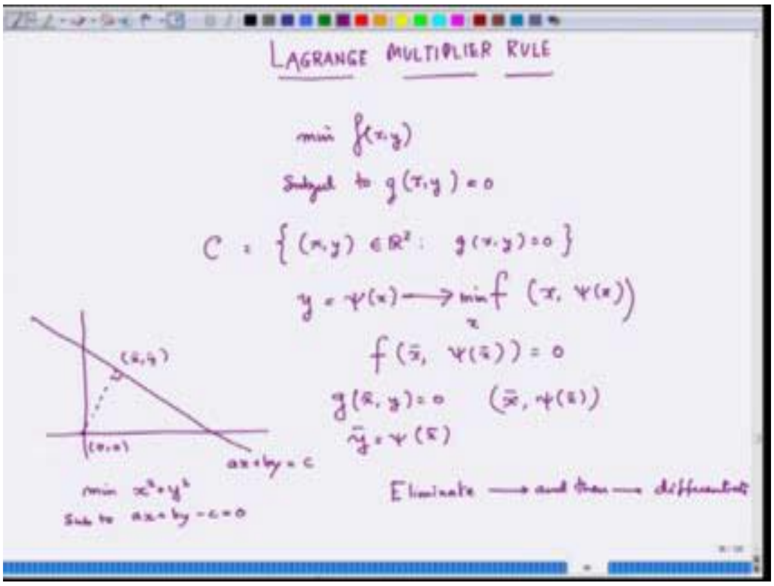

And we would to seek minimize a function of two variables, even maximize also do not worry subject to by g (x, y) is another function of two variables. So, basically what we are trying to say that only those x, y. If you form a set, you call it C, so this is called the feasible set. Take only those - x, y in order to which satisfies how would you go ahead and solve this. The key idea of solving this is the following and that is quite natural.

Suppose by a stroke of luck the function g (x, y) is such that y can be written as some function psi of x. Then you can immediately do one immediate thing that you can now put back instead of in f, you can now put back instead of y, you can put psi of x. Now your original problem now becomes to minimize this over only x, and now it becomes an unconstraint optimization problem, which you can minimize.

Now what we have? We are going to do we are now going to just first compute a point x bar which satisfies this and check whether by using the second order (Refer Time: 02:16) standard second order condition whether is x bar is minimize of this function. Now once you have done that you can obviously, use you can plug in the value of x bar here and obviously, you know y is psi x bar. So, my y bar is psi x bar and so this x bar and psi x bar are actually the solution of the problem. So, x bar and psi y by psi x bar is a solution of the problem.

If this happens, it would be pretty nice. For example, if you say ok, say I want to find a point on the plane on the line. I take the straight line. I am giving an example where such a thing can be done, but it might not be the case that you can always do it. So, what has been the step? So our step has been eliminating, and then differentiates.

Now, for example, I am asking that find the point on this plane - a x plus b y equal to c, such that the distance from the origin of that point is minimum that is you take find the point basically you from the origin you are dropping a perpendicular on this plane geometrically. This is my origin and you are trying to find this point x bar y bar. So, from a point of view a perspective of among minimization problem, I can setup the problem of minimizing x square plus y square that is the distance of any point from the origin subject to for this straight line a x plus b y minus c is equal to 0.

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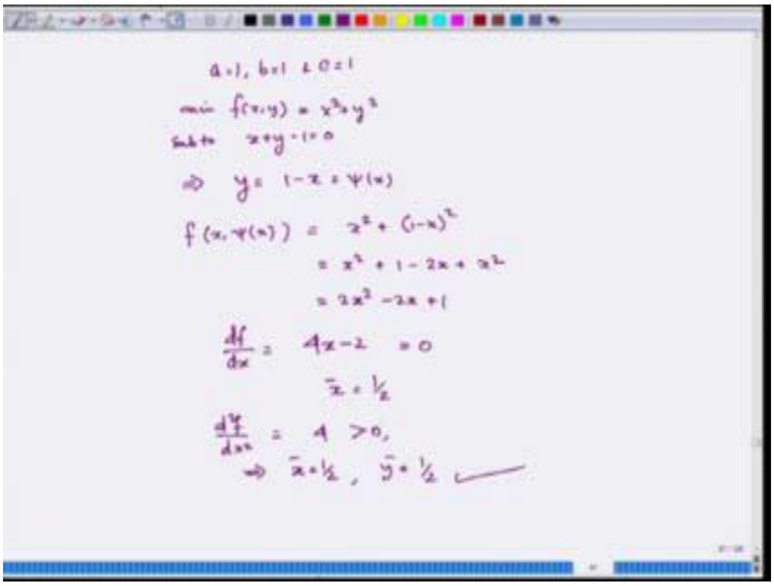

Here suppose I take a equal to 1 b equal to 1 and c equal to 1, so suppose I take a equal to 1, b equal to 1 and c equal to 1, basically x plus y is equal to 1 that is that line which passes through 1, 0, and 0, 1 thing which you know very well. Now, so my problem becomes minimize f of x y subject to x plus y minus 1 is equal to 0. Now what is y. So, from here it implies that y is equal to 1 minus x, where f of x square or f of x y is x square plus y square.

Basically now my original problem in the form of this is my psi of x. So, f x psi of x now becomes x square plus 1 minus x whole square. This is equal to x square plus 1 minus 2 x plus x square that is 2 x square minus 2 x plus 1. Basically now I have a single function of one real variable; here I wrote differentiation basically you have to find df dx of these, which is 4 x minus 2 and find.

So, put this is equal to 0 and find x is equal to half. So, once you got x equal to half, you try out on these function again, you take second derivative of this that is 4, which is always strictly greater than 0. This would imply that x is equal to half is a true minimizer of this problem; and so what is y, y is 1 minus x, so y is equal to half. So, x bar equal to half, and y bar equal to half are the solutions of this problem, which is quite obvious from even the geometry.

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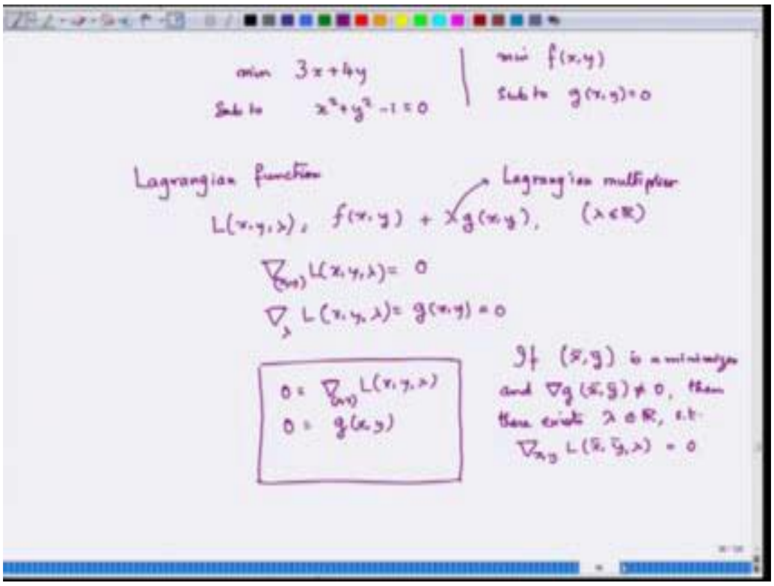

But this is not the only kind of problem that you are facing. Suppose you are unable eliminate in a proper way or if you eliminate thing can be slightly problematic. Because for example, if you have a problem of this form, minimize 3 x plus 4 y subject to x square plus y square plus 1 is equal to 0 that is find the minimum value of the function 3 x plus 4 y, when x and y are lying on the unit circle. Now you can say ok, why I cannot eliminate, I can write y is equal to root over 1 minus x square, no you cannot write y actually as a function of x. You have to write y as a plus minus root over 1 minus x square. It is not a function.

And in such a case, you have no choice to eliminate and then differentiate. Lagrange taught us some effective thing; Lagrange say do not worry; there is a different way; you first differentiate and then eliminate. So, you construct what is called a Lagrangian function; and this Lagrangian function, you write as L x, y, lambda which is essentially define like this f of x, y. So, you have a problem like this that is what (Refer Time: 07:51) this is the abstract version of the problem.

And here is the concrete example which I have written on the left sub to means subject to, so f(x, y) plus lambda g(x, y) where lambda is some element in real line. Now, this L x lambda is a function of two variables. What he says is that first you if x what is the solution then what I should do to find the solution, what should I do, the first step I would differentiate, so it is L (x, y, lambda) x.

And then differentiate that Lagrangian function with respect to lambda and that will only give you g(x, y) and that equated to 0. And basically it is telling that you have to first do the following. Find those, so here you have L(x, y, lambda) and also. first you construct this Lagrangian function differentiate and now when you have this you now try to eliminate out lambda and get the value of x and y that is the key idea. Lambda here is called the Lagrangian multiplier.

This is essentially a necessary condition what if you think that this is a trick by which you can you solve this and get an x, y, you get a minimize, our answer is no. It is essentially a necessary condition; it says that if x bar, y bar this is the thing which is not taught in the calculus classes. In general, let me tell you the real fact - if x bar, y bar is a minimizer and the gradient of g at x bar, y bar is a nonzero vector then let both the components of the variable del f del x or del f del y g del x del g del y both cannot be 0 at the same time.

Then there exists lambda in R - a real number such that L (x, y, lambda) is equal to 0 that is L (x bar, y bar, lambda) this is 0 that is what Lagrangians the multiplier rule says that is exactly what it is. So, you really have to know this condition that if this condition holds and if x bar, y bar has a minimizer then this will happen. See in order to find to minimize, I can actually start doing this.

But the issue is that if I find a point that is only a point which is called a Lagrange's critical point, it does not at all tell you that that is the minimize, the most books will write that as a minimizer there lot of hidden things which they do not tell you. And for that you really had to go and study branch of mathematics called optimization theory, but we do not do this here.

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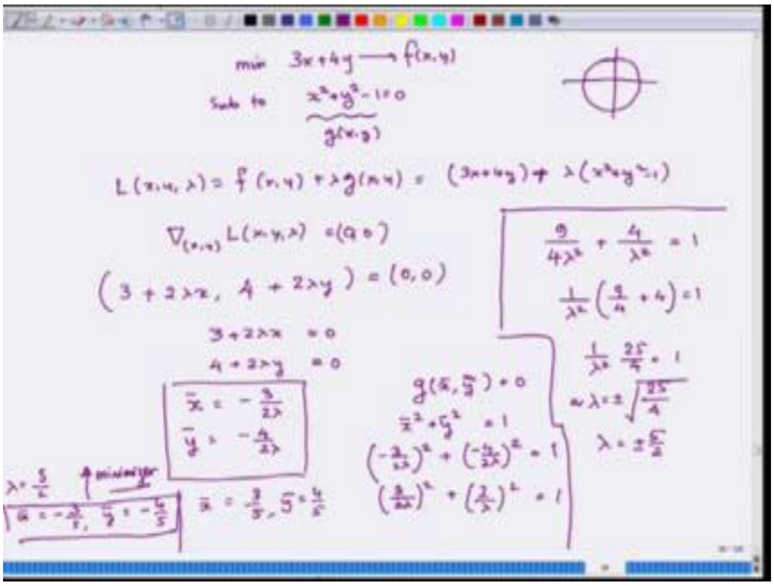

We now really take an example, this concrete example that we had taken; we now try to do it step-by-step. So, minimize 3 x plus 4y subject to, so this is my f (x, y) and this is my g (x, y). I construct the Lagrangian function f (x, y) plus lambda g (x, y) is equal to 3 x plus 4 y plus lambda x square plus y square minus 1.

Now what is the meaning of, so let us write down the gradient vector, take that del f del x with respect to x. It will become 3 plus 2 lambda x that here if you write minus does not matter much in this particular case, but does not matter we just take per the convention we are using. Here it you have 3 as derivative here you have 2 x, so that is all this is the gradient with respect del f del x basically del f del x, and then with respect to y. And with respect to y, it will be 4 plus 2 lambda y, these vector is actually the 0, 0 vector - 0 vector.

When I am writing this, this essentially means these. So, which means that 3 plus 2 lambda x is equal to 0, which means 4 plus 2 lambda y is equal to 0. So, what is that, what does this mean, can we eliminate lambda and do something. From here, we can have x is equal to what do we have x is equal to minus 3 by 2 lambda; and y is equal to minus 4 by 2 lambda.

Now I have to get rid of this lambda and get something, so but I know that this is my x bar and y bar, solved this and this, this is my required x bar and y bar, but this lambda is something undetermined. So, what I do, I go to the fact, I use the fact that g (x bar, y bar) is equal to 0. If this was a truly a minimizer then it is true satisfy this basic constraint. I will have x bar whole square plus y bar whole square is equal to 1, which will give me minus 3 by 2 lambda whole square plus minus 4 by 2 lambda whole square is equal to 1.

It is giving me basically same as writing 3 by 2 lambda whole square plus 2 y lambda whole square is equal to 1 from here we can really figure out lambda. So, what do we get from if I calculate this out further, let me calculate it out here. So, we remain as the same page with all problems. I calculate this out further it will be 9 by 4 lambda square plus 4 by lambda square would be equal to 1, so that means, 1 by lambda square 9 by 4 plus 4 is equal to 1. It is becoming 1 by lambda square 25 by 4 is equal to 1 or lambda is equal to plus minus 25 by 4 square root. Lambda is equal to 5 by 2 plus minus, we have two values now.

So, what are you going to do? Now, you will get two sets of x bar and y bar. So, here x bar first take lambda equal to 5 by 2. In that case, your x bar would be equal to what would be x bar, x bar would be equal to lambda I am putting 5 by 2, it will become plus minus 3 by. So, here first if I put lambda going to 5 by 2, it will become minus 3 by 5; and y bar would become minus 4 by 5; similarly, x bar, if I put this as minus would become 3 by 5; and y bar would become 4 by 5.

Now the question is which one gives me the maximum, which one gives me the minimum; do at all give me maximum and minimum, how do I know. See this is the question, which is never answered to you in the calculus class. The interesting fact is that whether such a problem would actually have a maximizer and minimizer. If you look at the set, the feasible region here, this is x square plus y square equal to 1, and this is the close set and a bounded set.

It is essentially what is called the compact set in two dimensions we have not discussed all this. I am telling you where you are not told, and you are just told ok now somehow put x bar and y bar into that top one, the objective function, you put both the sets of value whichever is the lower one you take that as a minimum value. But how do you guarantee that a minimum would be they are. So, because this function is a continuous function, there is a minimizer and a miximizer over compact set that is the result in two dimensions which we have not spoken about.

Now that is the very, very important result when are on this sort of close bounded set, so this problem has a minimizer and maximize; and both of them should be at least has one minimizer and one maximizer and which has a global maximum minimizer and a maximize. And you will immediately know that here at least, so which means they should give me a maximum value and a minimum value. So, both of these maximizers and minimizers must be actually satisfying some sort of these, these lagrange multiplier rule this is also something you have to keep in mind.

So, you have to now do, once you have guaranteed a maximizer and the minimizer exists, now you plug in the values and you figure out which is the maximizer and which is the minimize. You put in the minus values, you will get a minus something; if you put in the plus values you will get plus something. This is minimize, this is minimize, this, this one. So, you see how nicely we have been able to at least get a point where we can speak about whether it is a maximizer or a minimizer or not, but in these you would really want to determine or to guarantee that this is a minimizer, maximize, you need to use second order condition (Refer Time: 19:48) beyond the scope of this course.

Here by arguing the way I have just argued, we can say whether we can choose from the Lagrangian the critical points that you get the points that satisfy the Lagrange multiplier rule that which is the minimizer or which is the maximizer.

With this very brief idea, what the Lagrangian multiplier rule, I end the course here; please look at the examples in your assignments. And in the last week, we will just have two topics which we will break it up into three and three classes; one is multiple integrals, and another is infinite series and both are important in understanding calculus.

Thank you very much.