Basic Calculus for Engineers, Scientists and Economists
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Lecture – 23
Unconstrained Minimization of Functions of Two Variables

Good morning, we are in the last phase of the last, but one week. So, we are going to discuss Maxima and Minima functions of Two Variables. This is very, very important because maximization and minimization comes everywhere, whether you study physics or engineering or even medicine or biology, it does not matter, you have to deal with the issue of maximizing or minimizing function.

In fact, one of the all-time greatest mathematician Euler, whose notation dy, dx, how actually used and whose formalism that you see in many, many places are due to him, had a statement that nothing in this world happens without a function been maximized or minimized. This is a very, very important thing, because real life modeling is largely with functions of more than one variable.

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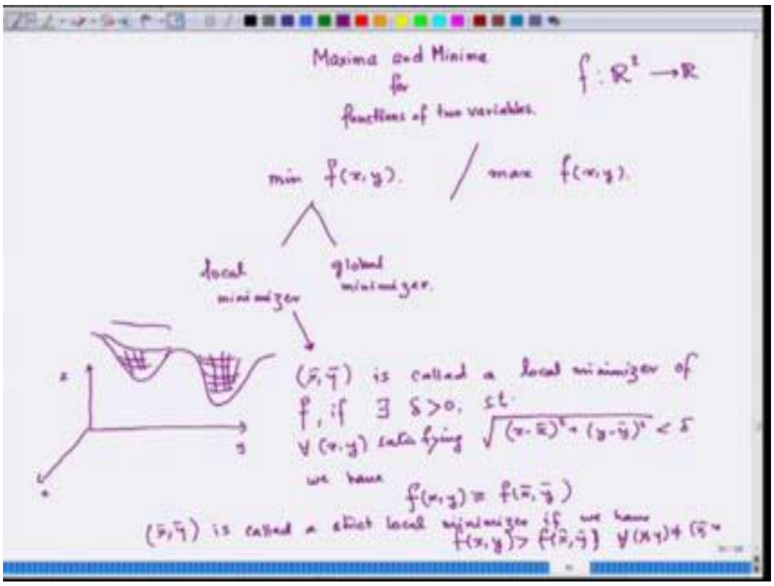

Basically, we are talking about I will just talking about I will just talk in terms of minimization, you can also talk in terms of minimization, it does not matter. I will write everything for minima, but you can write everything for maxima, if you want as a game. So, just like in the case for real variables, one real variable, here when I am talking about a minimizer I am missing. What I have to be clear, what I am essentially looking for, whether I am looking for a local minimizer any x, which gives you the minima, (x, y) pair which gives you the minimum value of this function of a local minimize or you looking for the global minimizers.

Similarly, in this case global or local maximizers, so what do you mean by a local minimizer. For example, if you have function like this say it is not so easy to draw a graph of three-dimensional graph here. So, somewhere like this. Think that this is a three dimensional contour and this is the xyz axis.

So, how you see here also this point is some sort of a minimizer, but about some given region, here this is a mini minimizer for this region, but this could be a global minimizer of this function also. So, we essentially have to deal with local minimizers some cases were become global minimizers. So, a local minimizer is what. So, x bar y bar is called a local minimizer. This function f sometimes if you want to write a function of two variables you can write it as a function from R 2 to R. Where, R 2 is nothing but a symbolism of the plane just telling you that every point in that plane is given by a pair of real numbers called a local minimizer of f.

If there exists a delta greater than 0, such that for all x, y satisfying x, minus x bar whole square plus y minus y bar whole square the distance between x y and x bar y bar satisfying this, we have f of x, y is greater than or equal to f of x bar y bar. If this relation holds, so x bar y bar is called strict local minima.

This concept you will not find in standard calculus textbooks and that is what actually happens when you do a second order check, which you will soon see. This called a strict local minimizer if everything else is satisfied. If we have f of x, y strictly greater than f of x bar y bar of course, local means they have to satisfy this, there must be some delta for which this happens, for all x, y, which is not equal to x bar y bar.

Now, how do I detect how do I really start trying to find a minima we are not going to draw a graphs and look at the graphs not even the modern mathematical software's can actually give you a vision of the graph from R 2 to R. They cannot play map it for a whole what to you essentially mapping for the part of the R 2 (Refer Time: 05:49) part of the domain and then you look at it. So, what you can find even if you declare some point of the global minima that could be actually a local minimizer as a whole.

So, essentially there must be some other ways an analytic ways of finding them. We have already studied analytical ways of finding them in our course in or class in the calculus of the when we are talking about the maxima minima of functions of one variables. So, we know that if a function of one variable has a minimizer at x bar then f dash or x bar must be equal to 0.

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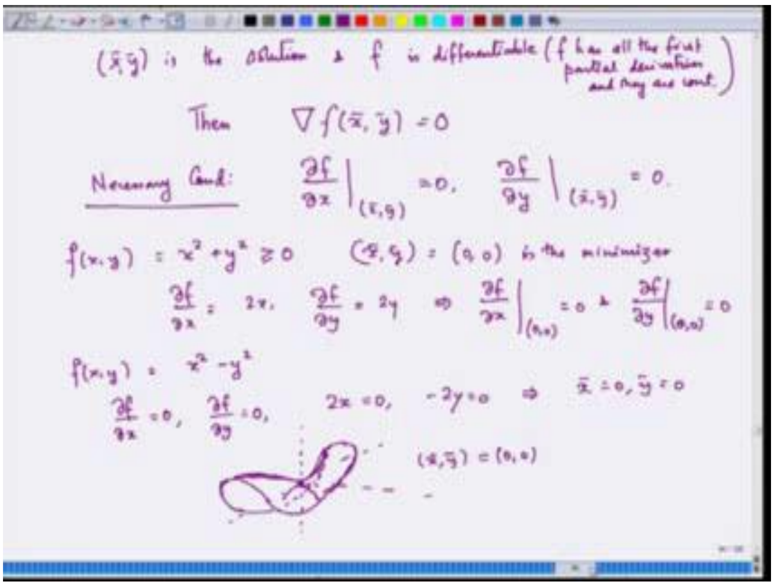

Could we try out some sort of thing like that? Now, suppose x bar y bar is the solution then and f is differentiable. So, when f is differentiable somebody says it is differentiable this term differentiability is not so simple when you functions of two variable. But for all our work and all of the usual practical work in calculus; this means f has the partial derivatives f has all the partial derivatives and they are continuous.

All the first partial derivatives and they are continuous means the first partial derivatives are continuous as the function of x, y cont is short for continuous. So, essential as the meaning, so when then if this happens then x bar y bar must follow this condition, x bar y bar must follow this condition that is you should have del of del x compute at x bar y bar that must give you 0 and del f del y at x bar y bar that must give you 0.

Let us consider the function f of x, y. Let us test this that whether it is actually happening x square plus y square you can understand that the x square plus y square is always this function is anyway always greater than or equal to 0 and if put x equal to 0 and y equal to 0 the function of value would be 0. So, here x bar y bar is equal to 0, 0 is the minimizer. But you see if you take del of del f x it is 2 x and del of del f (Refer Time: 08:58) f of y is what del y of f is 2y and it is implies that del of del x at the point (0,0) is 0. And del f del y, you just put x equal to 0, y equal to 0 there at the point (0,0) is also 0, which validates for what we have said. But these are just necessary conditions and need not be sufficient.

For example, if you take f (x, y) equal to x square minus y square, and then find the points where this would be satisfied. So, we are trying to find points where this will hold. This will again lead me to the equation 2 x equal to 0 and minus 2 y equal to 0, which would again imply that the only point, where the function partial derivative is vanished is x bar equal to 0 and y bar equal to 0. But if you draw the graph of this function which this is called as saddle graph. I have this. So, what do you see? I have obviously, not drawn the graph very well. This is called as the saddle graph.

This point x 0, y 0 you know this is minimizing this curve along the y axis, but maximizing the curve along the x axis. It is maximizing the function along the x axis and minimizing the curve along the y axis. So, such have saddle points, which may also occur sometimes. So, you see x bar, x 0 y 0 here x bar y bar equal to (0,0) is not truly a minimizer, but it satisfies a necessary condition, satisfies this condition of the gradient mini equal to 0. This condition is only a necessary condition and not a sufficient condition.

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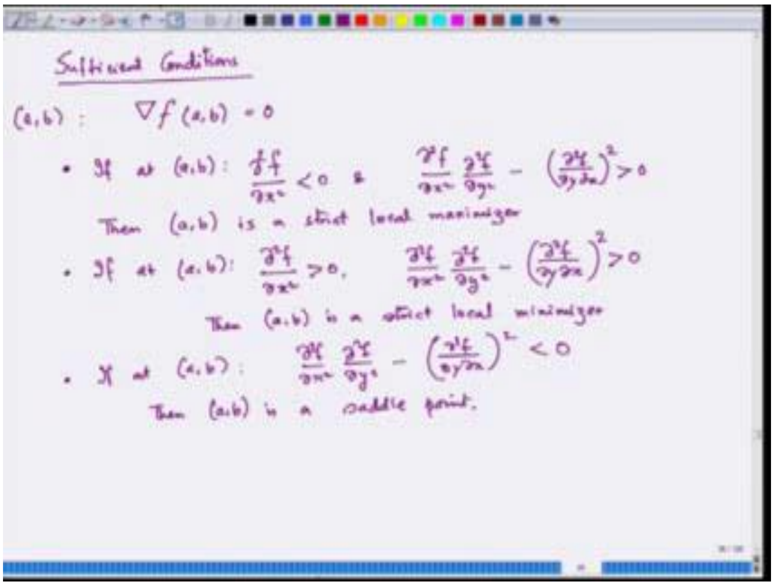

Now, is there any way is there a way to talk about sufficiency, when if you invoke second order derivatives in the case of function so one real variable. Now, we look at these sufficient conditions. So, what we are going to do is we are going to list down the second sort of conditions, which are required to be check at a critical point.

Now suppose you have a point, where say point (a, b) such that the derivative of (a, b) equal to 0 the partial derivative is 0. Now I am going to make a check assuming that the function is now twice differentiable or twice continuously differentiable if you want, whether I do have some we whether we can have some conclusion about what is the true nature of ab. Number one, so if you have this, so if at a b you have the following that del 2 f del x 2 is strictly less than 0 and del 2 f del x 2 del 2 f del y 2.

I am assuming all of these are continuous, because then I can have that del 2 f del x del y and del 2 f del y del x is same like we have discussed in the last class. If we if at (a, b) these quantities are found to be greater than 0, then (a, b) is a strict local maximizer this word strict is not mentioned in the calculus books, but it is actually like that, if at (a, b) you have this one. I am not going to the details of proves. So, you just learn it as a working rule for the moment. Then (a, b) is a strict local minimizer.

Now if at (a, b) you have no information on this. It could be 0 for example, you just have the following information for example, and this could be 0. But you have if you, now if you have the following that and no other information at (a, b), then (a, b) is a saddle point.

So, what do you mean by a saddle point. Saddle point means, so if I instead of, if I fixed my y and I put the value of y as b. Then this becomes the function of x and then as a function of x at the point a, the function is either maximized or minimized and conversely, where put form a in the place of x in the function and then it becomes the function of y. Then at the point b it is either maximized or minimized. If it is maximized along x it will be minimized along y, if it is minimized along x it will be maximized along y such points are called saddle points.

The usual definition given, for example, Thomas and Finney that you will take a point around in one side you will the function value is going down and one side it is fine the function value is going down up is not really a very logical definition at least to me I do not find that definition very interesting.

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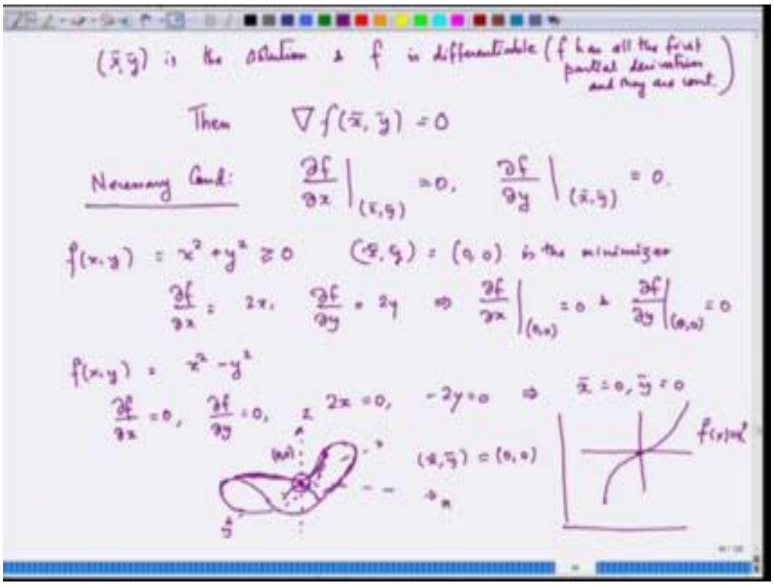

In that sense, every point will really become a saddle point. The true nature of the saddle if you look at the thing here this this is my x axis this is my x axis and this is my z axis. So, you see if I fix up the value. Take this point (0,0). So, at the point (0,0), this is very crucial point at this point (0,0) at the origin here, if I this is my origin, if I take, if I fix the value 0 and put it here put y equal to 0. If I put y equal to 0 then what is happening at x equal to 0, x square is getting the function x square is getting minimized. If I put now x equal to 0, then at (x, y) equal to 0, at y equal to 0 the function minus y square is getting maximized.

If you look at this function, so (Refer Time: 17:19) is the very, very important thing that you have, now put x equal to 0, x is now look at what is the nature of the function. Then then that is of this form minus y square and then the z value that nothing but z is, z becomes minus of y square. It is of this form and it is getting maximized at 0. It usually what happens is a saddle point in optimization is always said to be a point, where when you are fixing up the x value then it is maximized along y and if you fix up the y value it is minimized along x.

This is a very crucial point. The saddle point has to have that property that you fix the x, that is you put one of the value critical point values, and f of x and then as a function of y, it will get minimized as a maximized in the other critical value. And if you put a function of, if you put say y equal to for example, y equal to is if you put the value of y, the critical value of y then at the function of x it will get minimized at the critical value of x. And this is a very, very crucial point to understand; not the definition, where they have given that you have to look around and see whether it is at one point it is increasing; at one point it is decreasing, but the crucial fact is that this sort of thing is happening at a saddle point.

So, every pointy is really not a saddle point, when you are coming to functions of two variables. But when you are in functions of one variable, for example, like this f of f x is equal to x cube. So, point is either a local minima or global minima or point of this form, but the function is changing it is I mean a curve is changing it is shape. That this sort of points of a points of inflection and this mean a critical point has to be any of the three; local global minima or local global or maxima or a point of inflection it cannot be anything else. But such a thing can happen if your function, if where in the function of two variables.

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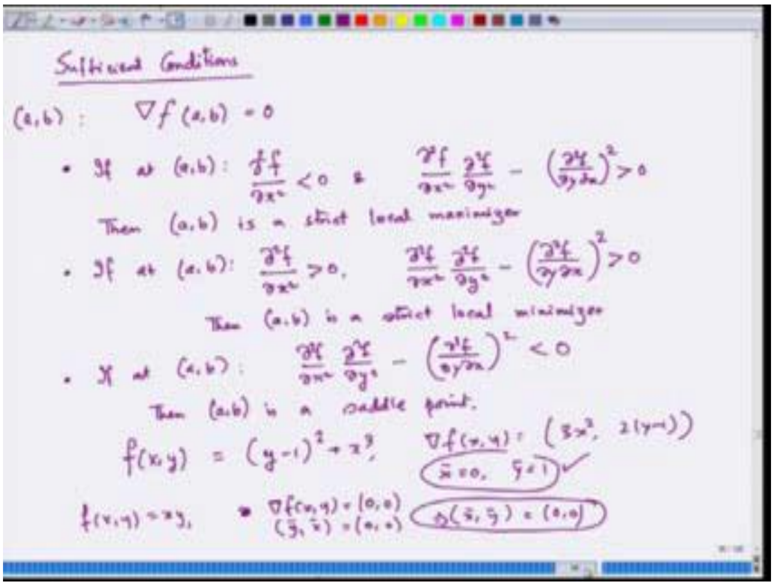

If where in the function of two variables see if you have a function of two variables take an example f of x, y is equal to y minus 1 whole square plus x cube. So, here if you do grad f (x, y) equal to grad f (x, y) you simply get 3 x square and 2 y minus 1 and so x bar is equal to 0 and y bar equal to 1 is the only critical point.

But this critical point suppose you put it is neither a local minima or a global minima or a local maxima or global maxima, now you put x bar equal to 0. So, you have y minus 1 square now. So, y minus 1 square does not get maximized at y equal, y bar equal to 1 it get minimized at y bar equal to 1. And if you put say x bar I mean y bar equal to 1 and you have x cube and you know that x cube does not get minimized at x bar equal to 0 we have just drawn the graph of x cube in the last page.

So, you see immediately that this point cannot be a saddle point. There are functions where a, where there are critical points and the only critical point, which are neither local minima nor local maxima or global minima global maxima or even a saddle point is just a critical point. This is the major difference between the study of maximization and minimization in one variable and to maximization and minimization in two variables and this is absolutely critical I am just giving you a saddle point ideas critical, I am sure that you can compute out these things and look at look at any examples from the book.

And if you have any questions my (Refer Time: 21:13) will answer you on the on the forum. But for example, if you take the function f (x, y) equal to (x, y), so, here x is, so if you take grad f (x, y) equal to 00. Then you will have y x, y bar x bar equal to (0,0) right. So, here what will you have? So, here what is the issue here you have y, here you have x. So, here also you have (0,0) as the critical point.

Now, what would happen if I put 0 you know very well if you draw this graph this graph is a slightly change strange graph; it is also, it does not again, you can see the picture in the book of Thomas' calculus. You will observe that at the point (0,0), if I put x equal to 0, then the function becomes 0. So, basically then it is a 0 function, it will be maximized at 0, it is a trivial thing.

But it is again it will be minimized if you put y equal to x equal to 0, then it will again be maximized at y, at y equal to 0; you can say I can put any value of y and that will that will maximize it, but no I am just looking at the point (0,0). So, at the point (0,0), it will be doing both the things, but if you put (0,0) it is 0, but if I put 0 minus 1, this will, if I put 1 and minus 1 it will have a negative value. So, you see this are not local maximizer or global maximizer, but is a saddle point.

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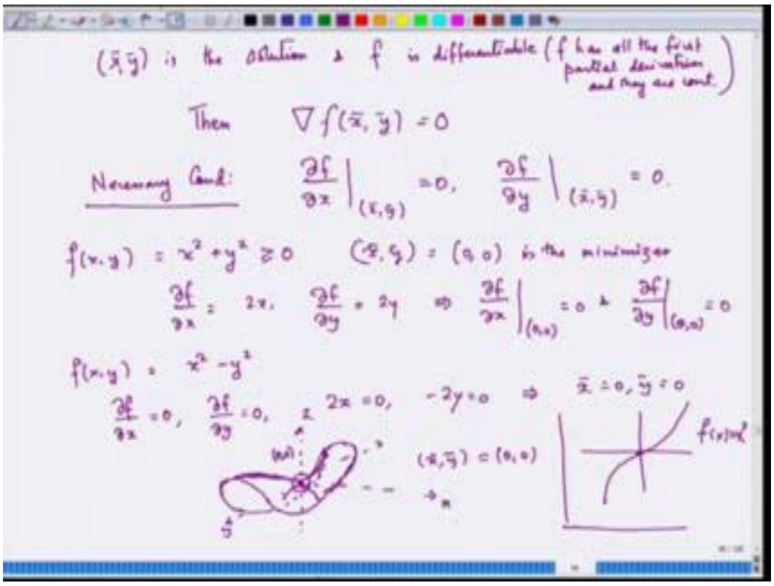

So, very important to keep in view, there for example here that this could be a critical point without being a saddle point, but this is an obvious saddle point. While in the earlier case here this saddle point had some more information. It was giving you more information about how it is not a, this is not a trivial saddle point. While the other one, this one, well this case f of x, y, f of f (x, y) is equal to (x, y), (0,0) is a trivial saddle point. But here again you have to concentrate on these example, which I have given where the only critical point is neither a saddle point nor a global or local maxima. So, we would end it here.

Thank you very much.